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Lifting to Passivity for $\mathcal{H}_2$-Gain-Scheduling Synthesis with Full Block Scalings ⋆

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Abstract: We focus on the $\mathcal{H}_2$-gain-scheduling synthesis problem for time-varying parametric scheduling blocks with scalings. Recently, we have presented a solution of this problem for D- and positive real scalings by guaranteeing finiteness of the $\mathcal{H}_2$-norm for the closed-loop system with suitable linear fractional plant and controller representations. In order to reduce conservatism, we extend these methods to full block scalings by designing a triangular scheduling function and by introducing a new lifting technique for gain-scheduled synthesis that enables convexification.

Keywords: Linear parameter-varying systems, Controller constraints and structure, Convex optimization

1. INTRODUCTION

The design of linear parametrically-varying (LPV) systems is widely spread over the control literature and can be roughly divided into two classes. On the one hand, parameter-dependent Lyapunov functions, as in Becker (1995), Wu et al. (1996), Apkarian and Adams (1997), Wu and Dong (2005), de Souza and Trofino (2006), and Sato (2011), are used for synthesis with linear matrix inequalities (LMIs) by approximating the parameter space of the scheduling variable. On the other hand, the so-called scaling approach can directly handle rational parameter dependence, as in Packard (1994), Apkarian and Gahinet (1995) for D-scalings, Helmersson (1998) for positive real-scalings, Scorletti and El Ghaoui (1998) for D/G-scalings, and Scherer (2000), Veenman and Scherer (2014) for the least conservative full block scalings. These approaches are as well of interest because of their link to distributed controller design (see Langbort et al. (2004)) and their flexibility for handling more complex scheduling blocks such as delays as considered in Rösinger and Scherer (2019).

In this work, we look at the concrete configuration in Fig. 1 which has shown to be well-suited for analysis and synthesis of LPV controllers (see Packard (1994), Apkarian and Gahinet (1995)). For an uncertain plant $G(\hat{\Delta})$ with $\hat{\Delta}$ being an arbitrary fast time-varying matrix-valued parametric uncertainty, we employ constant full block scalings to synthesize a controller $K(\hat{\Delta})$ which achieves an $\mathcal{H}_2$-cost criterion imposed on $w_p \to z_p$. Concrete applications of LPV design with $\mathcal{H}_2$-performance guarantees are, e.g., the control of autonomous cars and helicopters in Mustaki et al. (2019) and Guerreiro et al. (2007), respectively.
Recently, Rösinger and Scherer (2019) present the first scaling solution to this problem with D-scalings in case that the uncertainty takes values in the unit disk or with positive-real scalings in case that the uncertainty

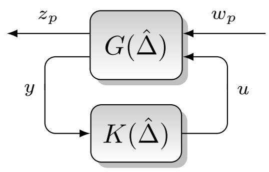

Fig. 1. Feedback-loop for gain-scheduling.

is passive. Technically, this approach uses a convexifying transformation for controller and scaling parameters based on Masubuchi et al. (1998), Scherer et al. (1997), while suitable structured plant and controller descriptions guarantee well-posedness for the closed-loop $\mathcal{H}_2$-norm by design. However, these results heavily rely on the particular structure of D- and positive real scalings and cannot be easily extended to the less conservative full block scalings.

As the main contribution of this work, we present a complete solution for the $\mathcal{H}_2$-gain scheduling problem with full block scalings in terms of LMIs. For this purpose, we introduce a new design approach based on what we call lifting to passivity. This amounts to a loss-less embedding of the original synthesis problem into a passivity framework involving a suitable structural extension (or lifting) of the plant and the controller, and is the enabling factor for being able to convexify the problem through a transformation that operates on both the controller and the scaling parameters. The use of a related passivation step has been beneficial already for a completely different objective in robustness analysis and synthesis involving integral quadratic constraints in Veenman and Scherer (2013), Veenman and Scherer (2014). As a novel feature of this paper, we develop a systematic approach for using such a procedure in the context of gain-scheduled synthesis. As a further contribution, we reveal how suitable structured plant and controller representations can be exploited in our designs to render the $\mathcal{H}_2$-norm finite.

Outline. After introducing the notation used in this work, Section 2 formulates the $\mathcal{H}_2$-gain scheduling problem under investigation, while Section 3 presents the lifting

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design technique. The resulting specifically structured design problem is solved in Section 4. Finally, a short example clarifies that our results are less conservative than those in Rösinger and Scherer (2019).

Notation. Let $\mathbb{S}^n$ denote the set of real symmetric matrices of dimension $n \times n$. For some matrices $M \in \mathbb{R}^{r\times s}$ and $P \in \mathbb{R}^{r\times r}$ we abbreviate M^TPM by $(*)^TPM$ and $P + P^T$ by $\mathrm{He}(P)$ and denote by $\mathrm{tr}(P)$ the trace of P. Matrix entries that can be inferred by symmetry are indicated by $*$. We drop superscripts specifying partitions and dimensions of matrices if they are clear from the context. Further, I and I_m denote identity matrices (with m specifying the dimension if not clear from the context) and $\mathrm{col}(u_1, u_2) := \begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T$ is used for vectors. If X, R, S and A_{ij}, B_i, C_j, D are some suitable matrices for $i, j = 1, 2$, we abbreviate

$$\mathcal{L}\Big(X, R, S, \left(\begin{smallmatrix} A_{ij} & B_i \\ C_j & D \end{smallmatrix}\right)\Big) := (*)^T \left(\begin{array}{c:c:c} X & 0 & 0 \\ \hdashline 0 & R & 0 \\ \hdashline 0 & 0 & S \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ A_{11} & A_{12} & B_1 \\ \hdashline 0 & I & 0 \\ A_{21} & A_{22} & B_2 \\ \hdashline 0 & 0 & I \\ C_1 & C_2 & D \end{array}\right)$$

and refer to its left upper sub-block as

$$\mathcal{L}_{\mathrm{sub}}\Big(X, R, S, \left(\begin{smallmatrix} A_{ij} \\ C_j \end{smallmatrix}\right)\Big) := (*)^T \left(\begin{array}{c:c:c} X & 0 & 0 \\ \hdashline 0 & R & 0 \\ \hdashline 0 & 0 & S \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A_{11} & A_{12} \\ \hdashline 0 & I \\ A_{21} & A_{22} \\ \hdashline 0 & 0 \\ C_1 & C_2 \end{array}\right).$$

2. PROBLEM FORMULATION

In the sequel, we introduce the $\mathcal{H}_2$-gain scheduling problem for full block scalings.

2.1 Structured plant and controller representations

For some full block time-varying uncertainty $\hat{\Delta}$ taking values in some polytope, let us consider the standard LPV configuration in Fig. 1 with a $\hat{\Delta}$-dependent LPV system $G(\hat{\Delta})$ and a corresponding controller $K(\hat{\Delta})$. To systematically guarantee finiteness of the closed-loop $\mathcal{H}_2$-norm, we use specifically structured linear fractional representations (LFRs) for $G(\hat{\Delta})$, $K(\hat{\Delta})$. Let $G(\hat{\Delta})$ be structured as in

$$\begin{pmatrix} \dot{x} \\ \hline z_p \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A(\hat{\Delta}) & B^p(\hat{\Delta}) & B^u(\hat{\Delta}) \\ \hline C^p(\hat{\Delta}) & 0 & D^u(\hat{\Delta}) \\ C^y(\hat{\Delta}) & D^y(\hat{\Delta}) & D(\hat{\Delta}) \end{array}\right) \begin{pmatrix} x \\ \hline w_p \\ u \end{pmatrix}, \tag{1}$$

with $D(0) = 0$, performance channel $w_p \to z_p$, control channel $u \to y$, and let us describe the controller $K(\hat{\Delta})$ by

$$\begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A^c(\hat{\Delta}) & B^c(\hat{\Delta}) \\ C^c(\hat{\Delta}) & 0 \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix} \tag{2}$$

such that all $\hat{\Delta}$-dependent operator blocks in (1), (2) are LFRs in $\hat{\Delta}$. Analogous to the approach for one repeated block in Rösinger and Scherer (2019), the zero block structures in (1), (2) guarantee that the performance channel in Fig. 1 has an identically vanishing direct feedthrough term. Since $w_p \to z_p$ is zero in (1), standard techniques for linear fractional transformations (LFTs) show that $G(\hat{\Delta})$ can be expressed as the LFR

$$\begin{pmatrix} \dot{x} \\ \hline \hat{z}_1 \\ \hline \hat{z}_2 \\ \hdashline z_p \\ y \end{pmatrix} = \left(\begin{array}{c|c:cc} A_{11} & \hat{A}_{12} & B_1^p & B_1 \\ \hline \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2^p & \hat{B}_2 \\ \hdashline C_1^p & \hat{C}_2^p & D^p & D_1 \\ C_1 & \hat{C}_2 & D_2 & D_3 \end{array}\right) \begin{pmatrix} x \\ \hline \hat{w}_1 \\ \hline \hat{w}_2 \\ \hdashline w_p \\ u \end{pmatrix} =$$

$$= \left(\begin{array}{c|c|c:cc} A_{11} & \bar{A}_{12} & \bar{A}_{13} & B_1^p & B_1 \\ \hline \bar{A}_{21} & \bar{A}_{22} & 0 & 0 & \bar{B}_2 \\ \hline \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & \bar{B}_3^p & \bar{B}_3 \\ \hdashline C_1^p & \bar{C}_2^p & 0 & 0 & D_1 \\ C_1 & \bar{C}_2 & \bar{C}_3 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline \hat{w}_1 \\ \hline \hat{w}_2 \\ \hdashline w_p \\ u \end{pmatrix}, \quad \hat{w} = \Delta\hat{z} \tag{3}$$

with matrices $A_{11} \in \mathbb{R}^{n^s\times n^s}$, $B_1 \in \mathbb{R}^{n^s\times m}$, $C_1 \in \mathbb{R}^{k\times n^s}$, as well as with a structured uncertainty channel $\hat{w} \to \hat{z}$ for $\hat{z} := \mathrm{col}(\hat{z}_1, \hat{z}_2)$ and $\hat{w} := \mathrm{col}(\hat{w}_1, \hat{w}_2)$; the matrices associated to $\hat{w} \to \hat{z}$ are indicated with the symbols $\wedge$ or $-$ in (3). W.l.o.g., the LFT manipulations can be always performed such that $\Delta = \mathrm{diag}(\hat{\Delta}, \hat{\Delta})$ has a diagonal structure which is compatible with the partition of $\hat{A}_{22}$. Since we only work with Δ in the sequel, we write $G(\Delta)$ for (3) and assume that $\Delta \in \mathbf{\Delta}$ where $\mathbf{\Delta} := C([0,\infty), \mathbf{V})$ is the corresponding class of full block time-varying uncertainties for some given value set $\mathbf{V} = \mathrm{Co}\{\Delta_1, \ldots, \Delta_N\} \ni 0$ represented as the convex hull of finitely many real matrices $\Delta_i \in \mathbb{R}^{\hat{u}\times\hat{v}}$. We hence consider (3) with $\Delta \in \mathbf{\Delta}$ as the precise mathematical description for (1).

As the zero block structure for $K(\hat{\Delta})$ in (2) resembles that in (1), the above LFT manipulations motivate to look at the following structured controller LFR

$$\begin{pmatrix} \dot{x}_c \\ \hline \hat{z}_{c,1} \\ \hline \hat{z}_{c,2} \\ \hdashline u \end{pmatrix} = \left(\begin{array}{c|c:c} A_{11}^c & A_{12}^c & B_1^c \\ \hline A_{21}^c & A_{22}^c & B_2^c \\ \hdashline C_1^c & C_2^c & D^c \end{array}\right) \begin{pmatrix} x_c \\ \hline \hat{w}_{c,1} \\ \hline \hat{w}_{c,2} \\ \hdashline y \end{pmatrix} =$$

$$= \left(\begin{array}{c|c|c:c} A_{11}^c & \bar{A}_{12}^c & \bar{A}_{13}^c & B_1^c \\ \hline \bar{A}_{21}^c & \bar{A}_{22}^c & 0 & 0 \\ \hline \bar{A}_{31}^c & \bar{A}_{32}^c & \bar{A}_{33}^c & \bar{B}_3^c \\ \hdashline C_1^c & \bar{C}_2^c & 0 & 0 \end{array}\right) \begin{pmatrix} x_c \\ \hline \hat{w}_{c,1} \\ \hline \hat{w}_{c,2} \\ \hdashline y \end{pmatrix}, \quad \hat{w}_c = \Delta_c(\Delta)\hat{z}_c \tag{4}$$

with $\hat{z}_c := \mathrm{col}(\hat{z}_{c,1}, \hat{z}_{c,2})$, $\hat{w}_c := \mathrm{col}(\hat{w}_{c,1}, \hat{w}_{c,2})$ and the matrices $A_{11}^c \in \mathbb{R}^{n^c\times n^c}$, $B_1^c \in \mathbb{R}^{n^c\times k}$, $C_1^c \in \mathbb{R}^{m\times n^c}$. We refer to (4) as $K(\Delta)$ in order to display the dependence on Δ. In order to have large enough flexibility in synthesis, we search for a lower block-triangular scheduling function

$$\Delta_c : \mathbf{V} \to \mathbb{R}^{r^c\times r^c} \quad \text{with} \quad \Delta_c(V) := \left(\begin{smallmatrix} \Delta_{11}^c(V) & 0 \\ \Delta_{21}^c(V) & \Delta_{22}^c(V) \end{smallmatrix}\right) \tag{5}$$

of partition $r^c := r_1^c + r_2^c$. Indeed, for such a triangular $\Delta_c(.)$, the controller LFR (4) still ensures the structure in (2). Note that $\Delta_c(\Delta)$ might depend in a nonlinear fashion on $\Delta \in \mathbf{\Delta}$, while the choice of r^c, n^c is part of the design problem. The closed-loop system for the plant (3) interconnected with (4) is then given by

$$\begin{pmatrix} \dot{x}_e \\ \hdashline \hat{z} \\ \hat{z}_c \\ \hdashline z_p \end{pmatrix} = \left(\begin{array}{c:c:c} \hat{\mathcal{A}}_{11} & \hat{\mathcal{A}}_{12} & \hat{\mathcal{B}}_1 \\ \hdashline \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22} & \hat{\mathcal{B}}_2 \\ \hdashline \hat{\mathcal{C}}_1 & \hat{\mathcal{C}}_2 & \hat{\mathcal{D}} \end{array}\right) \begin{pmatrix} x_e \\ \hdashline \hat{w} \\ \hat{w}_c \\ \hdashline w_p \end{pmatrix}, \; \begin{pmatrix} \hat{w} \\ \hat{w}_c \end{pmatrix} = \Delta_{ex}(\Delta) \begin{pmatrix} \hat{z} \\ \hat{z}_c \end{pmatrix} \tag{6}$$

with extended state $x_e := \mathrm{col}(x, x_c)$, extended scheduling block $\Delta_{ex}(V) := \left(\begin{smallmatrix} V & 0 \\ 0 & \Delta_c(V) \end{smallmatrix}\right)$, and suitable closed-loop matrices $\hat{\mathcal{A}}_{ij}$, $\hat{\mathcal{B}}_i$, $\hat{\mathcal{C}}_j$, $\hat{\mathcal{D}}$ for $i, j = 1, 2$.

Definition 1. The controlled system (6) is *well-posed* if $I - \Delta_{ex}(V)\hat{\mathcal{A}}_{22}$ is non-singular for all $V \in \mathbf{V}$. It is *stable* if there exist constants K and $\alpha > 0$ such that every solution of (6) which is obtained for $w_p = 0$ and any $\Delta \in \mathbf{\Delta}$ fulfills

$$\|x_e(t)\| \leq K e^{-\alpha(t-t_0)} \|x_e(0)\| \quad \text{for all} \quad t \geq 0.$$

If (6) is well-posed, we can close the loop with $\Delta_{ex}(\Delta)$ to get $\left(\begin{smallmatrix} \dot{x}_e \\ z_p \end{smallmatrix}\right) = \left(\begin{smallmatrix} \star & \star \\ \star & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_e \\ w_p \end{smallmatrix}\right)$ where the entries with $\star$ depend on Δ and $\Delta_c(\Delta)$; note that the structured LFRs (3), (4) imply (1), (2) which lead to the desired zero block for $w_p \to z_p$ to render the $\mathcal{H}_2$-norm finite. Hence, the *$\mathcal{H}_2$-gain-scheduling problem* involves a nontrivial structural requirement.

Problem 2. For a given bound $\gamma > 0$, determine a controller $K(\Delta)$ structured as in (4)-(5) such that

(G1) the controlled LFR (6) is well-posed and stable,
(G2) the squared $\mathcal{H}_2$-norm of $w_p \to z_p$ for linear time-varying systems (in the stochastic setting as in Paganini and Feron (2000)) is smaller than γ for $x_e(0) = 0$ and for all $\Delta \in \mathbf{\Delta}$.

2.2 Analysis conditions for the original system

As well-known by the full block S-procedure, the conditions (G1)-(G2) are achieved if some matrix inequalities are feasible. This is formulated in the following standard analysis result from Scherer (2000) based on the class $\hat{\mathbf{P}}$ of full block scalings $\hat{\mathcal{P}} \in \mathbb{S}^{(\hat{u}+r^c+\hat{v}+r^c)}$ satisfying

$$(*)^T \hat{\mathcal{P}} \begin{pmatrix} \Delta_{ex}(V) \\ I_{\hat{v}+r^c} \end{pmatrix} \succ 0 \quad \text{for all} \quad V \in \mathbf{V}. \tag{7}$$

Theorem 3. The design goals (G1)-(G2) are reached for the structured controller $K(\Delta)$ with (4)-(5) if there exist $\mathcal{X}_1 \succ 0$, $Z \succ 0$ with $\operatorname{tr}(Z) < 1$ as well as $\hat{\mathcal{P}} \in \hat{\mathbf{P}}$ such that

$$\begin{aligned} \mathcal{L}_{\text{sub}}\left(\left(\begin{smallmatrix} -\mathcal{X}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \hat{\mathcal{P}}, P_Z, \begin{pmatrix} \hat{\mathcal{A}}_{ij} \\ \hat{\mathcal{C}}_j \end{pmatrix}\right) &\prec 0, \\ \mathcal{L}\left(\left(\begin{smallmatrix} 0 & \mathcal{X}_1 \\ \mathcal{X}_1 & 0 \end{smallmatrix}\right), \hat{\mathcal{P}}, P_\gamma, \begin{pmatrix} \hat{\mathcal{A}}_{ij} & \hat{\mathcal{B}}_i \\ \hat{\mathcal{C}}_j & \hat{\mathcal{D}} \end{pmatrix}\right) &\prec 0 \end{aligned} \tag{8}$$

hold for the closed-loop system (6) with

$$P_Z := \left(\begin{smallmatrix} 0 & 0 \\ 0 & Z^{-1} \end{smallmatrix}\right), \quad P_\gamma := \left(\begin{smallmatrix} -\gamma I & 0 \\ 0 & 0 \end{smallmatrix}\right). \tag{9}$$

Since (8) involve two inequalities with specific outer factors and $\hat{\mathcal{P}}$ is unstructured, we cannot directly eliminate or substitute the controller parameters for convexification. In the sequel, we thus introduce a novel design procedure, while, in view of Scherer (2000), we anticipate the synthesis result to be formulated with the full block scaling class

$$\mathbf{P}_p := \left\{ P \in \mathbb{S}^{\hat{u}+\hat{v}} \;\middle|\; (*)^T P \left(\begin{smallmatrix} I_{\hat{u}} \\ 0 \end{smallmatrix}\right) \prec 0 \text{ and } (*)^T P \left(\begin{smallmatrix} V \\ I_{\hat{v}} \end{smallmatrix}\right) \succ 0 \text{ for all } V \in \mathbf{V} \right\} \tag{10}$$

related to Δ and the corresponding dual scaling class

$$\mathbf{P}_d := \left\{ \tilde{P} \in \mathbb{S}^{\hat{u}+\hat{v}} \;\middle|\; (*)^T \tilde{P} \left(\begin{smallmatrix} 0 \\ I_{\hat{v}} \end{smallmatrix}\right) \succ 0 \text{ and } (*)^T \tilde{P} \begin{pmatrix} I_{\hat{u}} \\ -V^T \end{pmatrix} \prec 0 \text{ for all } V \in \mathbf{V} \right\}. \tag{11}$$

3. LIFTING DESIGN PROCEDURE

If $\hat{\mathbf{P}}$ is restricted in Theorem 3 to the class of positive real scalings $\begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}$ satisfying the passivity condition related to (7), i.e. $\operatorname{He}[\delta Q] \succ 0$ for all real $\delta \geq 0$, the approach in Rösinger and Scherer (2019) shows that the anti-diagonal scaling block is a fundamental stumbling block for convexification by transformation. This motivates to replace the intractable inequalities (8) by a suitable, sufficient analysis condition for a certain class of passive scalings.

3.1 Lifted plant and closed-loop formulation

First, let us define a new LFR by reformulating the equations for $G(\Delta)$ in (3). Note that $\hat{w} = \Delta\hat{z}$ is equivalent to $\hat{w} = -\hat{w} + 2\Delta\hat{z}$ and thus to $w = \Delta_l(\Delta)z$ for $\Delta \in \mathbf{\Delta}$ where

$$w := z := \begin{pmatrix} \hat{w} \\ \hat{z} \end{pmatrix}, \quad \Delta_l(V) := \begin{pmatrix} -I_{\hat{u}} & 2V \\ 0 & I_{\hat{v}} \end{pmatrix} \quad \text{for } V \in \mathbf{V}. \tag{12}$$

Similarly, we can rearrange the matrices in (3) related to the uncertainty channel $\hat{w} \to \hat{z}$ to infer that (3) is true iff

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \hline z \\ \hdashline z_p \\ y \end{pmatrix} &= \left(\begin{array}{c|c:cc} A_{11} & A_{12} & B_1^p & B_1 \\ \hline A_{21} & A_{22} & B_2^p & B_2 \\ \hdashline C_1^p & C_2^p & D^p & D_1 \\ C_1 & C_2 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ \hdashline w_p \\ u \end{pmatrix} \\ &:= \left(\begin{array}{c|cc:cc} A_{11} & \hat{A}_{12} & 0 & B_1^p & B_1 \\ \hline 0 & I_{\hat{u}} & 0 & 0 & 0 \\ 2\hat{A}_{21} & 2\hat{A}_{22} & -I_{\hat{v}} & 2\hat{B}_2^p & 2\hat{B}_2 \\ \hdashline C_1^p & \hat{C}_2^p & 0 & D^p & D_1 \\ C_1 & \hat{C}_2 & 0 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ \hdashline w_p \\ u \end{pmatrix}, \quad w = \Delta_l(\Delta)z \end{aligned} \tag{13}$$

holds for $\Delta \in \mathbf{\Delta}$. This construction results in a specifically structured uncertainty channel $w \to z$ of dimension $(\hat{u} + \hat{v}) \times (\hat{u} + \hat{v})$; in the sequel, we abbreviate (13) by $G_l(\Delta)$ and refer to $G_l(\Delta)/\Delta_l(\Delta)$ as *lifted* LFR/*lifted block*.

For the lifted LFR (13), let us describe the associated controller $K(\Delta)$ again by (4)-(5) with the difference that $\Delta_c(.)$ is scheduled by $\Delta_l(\Delta)$ which, in general, leads to a larger size of the scheduling channel. For reasons of space, let use $\Delta_c(.)$ instead of $\Delta_c(\Delta_l(.))$ in the sequel.

By interconnecting (13) with (4), we get the closed-loop system

$$\begin{pmatrix} \dot{x}_e \\ \hdashline z \\ \hat{z}_c \\ \hdashline z_p \end{pmatrix} \left(\begin{array}{c:c:c} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1 \\ \hdashline \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{B}_2 \\ \hdashline \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \end{array}\right) \begin{pmatrix} x_e \\ \hdashline w \\ \hat{w}_c \\ \hdashline w_p \end{pmatrix}, \quad \begin{pmatrix} w \\ \hat{w}_c \end{pmatrix} = \Delta_{lc}(\Delta) \begin{pmatrix} z \\ \hat{z}_c \end{pmatrix} \tag{14}$$

with the corresponding scheduling block being defined as

$$\Delta_{lc}(V) := \begin{pmatrix} \Delta_l(V) & 0 \\ 0 & \Delta_c(V) \end{pmatrix} \in \mathbb{R}^{(r^s+r^c)\times(r^s+r^c)} \tag{15}$$

for some $V \in \mathbf{V}$ and for the relevant dimensions $n := n^s + n^c$, $r^s := \hat{u} + \hat{v}$, $r := r^s + r^c = (\hat{u}+\hat{v}) + (r_1^c + r_2^c)$; the closed-loop matrices can be routinely expressed as

$$\left(\begin{array}{c:c} \mathcal{A}_{ij} & \mathcal{B}_i \\ \hdashline \mathcal{C}_j & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc:c} A_{ij} & 0 & B_i^p \\ 0 & 0 & 0 \\ \hdashline C_j^p & 0 & D^p \end{array}\right) + \left(\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hdashline 0 & D_1 \end{array}\right) \begin{pmatrix} A_{ij}^c & B_i^c \\ C_j^c & D^c \end{pmatrix} \left(\begin{array}{cc:c} 0 & I & 0 \\ C_j & 0 & D_2 \end{array}\right).$$

3.2 Lifted analysis conditions with passive scaling classes

As a first observation, the scalings of $\mathbf{P}_p$, $\mathbf{P}_d$ in (10)-(11) already fulfill a passivity condition for the lifted block, i.e.

$$\begin{aligned} \mathbf{P}_p &= \left\{ P \in \mathbb{S}^{r^s} \;\middle|\; \operatorname{He}[P\Delta_l(V)] \succ 0 \text{ for all } V \in \mathbf{V} \right\}, \\ \mathbf{P}_d &= \left\{ \tilde{P} \in \mathbb{S}^{r^s} \;\middle|\; \operatorname{He}\left[\tilde{P}\Delta_l(V)^T\right] \succ 0 \text{ for all } V \in \mathbf{V} \right\}; \end{aligned} \tag{16}$$

this can be seen, e.g., for $\mathbf{P}_p$ by applying a congruence transformation with the invertible $\left(\begin{smallmatrix} I_{\hat{u}} & V \\ 0 & I_{\hat{v}} \end{smallmatrix}\right)$ to the condition $\operatorname{He}[P\Delta_l(V)] \succ 0$ for some $P \in \mathbb{S}^{\hat{u}+\hat{v}}$ and $V \in \mathbf{V}$. Secondly, if we replace V by the lifted block $\Delta_l(V)$, the extended block $\Delta_{ex}(V)$ from Section 2.1 becomes $\Delta_{lc}(V)$ in (15).

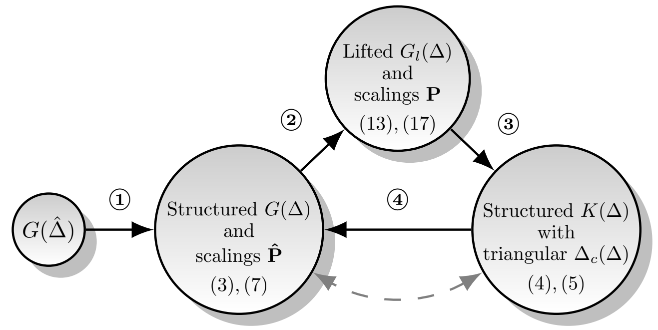

Fig. 2. Steps of lifting technique: Build plant LFR $G(\Delta)$ in ① and lifted plant LFR ②, design controller $K(\Delta)$ for lifted LFR ③ and interconnect with $G(\Delta)$ in ④.

Hence, this motivates to define an appropriate scaling class $\mathbf{P}$ for the lifted $\Delta_{lc}(V)$ by some passivity condition as

$$\mathbf{P} := \left\{\mathcal{P} \in \mathbb{S}^r \;\middle|\; \mathrm{He}[\mathcal{P}\Delta_{lc}(V)] \succ 0 \quad \text{for all} \quad V \in \mathbf{V}\right\}. \quad (17)$$

It will be insightful to see in Section 3.5 that the specific choice of $\mathbf{P}$ causes no restriction if compared to the full block scaling class of Scherer (2000). Moreover, it will be crucial to see that a solution for Problem 2 can be obtained by solving the $\mathcal{H}_2$-gain-scheduling problem for the lifted LFR. This is achieved by starting, analogously to Section 2.2, with the analysis inequalities

$$\begin{aligned} \mathcal{L}_{\mathrm{sub}}\Big(\big(\begin{smallmatrix}-\mathcal{X}_1 & 0\\ 0 & 0\end{smallmatrix}\big), \big(\begin{smallmatrix}0 & \mathcal{P}\\ \mathcal{P} & 0\end{smallmatrix}\big), P_Z, \big(\begin{smallmatrix}\mathcal{A}_{ij}\\ \mathcal{C}_j\end{smallmatrix}\big)\Big) &\prec 0,\\ \mathcal{L}\Big(\big(\begin{smallmatrix}0 & \mathcal{X}_1\\ \mathcal{X}_1 & 0\end{smallmatrix}\big), \big(\begin{smallmatrix}0 & \mathcal{P}\\ \mathcal{P} & 0\end{smallmatrix}\big), P_\gamma, \big(\begin{smallmatrix}\mathcal{A}_{ij} & \mathcal{B}_i\\ \mathcal{C}_j & \mathcal{D}\end{smallmatrix}\big)\Big) &\prec 0 \end{aligned} \quad (18)$$

for the controlled system (14) and scalings $\mathcal{P} \in \mathbf{P}$ with a passivity structure. As a crucial advantage over the original conditions for full block scalings in (8), we show that (18) can be indeed convexified.

3.3 Steps of lifting

Let us now summarize the concrete lifting design technique which is visualized in Fig. 2 and consists of four steps: For the first step ①, we have described in Section 2 the uncertain plant $G(\hat{\Delta})$ of (1) by the structured LFR $G(\Delta)$ in (3) in order to formulate the analysis conditions (8) with the class of full block scalings $\hat{\mathbf{P}}$. Next, we have performed the lifting step ② in Section 3.1 to obtain the lifted LFR $G_l(\Delta)$ in (13). In the synthesis step ③, presented in Section 4, we solve the associated $\mathcal{H}_2$-gain scheduling problem for the lifted LFR to obtain a structured controller $K(\Delta)$ with triangular $\Delta_c(\Delta)$ as in (4)-(5). For this purpose, we rely on the analysis inequalities (18) for the lifted LFR and use the passive scaling class $\mathbf{P}$. The last step ④ is given in Section 3.4 and clarifies that the constructed controller also solves the desired gain-scheduling Problem 2 for the original LFR $G(\Delta)$. Note that the design approach for positive real scalings in Rösinger and Scherer (2019) is only based on ① and the dashed grey lines, while ②-④ are the core novel synthesis steps for full block scalings.

3.4 Consequences for the original system

The following result covers step ④ of Fig. 2.

Theorem 4. Suppose there exist a structured controller $K(\Delta)$ with (4)-(5) as well as $\mathcal{X}_1 \succ 0$, $Z \succ 0$ with $\mathrm{tr}(Z) < 1$, $\mathcal{P} \in \mathbf{P}$ such that the closed-loop system (14) for the lifted LFR (13) fulfills (18) with P_Z, P_γ structured as in (9). Then we can construct a full block scaling $\hat{\mathcal{P}} \in \hat{\mathbf{P}}$ with (7) such that the inequalities (8) of Theorem 3 are true for the closed-loop system (6) obtained for the initial plant LFR (3) and the same controller $K(\Delta)$.

Proof. For some matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $Q \in \mathbb{S}$, $R \in \mathbb{S}$ and S of suitable dimension, we first observe that

$$\mathrm{He}\left[\begin{pmatrix}\mathbb{A}\\ \mathbb{C}\end{pmatrix}^T \begin{pmatrix}Q & S^T\\ S & R\end{pmatrix}\begin{pmatrix}\mathbb{A}\\ \mathbb{B}\end{pmatrix}\right] = (*)^T \begin{pmatrix}2Q & S^T & S^T\\ S & 0 & R\\ S & R & 0\end{pmatrix}\begin{pmatrix}\mathbb{A}\\ \mathbb{B}\\ \mathbb{C}\end{pmatrix}. \quad (19)$$

Now, let the analysis inequalities in (18) be satisfied for some $\mathcal{P} \in \mathbf{P}$ and for the lifted LFR interconnected with a given controller $K(\Delta)$. By the definition of $\mathbf{P}$, we infer $\mathrm{He}[\mathcal{P}\Delta_{lc}(V)] \succ 0$ for all $V \in \mathbf{V}$. Applying for each $V \in \mathbf{V}$ a congruence transformation with

$$\begin{pmatrix}V & 0\\ I_{\hat{v}} & 0\\ 0 & I_{r^c}\end{pmatrix} \quad \text{yields} \quad \mathrm{He}\left[\begin{pmatrix}V & 0\\ I_{\hat{v}} & 0\\ 0 & I_{r^c}\end{pmatrix}^T \mathcal{P}\begin{pmatrix}V & 0\\ I_{\hat{v}} & 0\\ 0 & \Delta_c(V)\end{pmatrix}\right] \succ 0$$

for all $V \in \mathbf{V}$. Next, let us partition $\mathcal{P}$ according to the outer factors of the latter inequality as

$$\mathcal{P} = \begin{pmatrix}Q & S^T\\ S & R\end{pmatrix} = \begin{pmatrix}Q_{11} & Q_{12} & S_1^T\\ Q_{21} & Q_{22} & S_2^T\\ S_1 & S_2 & R\end{pmatrix}$$

to conclude with (19) after a suitable permutation that

$$(*)^T \underbrace{\begin{pmatrix}2Q_{11} & S_1^T & 2Q_{12} & S_1^T\\ S_1 & 0 & S_2 & R\\ 2Q_{21} & S_2^T & 2Q_{22} & S_2^T\\ S_1 & R & S_2 & 0\end{pmatrix}}_{=:\hat{\mathcal{P}}}\begin{pmatrix}V & 0\\ 0 & \Delta_c(V)\\ I_{\hat{v}} & 0\\ 0 & I_{r^c}\end{pmatrix} \succ 0. \quad (20)$$

Thus $\hat{\mathcal{P}} \in \hat{\mathbf{P}}$. It is essential that the analysis inequalities (8) obtained for (3) and for the same $K(\Delta)$ are also valid for the constructed $\hat{\mathcal{P}}$ from (20). This follows by applying suitable congruence transformations to (18) along with (19); we need to omit the details for reasons of space. ■

3.5 Comparison of scaling classes

Let $\hat{\mathbf{P}}_{\mathbf{F}}$ be the full block scaling class used for gain-scheduling in Scherer (2000). Note that $\hat{\mathbf{P}}_{\mathbf{F}}$ is a subset of $\hat{\mathbf{P}}$ from Section 2.2 and consists of all scalings $\hat{\mathcal{P}} \in \mathbb{S}^{(\hat{u}+r^c+\hat{v}+r^c)}$ satisfying in addition to (7) the constraints

$$(*)^T\hat{\mathcal{P}}\big(\begin{smallmatrix}I_{\hat{u}+r^c}\\ 0\end{smallmatrix}\big) \prec 0 \quad \text{and} \quad (*)^T\hat{\mathcal{P}}\big(\begin{smallmatrix}0\\ I_{\hat{v}+r^c}\end{smallmatrix}\big) \succ 0. \quad (21)$$

We emphasize that it is not at all clear how to convexify the synthesis problem based on (8) for the class $\hat{\mathbf{P}}_{\mathbf{F}}$. Still, let us briefly sketch that the choice of the specifically structured scalings $\mathbf{P}$ in (17) causes no conservatism, i.e., if γ_F is the optimal bound obtained for (8) with $\hat{\mathbf{P}}_{\mathbf{F}}$, and γ_l denotes the one for synthesis based on (18) with the lifted LFR and $\mathbf{P}$, the relation $\gamma_l \leq \gamma_F$ always holds.

For this purpose, let us perform the lifting step in Section 3.1 both for the plant $G(\Delta)$ and for $K(\Delta)$. This leads to the lifted plant LFR $G_l(\Delta)$ in (13) as well as to a lifted controller LFR $K_l(\Delta)$ with a scheduling channel resembling the structure of those for $G_l(\Delta)$, while being scheduled by the structured $\Delta_l(\Delta_c(\Delta))$ with $\Delta_l(.)$ from

(12). Note that the resulting LFR of $K_l(\Delta)$ can be always obtained by a structural restriction of the LFR matrices for $K(\Delta)$. By exploiting the scaling properties (21), (7) imposed for $\hat{\mathbf{P}}_{\mathbf{F}}$, it is crucial to see that the original analysis inequalities (8) hold for some $\hat{\mathcal{P}} \in \hat{\mathbf{P}}_{\mathbf{F}}$ if and only if the modified analysis inequalities (18) are satisfied for the closed-loop system obtained from interconnecting $G_l(\Delta)$ with the lifted controller LFR $K_l(\Delta)$, and for some scaling $\mathcal{P} \in \mathbb{S}$ satisfying the passivity constraint

$$\mathrm{He}\Big[\mathcal{P}\Big(\begin{smallmatrix} \Delta_l(V) & 0 \\ 0 & \Delta_l(\Delta_c(V)) \end{smallmatrix}\Big)\Big] \succ 0 \quad \text{for all} \quad V \in \mathbf{V}. \qquad (22)$$

We omit the details for reasons of space, but remark that, upon permutation, $\mathcal{P}$ in (22) equals $\hat{\mathcal{P}}$. We observe that (22) is exactly the condition that appears for the passive scalings $\mathbf{P}$ in (17) if replacing $\Delta_l(\Delta_c(V))$ by $\Delta_c(V)$. Since the class of all LFRs for $K(\Delta)$ encompasses that of all LFRs for $K_l(\Delta)$ as argued above, we infer $\gamma_l \leq \gamma_F$.

4. SYNTHESIS FOR LIFTED SYSTEM

In the following part we deal with the synthesis step ③ in Fig. 2, i.e., we use a structured controller parameter transformation combined with a suitable scaling factorization to solve the $\mathcal{H}_2$-gain-scheduling problem for the lifted LFR. In the context of structured $\mathcal{H}_2$-design, a related factorization is established for positive definite matrices in Scherer (2014) to design triangular, time-invariant controllers, as well as for positive real matrices in Rösinger and Scherer (2019) to synthesize gain-scheduled controllers with a diagonal scheduling function of scalar parameters. Technically, we show as a novel step that the passivity condition for $\mathbf{P}$ in (17) can be used to derive a structured factorization for possibly indefinite scalings which is used to guarantee the existence of a block-triangular scheduling function $\Delta_c(.)$ for matrix parameters.

Before formulating the main synthesis result, we present the corresponding variables which consist of the matrices $X_1, Y_1 \in \mathbb{S}^{n^s}$. Further we take

$$X_2 = \begin{pmatrix} Q_2 & Q_3 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} \tilde{Q}_1 & I_{r^s} \end{pmatrix} \qquad (23)$$

of dimension $r^s \times (r^s + r^s)$ with $Q_2, Q_3 \in \mathbf{P}_p$, $\tilde{Q}_1 \in \mathbf{P}_d$ of dimension $r^s \times r^s$ where the sets $\mathbf{P}_p$ and $\mathbf{P}_d$ are given in (16). Moreover, for a compact notation, we use

$$\left(\begin{array}{c:c:c} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \boldsymbol{L}_1 \\ \hdashline \boldsymbol{K}_{21} & \boldsymbol{K}_{22} & \boldsymbol{L}_2 \\ \hdashline \boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{N} \end{array}\right) := \left(\begin{array}{c:cc:c} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{L}_1 \\ \hdashline \bar{K}_{21} & \bar{K}_{22} & Q_2^T A_{22} & 0 \\ \bar{K}_{31} & \bar{K}_{32} & \bar{K}_{33} & \bar{L}_3 \\ \hdashline \bar{M}_1 & \bar{M}_2 & 0 & 0 \end{array}\right) \qquad (24)$$

of the partition $(n^s+(r^s+r^s)+m)\times(n^s+(r^s+r^s)+k)$ which includes Q_2 from (23) and the unstructured variables $\bar{K}_{ij}$, $\bar{L}_i$, $\bar{M}_j$. This leads to the following $\mathcal{H}_2$-gain scheduling synthesis result.

Theorem 5. Let $\gamma > 0$ be fixed. There exists a structured controller with triangular scheduling function $\Delta_c(.)$ as in (4)-(5) and some $\mathcal{X}_1 \succ 0$, $\mathcal{P} \in \mathbf{P}$, $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ such that the inequalities (18) (with (9)) hold for the closed-loop system (14) iff there exist $X_1, Y_1 \in \mathbb{S}^{n^s}$, structured X_2, Y_2 from (23), $\bar{K}_{ij}$, $\bar{L}_i$, $\bar{M}_j$ with (24), and some $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ such that

$$\begin{aligned} \mathcal{L}_{\mathrm{sub}}\Big(\big(\begin{smallmatrix} -\boldsymbol{X} & 0 \\ 0 & 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\big), P_Z, \Big(\begin{smallmatrix} \boldsymbol{A}_{ij} \\ \boldsymbol{C}_j \end{smallmatrix}\Big)\Big) \prec 0, \\ \mathcal{L}\Big(\big(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\big), \big(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\big), P_\gamma, \Big(\begin{smallmatrix} \boldsymbol{A}_{ij} & \boldsymbol{B}_i \\ \boldsymbol{C}_j & \boldsymbol{D} \end{smallmatrix}\Big)\Big) \prec 0 \end{aligned} \qquad (25)$$

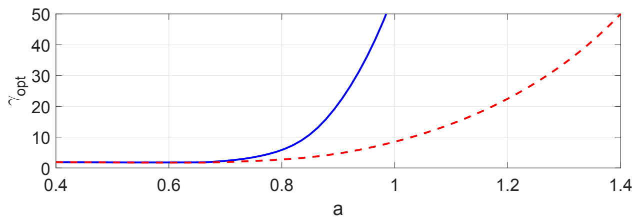

Fig. 3. Optimal bounds γ_{opt} for the lifted design (dashed red) and D/G-scalings (full blue) with $a \in [0.4, 1.4]$.

are satisfied after inserting for $i, j = 1, 2$ the blocks

$$\boldsymbol{X} := \begin{pmatrix} Y_1 & I_{n^s} \\ I_{n^s} & X_1 \end{pmatrix},$$

$$\left(\begin{array}{c:c} \boldsymbol{A}_{ij} & \boldsymbol{B}_i \\ \hdashline \boldsymbol{C}_j & \boldsymbol{D} \end{array}\right) := \left(\begin{array}{cc:c} A_{ij}Y_j & A_{ij} & B_i^p \\ 0 & X_i^T A_{ij} & X_i^T B_i^p \\ \hdashline C_j^p Y_j & C_j^p & D^p \end{array}\right) + \qquad (26)$$

$$+ \left(\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hdashline 0 & D_1 \end{array}\right)\begin{pmatrix} \boldsymbol{K}_{ij} & \boldsymbol{L}_i \\ \boldsymbol{M}_j & \boldsymbol{N} \end{pmatrix}\left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & C_j & D_2 \end{array}\right).$$

Since $\mathbf{V} = \mathrm{Co}\{\Delta_1, \ldots, \Delta_N\}$ and the sets $\mathbf{P}_p$, $\mathbf{P}_d$ can be expressed as in (10), (11), the conditions $Q \in \mathbf{P}_p$, $\tilde{Q} \in \mathbf{P}_d$ reduce to finitely many inequalities (see Scherer (2000)):

$$(*)^T Q\big(\begin{smallmatrix} I \\ 0 \end{smallmatrix}\big) \prec 0, \qquad (*)^T Q\big(\begin{smallmatrix} \Delta_i \\ I \end{smallmatrix}\big) \succ 0,$$
$$(*)^T \tilde{Q}\big(\begin{smallmatrix} 0 \\ I \end{smallmatrix}\big) \succ 0, \qquad (*)^T \tilde{Q}\Big(\begin{smallmatrix} I \\ -\Delta_i^T \end{smallmatrix}\Big) \prec 0 \quad \text{for} \quad i = 1, \ldots, N.$$

After applying the Schur complement to (25), we get a standard LMI test with finitely many constraints such that a direct minimization over γ is possible. We present the proof of Theorem 5 in Appendix A. Note that our proof is constructive, i.e., if the associated LMIs are feasible, a suitable $\mathcal{H}_2$-controller (4)-(5) can be constructed with McMillan degree of at most n^s and scheduling block size r^c of at most $2r^s$, while we give an explicit formula for $\Delta_c(.)$.

Remark 6. Analogously to Remark 5 and 6 in Rösinger and Scherer (2019), Theorem 5 can also handle gain-scheduling with quadratic performance and multiple objectives by properly modifying P_γ. Also $\bar{K}_{11}, \bar{K}_{12}, \bar{K}_{13}, \bar{L}_1$ can be partially eliminated to reduce the number of variables.

5. A NUMERICAL EXAMPLE

To present a short academic example, let the matrices of the structured LFR in (3) be given as in Section 4.2 of Rösinger and Scherer (2019) with $\hat{A}_{12}$ depending on some parameter $a \in [0.4, 1.4]$. Moreover, let $\Delta = \mathrm{diag}(\delta_1 I_2, \delta_2)$ be of size 3×3 with time-varying parametric uncertainties $\delta_1(t) \in [-0.8, 0.8]$, $\delta_2(t) \in [-0.6, 0.6]$. Based on implementations of our algorithms in the Matlab Robust Control Toolbox, we compare in Fig. 3 the optimal bounds γ_{opt} of the squared $\mathcal{H}_2$-norm for the lifted design (dashed red) obtained for the passive scaling class $\mathbf{P}$ from (17) with D/G-scalings (full blue). Note that $\mathcal{H}_2$-gain-scheduling synthesis for D/G-scalings with structured LFRs can be performed with the positive real scaling results from Rösinger and Scherer (2019) for the original LFR (3) by using the well-known Möbius transformation to map the uncertainty intervals for δ_i into $[0, \infty]$. To the best knowledge of the authors, there exist no alternative approaches that solve the underlying structured $\mathcal{H}_2$-design problem in this generality. The results confirm that the lifted approach is less conservative than D/G-scalings as expected from Section 3.5. In particular, beyond the shown parameter range for a, the synthesis LMIs get infeasible for D/G-scalings if a approaches 1.67, while the lifted design is feasible up to $a = 2.17$.

6. CONCLUSION AND OUTLOOK

In this work, we have introduced a new lifting technique to synthesize controllers for the $\mathcal{H}_2$-gain-scheduling problem with full block scalings. Especially, our design framework guarantees finiteness of the closed-loop $\mathcal{H}_2$-norm by relying on structured plant and controller LFRs, and by constructing a block-triangular scheduling function. We hope that these new methodologies offer manifold potential for refined synthesis results as the combination with parameter-dependent Lyapunov functions. A further task is the investigation of possible numerical advantages of the used scaling extension over existing approaches.

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Appendix A. PROOF OF THEOREM 5

Necessity. Let (18) be satisfied for (14), $\mathcal{X}_1 \succ 0$, $Z \succ 0$ with $\mathrm{tr}(Z) < 1$, and $\mathcal{X}_2 := \mathcal{P} \in \mathbf{P}$, i.e.

$$\begin{aligned} \mathcal{L}_{\mathrm{sub}}\left(\left(\begin{smallmatrix} -\mathcal{X}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), P_Z, \left(\begin{smallmatrix} \mathcal{X}_i\mathcal{A}_{ij} \\ \mathcal{C}_j \end{smallmatrix}\right)\right) \prec 0, \\ \mathcal{L}\left(\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), P_\gamma, \left(\begin{smallmatrix} \mathcal{X}_i\mathcal{A}_{ij} & \mathcal{X}_i\mathcal{B}_i \\ \mathcal{C}_j & \mathcal{D} \end{smallmatrix}\right)\right) \prec 0. \end{aligned} \tag{A.1}$$

Step 1 (*Factorizations*).
W.l.o.g., let us assume that $n^c \geq n^s$ to factorize $\mathcal{X}_1$ as

$$\mathcal{X}_i\mathcal{Y}_i = \mathcal{Z}_i \quad \text{with} \quad \mathcal{Y}_i := \begin{pmatrix} Y_i & I \\ V_i & 0 \end{pmatrix}, \; \mathcal{Z}_i := \begin{pmatrix} I & X_i \\ 0 & U_i \end{pmatrix} \tag{A.2}$$

for $i = 1$ such that $\mathcal{Y}_1$ has full column rank (see Scherer et al. (1997)).
Moreover, if we assume that $r_1^c \geq r^s$ and $r_2^c \geq r^s$, let us show that $\mathcal{X}_2$ can be also factorized as in (A.2) such that $\mathcal{Y}_2$ has full column rank where V_2 and U_2 are lower and upper block-triangular matrices, respectively, with respect to the partition $(r_1^c + r_2^c) \times (r^s + r^s)$, and where X_2, Y_2 are partitioned as in (23) for some suitable blocks Q_2, Q_3, $\tilde{Q}_1$.
For this purpose, let us first clarify that $\mathcal{X}_2 \in \mathbf{P}$ is invertible with some sub-blocks of full column rank, while we use the following partitions according to $r = r^s + r_1^c + r_2^c$:

$$\mathcal{X}_2 = \left(\begin{array}{c:cc} Q_3 & S_{13}^T & S_{23}^T \\ \hdashline S_{13} & R_{11} & R_{21}^T \\ S_{23} & R_{21} & R_{22} \end{array}\right), \; \mathcal{X}_2^{-1} = \left(\begin{array}{c:cc} \tilde{Q}_1 & \tilde{S}_{11}^T & \tilde{S}_{21}^T \\ \hdashline \tilde{S}_{11} & \tilde{R}_{11} & \tilde{R}_{21}^T \\ \tilde{S}_{21} & \tilde{R}_{21} & \tilde{R}_{22} \end{array}\right). \tag{A.3}$$

For the given partition of $\mathcal{X}_2$ in (A.3), we note that S_{13}, S_{23} are tall due to $r_j^c \geq r^s$ for $j = 1, 2$. Let us firstly perturb R_{11}, R_{21}, R_{22} to achieve invertibility of R_{22} and $\left(\begin{smallmatrix} R_{11} & R_{21}^T \\ R_{21} & R_{22} \end{smallmatrix}\right)$. This allows to perturb S_{13}, S_{23}, Q_3 such that

$$H := -\begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{21}^T \\ R_{21} & R_{22} \end{pmatrix}^{-1}\begin{pmatrix} S_{13} \\ S_{23} \end{pmatrix},\ \tilde{S}_{22} := -R_{22}^{-1}S_{23} \tag{A.4}$$

have full column rank and $Q_3 - \begin{pmatrix} S_{13}^T & S_{23}^T \end{pmatrix}\left(\begin{smallmatrix} R_{11} & R_{21}^T \\ R_{21} & R_{22} \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} S_{13} \\ S_{23} \end{smallmatrix}\right)$ is invertible. In particular, this implies invertibility of $\mathcal{X}_2$. Immediately, we infer that (A.2) is true for $i = 2$ with

$$\begin{pmatrix} X_2 \\ \hdashline U_2 \end{pmatrix} := \begin{pmatrix} Q_2 & Q_3 \\ \hdashline S_{12} & S_{13} \\ 0 & S_{23} \end{pmatrix}, \quad \begin{pmatrix} Y_2 \\ \hdashline V_2 \end{pmatrix} := \begin{pmatrix} \tilde{Q}_1 & I_{r^s} \\ \hdashline \tilde{S}_{11} & 0 \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix} \tag{A.5}$$

where $Q_2 := Q_3 - S_{23}^T R_{22}^{-1} S_{23}$ and $S_{12} := S_{13} - R_{21}^T R_{22}^{-1} S_{23}$. By the block-inversion formula, we note that $\tilde{Q}_1$ is invertible which, combined with (A.4), reveals that $\tilde{S}_{11} = H\tilde{Q}_1$ and $\tilde{S}_{22}$ have full column rank. Thus, V_2 has full column rank which implies the same for $\mathcal{Y}_2$ in (A.2).

Step 2 (*Proof that Q_2, $Q_3 \in \mathbf{P}_p$, $\tilde{Q}_1 \in \mathbf{P}_d$*).
For brevity, let us omit the argument of $\Delta_l(.)$, $\Delta_c(.)$ and $\Delta_{lc}(.)$. Further, let us split Δ_{lc} into two parts such that

$$0 \prec \mathrm{He}[\mathcal{X}_2\Delta_{lc}] = \mathrm{He}\left[\mathcal{X}_2\begin{pmatrix} \Delta_l & 0 \\ 0 & 0 \end{pmatrix}\right] + \mathrm{He}\left[\mathcal{X}_2\begin{pmatrix} 0 & 0 \\ 0 & \Delta_c \end{pmatrix}\right]. \tag{A.6}$$

Let us perform a congruence transformation with $\mathcal{Y}_2$ on (A.6) while using (A.2) for $i = 2$ and (A.5). This leads to

$$\begin{aligned} 0 \prec &\left(\begin{array}{cc:c} \mathrm{He}\left[\tilde{Q}_1\Delta_l^T\right] & * & * \\ \hdashline Q_2\Delta_l\tilde{Q}_1 + \Delta_l^T & \mathrm{He}[Q_2\Delta_l] & * \\ Q_3\Delta_l\tilde{Q}_1 + \Delta_l^T & Q_3\Delta_l + \Delta_l^T Q_2 & \mathrm{He}[Q_3\Delta_l] \end{array}\right) + \\ &+ \mathrm{He}\left[\begin{pmatrix} 0 \\ \hdashline U_2^T \end{pmatrix}\Delta_c\begin{pmatrix} V_2 & \vdots & 0 \end{pmatrix}\right]. \end{aligned} \tag{A.7}$$

Since U_2^T, V_2, Δ_c are lower block-triangular, the diagonal entries of (A.7) just read as Q_2, $Q_3 \in \mathbf{P}_p$, $\tilde{Q}_1 \in \mathbf{P}_d$.

Step 3 (*Derivation of synthesis inequalities (25)*).
Let us use the factorizations in (A.2) to apply congruence transformations with $\mathcal{Y}_i$ to (A.1) for $i = 1, 2$. We get

$$\begin{aligned} &\mathcal{L}_{\mathrm{sub}}\left(\begin{pmatrix} -\mathcal{Z}_1^T\mathcal{Y}_1 & 0 \\ 0 & 0 \end{pmatrix}, \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), P_Z, \begin{pmatrix} \mathcal{Z}_i^T\mathcal{A}_{ij}\mathcal{Y}_j \\ \mathcal{C}_j\mathcal{Y}_j \end{pmatrix}\right) \prec 0, \\ &\mathcal{L}\left(\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right), P_\gamma, \begin{pmatrix} \mathcal{Z}_i^T\mathcal{A}_{ij}\mathcal{Y}_j & \mathcal{Z}_i^T\mathcal{B}_i \\ \mathcal{C}_j\mathcal{Y}_j & \mathcal{D} \end{pmatrix}\right) \prec 0. \end{aligned} \tag{A.8}$$

By matching (A.8) to (25), the necessity part can then be finished similarly to Rösinger and Scherer (2019): By symmetry, $\mathcal{Z}_1^T\mathcal{Y}_1$ equals $\boldsymbol{X}$ from (26). Further, some calculations reveal that

$$\begin{aligned} \left(\begin{array}{c:c} \mathcal{Z}_i^T\mathcal{A}_{ij}\mathcal{Y}_j & \mathcal{Z}_i^T\mathcal{B}_i \\ \hdashline \mathcal{C}_j\mathcal{Y}_j & \mathcal{D} \end{array}\right) = &\left(\begin{array}{cc:c} A_{ij}Y_j & A_{ij} & B_i^p \\ 0 & X_i^T A_{ij} & X_i^T B_i^p \\ \hdashline C_j^p Y_j & C_j^p & D^p \end{array}\right) + \\ &+ \begin{pmatrix} 0 & B_i \\ I & 0 \\ \hdashline 0 & D_1 \end{pmatrix}\begin{pmatrix} K_{ij} & L_i \\ M_j & N \end{pmatrix}\left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & C_j & D_2 \end{array}\right) \end{aligned}$$

for $i, j = 1, 2$ after performing the substitution

$$\begin{aligned} \left(\begin{array}{c:c} K_{ij} & L_i \\ \hdashline M_j & N \end{array}\right) := &\left(\begin{array}{c:c} X_i^T A_{ij} Y_j & 0 \\ \hdashline 0 & 0 \end{array}\right) + \\ &+ \begin{pmatrix} U_i^T & X_i^T B_i \\ \hdashline 0 & I_m \end{pmatrix}\begin{pmatrix} A_{ij}^c & B_i^c \\ C_j^c & D^c \end{pmatrix}\left(\begin{array}{c:c} V_j & 0 \\ C_j Y_j & I_k \end{array}\right). \end{aligned} \tag{A.9}$$

Moreover, by exploiting the sparsity structure of the controller matrices and U_2, V_2, we can introduce

$$\left(\begin{array}{c:cc:c} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{L}_1 \\ \hdashline \bar{K}_{21} & \bar{K}_{22} & 0 & 0 \\ \bar{K}_{31} & \bar{K}_{32} & \bar{K}_{33} & \bar{L}_3 \\ \hdashline \bar{M}_1 & \bar{M}_2 & 0 & 0 \end{array}\right) := \left(\begin{array}{c:c:c} K_{11} & K_{12} & L_1 \\ \hdashline K_{21} & K_{22} - \left(\begin{smallmatrix} 0 & Q_2^T A_{22} \\ 0 & 0 \end{smallmatrix}\right) & L_2 \\ \hdashline M_1 & M_2 & 0 \end{array}\right) \tag{A.10}$$

which shows that (A.8) can be rewritten as (25) for (26).

Sufficiency. Let the inequalities in (25) be satisfied for (26) which comprises X_1, $Y_1 \in \mathbb{S}^{n^s}$, structured X_2, Y_2 from (23), $\bar{K}_{ij}$, $\bar{L}_i$, $\bar{M}_j$ with (24), and $Z \succ 0$ with $\mathrm{tr}(Z) < 1$.

Step 1 (*Construction of $\mathcal{X}_1$ and $\mathcal{X}_2$*).
To define $\mathcal{Y}_1$, $\mathcal{Z}_1$ by (A.2), we choose $U_1 := I_{n^s}$, $V_1 := I_{n^s} - X_1^T Y_1$. Hence, $\mathcal{Z}_1^T\mathcal{Y}_1 = \boldsymbol{X}$ and, since $\boldsymbol{X} \succ 0$ holds by (25), the matrices U_1, V_1 are invertible which implies the same for $\mathcal{Y}_1$, $\mathcal{Z}_1$. Thus (A.2) holds for $i = 1$ with $\mathcal{X}_1 := \mathcal{Z}_1\mathcal{Y}_1^{-1}$. To find some suitable $\mathcal{X}_2$, we can achieve invertibility of

$$T_1 := Q_2 - \tilde{Q}_1^{-1} \quad \text{and} \quad T_2 := Q_3 - Q_2$$

by perturbation. For any invertible matrices S_{13}, S_{23}, let

$$R_{21} = 0 \quad \text{and} \quad R_{11} := S_{13}T_1^{-1}S_{13}^T, \quad R_{22} := S_{23}T_2^{-1}S_{23}^T.$$

This shows the validity of $Q_2 = Q_3 - S_{23}^T R_{22}^{-1} S_{23}$ and

$$Q_3 - \begin{pmatrix} S_{13}^T & S_{23}^T \end{pmatrix}\begin{pmatrix} R_{11} & R_{21}^T \\ R_{21} & R_{22} \end{pmatrix}^{-1}\begin{pmatrix} S_{13} \\ S_{23} \end{pmatrix} = \tilde{Q}_1^{-1}.$$

Therefore, if we define $\mathcal{X}_2$ by the first relation in (A.3), the block-inversion formula reveals that $\mathcal{X}_2$ is invertible with its inverse satisfying the second relation in (A.3) for some suitable $\tilde{S}_{11}$, $\tilde{S}_{21}$, $\tilde{R}_{11}$, $\tilde{R}_{21}$, $\tilde{R}_{22}$. Further, let us take $\tilde{S}_{22} := -R_{22}^{-1}S_{23}$, $S_{12} := S_{13}$ to define U_2, V_2 by (A.5) and $\mathcal{Y}_2$, $\mathcal{Z}_2$ by (A.2). Hence, (A.2) is true for $i = 2$. Moreover, we identify $\tilde{S}_{11} = -S_{13}^{-T}T_1\tilde{Q}_1$ which shows that U_2, V_2 are invertible matrices having the right triangular structure. In particular, this shows that $\mathcal{Y}_2$ is invertible.

Step 2 (*Formula for the triangular Δ_c*).
For reasons of space we drop the argument of $\Delta_c(.)$ and $\Delta_l(.)$. Motivated by the necessity part, the goal is to find a suitable triangular Δ_c structured as in (5) such that (A.7) is true. We directly infer positive definiteness of the diagonal blocks in (A.7) since Q_2, $Q_3 \in \mathbf{P}_p$ and $\tilde{Q}_1 \in \mathbf{P}_d$. Thus, an explicitly formula for Δ_c can be obtained by rendering in (A.7) the off-diagonal blocks zero. Recall that U_2, V_2 are invertible, block-triangular matrices by construction which leads to the choice

$$\Delta_c := -U_2^{-T}\begin{pmatrix} Q_2\Delta_l\tilde{Q}_1 + \Delta_l^T & 0 \\ Q_3\Delta_l\tilde{Q}_1 + \Delta_l^T & Q_3\Delta_l + \Delta_l^T Q_2 \end{pmatrix}V_2^{-1}.$$

By reversing the congruence transformation with $\mathcal{Y}_2$ in the necessity part, (A.7) implies (A.6) and thus $\mathcal{X}_2 \in \mathbf{P}$.

Step 3 (*Construction of controller matrices*).
Let us now define K_{ij}, L_i, M_j by (A.10) and $N := \boldsymbol{N} = 0$. Since U_i, V_i are invertible for $i = 1, 2$, we can solve (A.9) for A_{ij}^c, B_i^c, C_j^c, D^c; these controller matrices have indeed the desired structure of (4) as can be seen analogously to Rösinger and Scherer (2019) by exploiting the structure of U_2, V_2 and (24). Hence, (A.8) is true and, by applying congruence transformations with $\mathcal{Y}_i^{-1}$ for $i = 1, 2$ along with the factorizations (A.2), we thus infer (18) for (14). ■