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ISS Small-Gain Theorem for Networked Discrete-Time Switching Systems ⋆

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Abstract: In this paper it is proved that a networked discrete-time switching system, equipped with a given switches digraph, is input-to-state stable, provided that there exist multiple Lyapunov functions (one for each mode) for each subsystem in the network, satisfying suitable standard inequalities, and provided that a set of suitable vector small-gain conditions are satisfied. The small-gain theorem here provided for the input-to-state stability takes into account the switches digraph. That is, the less is the number of edges in the switches digraph, the less is the number of involved Lyapunov inequalities and small-gain conditions which, if satisfied, guarantee the input-to-state stability of the entire switching system under study. The multiple Lyapunov functions for the entire system, guaranteeing the input-to-state stability, are determined by the multiple Lyapunov functions for each subsystem in the family. To the author's best knowledge, this is the first paper in the literature concerning small-gain theorems for the input-to-state stability of nonlinear discrete-time switching systems with given switches digraphs.

Keywords: Discrete-Time Switching Systems, Global Asymptotic Stability, Input-to-State Stability, Switches Digraphs.

1. INTRODUCTION

Switching systems have been extensively studied in the literature (see Liberzon, 2003, Liberzon & Morse, 1999, Sun & Ge, 2011, and the references therein). Sufficient Lyapunov conditions for the input-to-state stability of discrete-time switching systems are available in the literature (see Kundu & Chatterjee, 2016, Lian et al., 2017, Liu et al. 2016, and the references therein). In (Kundu & Chatterjee, 2016), under suitable Lyapunov conditions on each subsystem of the family, which allow for non input-to-state stable cases, a walk strategy on the switches digraph is studied such that the overall switching system is input-to-state stable. Stability issues for discrete-time switching systems with constraints in the switching-dwelling signal are studied in (Geromel & Colaneri, 2006, Kozyakin, 2014, Lu et al., 2018, Pepe, 2019, Philippe et al., 2016, Zhang et al., 2014). In particular, in (Pepe, 2019) the input-to-state stability property for discrete-time systems with given switches digraph is characterized by necessary and sufficient conditions by multiple Lyapunov functions. The switches digraph allows to reduce the number of involved necessary and sufficient Lyapunov inequalities, as selected by self-loops on modes (which are allowed to dwell on) and directed edges between modes (wich are allowed to switch from one to another). Small-gain theorems for continuous-time switching systems are available in the literature (see Dashkovskiy & Pavlichkov, 2017, Long, 2017, Long & Zhao, 2014, Yang & Liberzon, 2015). As far as the discrete-time case (see Jiang et al., 2004, Jiang & Wang, 2001 for non-switching nonlinear systems) is concerned, to our best knowledge, the small-gain methodology based on multiple Lyapunov functions for nonlinear switching systems with given switches digraph has not been exhaustively developed in the literature. The aim of this paper is to fill this gap. In this paper it is proved that a discrete-time switching system, with given switches digraph, is input-to-state stable, provided that there exist multiple Lyapunov functions, one for each mode (see Branicky, 1998, Daafouz et al., 2002, Jungers et al., 2017, Hetel et al., 2008, Pepe, 2019), for each subsystem in the network, satisfying suitable standard inequalities, and provided that a set of suitable vector small-gain conditions are satisfied. We remark that the small-gain theorem here provided for the input-to-state stability takes into account of the switches digraph. That is, the less is the number of edges in the switches digraph, the less is the number of involved Lyapunov inequalities and small-gain conditions which, if satisfied, guarantee the input-to-state stability at study. The small-gain conditions here provided are the discrete-time switching counterpart of well known small-gain conditions developed in the framework of networked systems described by ordinary differential equations in (Dashkovskiy et al., 2011). The small-gain conditions are obtained by inequalities on multiple Lyapunov functions (one for each mode of each subsystem) on the basis of the switches digraphs of each subsystem. From those multiple Lyapunov functions for each subsystem, and by the small-gain conditions (which take the switches digraph of each subsystem into account), a set of multiple Lyapunov functions (one for each mode of the entire system) are

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obtained for the entire discrete-time switching system. By this set of multiple Lyapunov functions for the entire system, equipped by an entire switches digraph as obtained from the ones for subsystems, the input-to-state stability is proved by means of results in (Pepe, 2019). To our best knowledge, this is the first result in the literature concerning small-gain theorems for the input-to-state stability of nonlinear discrete-time switching systems, with given switches digraphs. A simple academic example with an unstable subsystem in the family is studied, in order to show how the methodology here provided works.

Notation The symbol $\mathbb{R}$ denotes the set of real numbers, the symbol $\mathbb{R}_+$ denotes the set of non-negative real numbers, the symbol $\mathbb{N}$ denotes the set of non-negative integer numbers. For given positive integer n, $\mathbb{R}^n$ denotes the set of real vectors with n entries, $\mathbb{R}^n_+$ denotes the set of real vectors with n non-negative entries, $\mathbb{R}^n_{+,\neq 0}$ denotes the set of nonzero real vectors with n non-negative entries. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a real matrix. The symbol $\circ$ denotes the composition (of functions). Given $a, b \in \mathbb{R}^n$, we say $a > b$ if $a_i > b_i$ for all $i = 1, 2, \ldots, n$. Let us here recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is: of class $\mathcal{K}$ if it is zero at zero and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it is decreasing and converging to 0 as the argument tends to $+\infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $s \to \beta(s,t)$, $s \in \mathbb{R}_+$, is of class $\mathcal{K}$, and, for each fixed $s \in \mathbb{R}_+$, the function $t \to \beta(s,t)$, $t \in \mathbb{R}_+$, is of class $\mathcal{L}$. The standard acronyms GAS, ISS, and ODE stand for global asymptotic stability or globally asymptotically stable, input-to-state stability or input-to-state stable, and ordinary differential equation, respectively.

2. SWITCHING SYSTEMS WITH DIGRAPHS

For the reader's convenience, and for the self-containedness of the paper, we briefly recall here the results in (Pepe, 2019), which will be used in next sections concerning small-gain results for networked switching discrete-time systems. Let us consider the discrete-time switching system described by the following equation

$$x(k+1) = f_{\sigma(k)}(x(k), u(k)), \qquad k \in \mathbb{N},$$
$$x(0) = \xi, \tag{1}$$

where: $x(k) \in \mathbb{R}^n$, n is a positive integer; $u(k) \in \mathbb{R}^m$ is the input signal, m is a positive integer; σ is a function (the switching-dwelling signal) from $\mathbb{N}$ to S, $S = \{1, 2, \ldots, p\}$, p is a positive integer; for any $j \in S$, $f_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function, satisfying $f_j(0,0) = 0$; $\xi \in \mathbb{R}^n$. Let $E(S)$ (see Kundu & Chatterjee, 2016) be the finite set of all pairs $(i,j) \in S \times S$ such that it is allowed to switch (or to dwell if $i = j$) from system described by f_i to system described by f_j. A nonblocking switches digraph $G(S, E(S))$ is associated to the switching system (1) as follows (see Kundu & Chatterjee, 2016): 1) the set of vertices is the set of indexes in S; 2) the set of edges $E(S)$ consists of a directed edge (i,j) whenever it is allowed to switch from vertex (system) i to vertex (system) j, $i, j \in S$, $i \neq j$, and of a self-loop (j,j) at vertex j whenever it is allowed to dwell on vertex (system) j for two or more (even ∞) consecutive time-steps. Let us define the following sets (of switching-dwelling and input sequences)

$$\mathcal{M}_S = \{\sigma : \mathbb{N} \to S \mid (\sigma(k), \sigma(k+1)) \in E(S),\ \forall k \in \mathbb{N}\},$$
$$\mathcal{M}_u = \{u : \mathbb{N} \to \mathbb{R}^m\} \tag{2}$$

Notice that any switching-dwelling signal $\sigma \in \mathcal{M}_S$ is constrained to adhere to the provided switches digraph. For $\xi \in \mathbb{R}^n$, $\sigma \in \mathcal{M}_S$, $u \in \mathcal{M}_u$, we denote with $x(k, \xi, \sigma, u)$, $k \in \mathbb{N}$, the solution of (1) corresponding to initial condition ξ, switching-dwelling signal σ, and input signal u. We recall here the 0-GAS notion and Sontag's notion of ISS (see Jiang & Wang, 2001, Jiang & Wang, 2002, Pepe, 2019, Sontag, 1995).

Definition 1. The system described by (1), with $u(\cdot) \equiv 0$, is said to be 0-GAS if there exists a function β of class $\mathcal{KL}$ such that, for any initial condition $\xi \in \mathbb{R}^n$ and for any switching-dwelling signal $\sigma \in \mathcal{M}_S$, the corresponding solution $x(k, \xi, \sigma, 0)$ of (1) satisfies, for $k \in \mathbb{N}$, the inequality

$$|x(k, \xi, \sigma, 0)| \leq \beta(|\xi|, k) \tag{3}$$

Definition 2. The system described by (1) is said to be ISS if there exists a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial condition $\xi \in \mathcal{R}^n$, for any input signal $u \in \mathcal{M}_u$, and for any switching-dwelling signal $\sigma \in \mathcal{M}_S$, the corresponding solution $x(k, \xi, \sigma, u)$ of (1) satisfies, for $k \in \mathbb{N}$, the following inequality

$$|x(k, \xi, \sigma, u)| \leq \beta(|\xi|, k) + \gamma\left(\sup_{j=0,1,\ldots,k-1} |u(j)|\right), \tag{4}$$

where the second term of the sum in the right-hand side of (4) is taken equal to 0 for $k = 0$.

Theorem 3. (Pepe, 2019) The following statements are equivalent:

a) the system described by (1) is 0-GAS;
b) there exist p continuous functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i \in S$, functions α_i, $i = 1, 2$, of class $\mathcal{K}_\infty$, and a function α_3 of class $\mathcal{K}$, such that the following inequalities hold, for all $i \in S$, $(j, l) \in E(S)$, $x \in \mathbb{R}^n$,
   b_1) $\alpha_1(|x|) \leq V_i(x) \leq \alpha_2(|x|)$,
   b_2) $V_l(f_j(x, 0)) - V_j(x) \leq -\alpha_3(|x|)$

Theorem 4. (Pepe, 2019) The following statements are equivalent:

a) the system described by (1) is ISS;
b) there exist p continuous functions $V_i : \mathbb{R}^n \to \mathbb{R}^+$, $i \in S$, functions α_i, $i = 1, 2, 3$, of class $\mathcal{K}_\infty$, and a function α_4 of class $\mathcal{K}$, such that the following inequalities hold, for all $i \in S$, $(j, l) \in E(S)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$,
   b_1) $\alpha_1(|x|) \leq V_i(x) \leq \alpha_2(|x|)$,
   b_2) $V_l(f_j(x, u)) - V_j(x) \leq -\alpha_3(|x|) + \alpha_4(|u|)$.

3. NETWORK OF SWITCHING SYSTEMS

Let us consider the network of discrete-time switching subsystems described by the following equations

$$x_i(k+1) = f_{i,\sigma_i(k)}(x(k), u(k)),\ k \in \mathbb{N},$$
$$x_i(0) = x_{i,0}, \tag{5}$$

where: $x_i(k) \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$; N, n_i, $i = 1, 2, \ldots, N$, are positive integers; $x(k) = \left[x_1^T(k) \; x_2^T(k) \; \cdots \; x_N^T(k) \right]^T$, $k \in \mathbb{N}$; $u(k) \in \mathbb{R}^m$ is the input signal, $k \in \mathbb{N}$; m is a positive integer; $\sigma_i : \mathbb{N} \to S_i$, $S_i = \{1, 2, \ldots, p_i\}$, is the switching-dwelling signal related to sub-system i, p_i is a positive integer, $i = 1, 2, \ldots, N$; $f_{i,j} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$, $j \in S_i$, are locally Lipschitz functions satisfying $f_{i,j}(0, 0) = 0$; $n = \sum_{i=1}^N n_i$. Let $E_k(S_k)$, $k = 1, 2, \ldots, N$, be the finite set of all pairs $(i, j) \in S_k \times S_k$ such that it is allowed to switch (or to dwell if $i = j$) from subsystem described by $f_{k,i}$ to subsystem described by $f_{k,j}$. A nonblocking switches digraph $G_k(S_k, E_k(S_k))$, $k = 1, 2, \ldots, N$, is associated to each subsystem as follows: 1) the set of vertices is the set of indexes in S_k; 2) the set of edges $E_k(S_k)$ consists of a directed edge (i, j) whenever it is allowed to switch from vertex i (subsystem k at mode i) to vertex j (subsystem k at mode j), $i, j \in S_k$, $i \neq j$, and of a self-loop (j, j) at vertex j whenever it is allowed to dwell on vertex j for two or more (even ∞) consecutive time-steps. Let $p = \Pi_{i=1}^N p_i$ and let $S = \{1, 2, \ldots, p\}$. Let $L : S \to S_1 \times S_2 \times \cdots \times S_N$ be a bijective function, and let $L_i : S \to S_i$, $i = 1, 2, \ldots, N$, be the function defined, for $j \in S$, as the $i - th$ entry of $L(j)$. We can write the entire system with subsystems (5), with a unique switches digraph, as follows

$$
\begin{aligned}
& x(k+1) = f_{\sigma(k)}(x(k), u(k)), \; k \in \mathbb{N}, \\
& x(0) = x_0,
\end{aligned} \tag{6}
$$

with $\sigma : \mathbb{N} \to S$ and, for $j \in S$, $x = \left[x_1^T \; x_2^T \; \cdots \; x_N^T \right]^T \in \mathbb{R}^n$, $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$, $u \in \mathbb{R}^m$,

$$
\begin{aligned}
& f_j(x, u) = \\
& \left[f_{1,L_1(j)}^T(x, u) \; f_{2,L_2(j)}^T(x, u) \; \ldots \; f_{N,L_N(j)}^T(x, u) \right]^T
\end{aligned} \tag{7}
$$

Let $E(S)$ be the finite set of all pairs $(j, l) \in S \times S$ such that, $(L_k(j), L_k(l)) \in E_k(S_k)$ for all $k = 1, 2, \ldots, N$, (i.e., it is allowed to switch, or to dwell if $j = l$, from subsystem (5) described by $f_{k,L_k(j)}$ to subsystem (5) described by $f_{k,L_k(l)}$, for all $k = 1, 2, \ldots, N$). A nonblocking switches digraph $G(S, E(S))$ is then associated to the entire system (6) as follows: 1) the set of vertices is the set of indexes in S; 2) the set of edges $E(S)$ consists of a directed edge (j, l), for $j, l \in S$, $j \neq l$, whenever $(L_k(j), L_k(l)) \in E_k(S_k)$ for all $k = 1, 2, \ldots, N$, and of a self-loop (j, j), for $j \in S$, whenever $(L_k(j), L_k(j)) \in E_k(S_k)$ for all $k = 1, 2, \ldots, N$.

4. SMALL-GAIN ASSUMPTIONS

We introduce the following Assumptions 5, 6, which will be used in forthcoming Theorem 7.

Assumption 5. There exist locally Lipschitz functions $V_{i,j} : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, p_i$, functions $\alpha_{i,j,l}$ of class $\mathcal{K}_\infty$, $i = 1, 2, \ldots, N$, $(j, l) \in E(S)$, functions $\gamma_{i,k,j,l}$, $i, k = 1, 2, \ldots, N$, $(j, l) \in E(S)$, either of class $\mathcal{K}$ either identically zero, with $\gamma_{i,i,j,l} = 0$, $i = 1, 2, \ldots, N$, $(j, l) \in E(S)$, functions $\underline{\gamma}$, $\overline{\gamma}$ of class $\mathcal{K}_\infty$, functions $\rho_{i,j,l}$ of class $\mathcal{K}$, $i = 1, 2, \ldots, N$, $(j, l) \in E(S)$, such that the following inequalities hold:

a) $\underline{\gamma}(|x_i|) \leq V_{i,j}(x_i) \leq \overline{\gamma}(|x_i|)$, $\forall \; x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$, $j \in S_i$;

b) $V_{i,L_i(l)}(f_{i,L_i(j)}(x, u)) - V_{i,L_i(j)}(x_i) \leq$

$$
\begin{aligned}
& -\alpha_{i,j,l}(V_{i,L_i(j)}(x_i)) + \sum_{k=1}^N \gamma_{i,k,j,l}(V_{k,L_k(j)}(x_k)) + \rho_{i,j,l}(|u|), \\
& \forall \; x = \left[x_1^T \; x_2^T \; \cdots \; x_N^T \right]^T \in \mathbb{R}^n, \\
& \forall \; u \in \mathbb{R}^m, \; i = 1, 2, \ldots, N, \; (j, l) \in E(S)
\end{aligned} \tag{8}
$$

Let us define, for $j \in S$, the function $V_{vec,j} : \mathbb{R}^n \to \mathbb{R}_+^N$ as, for $x = \left[x_1^T \; x_2^T \; \ldots \; x_N^T \right]^T \in \mathbb{R}^n$, $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$,

$$
\begin{aligned}
& V_{vec,j}(x) = \\
& \left[V_{1,L_1(j)}(x_1) \; V_{2,L_2(j)}(x_2) \; \ldots \; V_{N,L_N(j)}(x_N) \right]^T
\end{aligned} \tag{9}
$$

Let us define, for $(j, l) \in E(S)$, the functions $A_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N$, $A_{j,l}^{-1} : \mathbb{R}_+^N \to \mathbb{R}_+^N$, $\Gamma_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N$, $\Omega_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N \times \mathbb{R}_+^N$, $\rho_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N$, for $s = [s_1 \; s_2 \; \ldots \; s_N]^T \in \mathbb{R}_+^N$, $\eta \in \mathbb{R}_+$, as follows

$$
A_{j,l}(s) = \begin{bmatrix} \alpha_{1,j,l}(s_1) \\ \alpha_{2,j,l}(s_2) \\ \vdots \\ \alpha_{N,j,l}(s_N) \end{bmatrix}, \; A_{j,l}^{-1}(s) = \begin{bmatrix} \alpha_{1,j,l}^{-1}(s_1) \\ \alpha_{2,j,l}^{-1}(s_2) \\ \vdots \\ \alpha_{N,j,l}^{-1}(s_N) \end{bmatrix}, \tag{10}
$$

$$
\begin{aligned}
& \Gamma_{j,l}(s) = \\
& \left[\sum_{k=1}^N \gamma_{1,k,j,l}(s_k) \; \sum_{k=1}^N \gamma_{2,k,j,l}(s_k) \; \cdots \; \sum_{k=1}^N \gamma_{N,k,j,l}(s_k) \right]^T,
\end{aligned} \tag{11}
$$

$$
\Omega_{j,l}(s) = \begin{bmatrix} \gamma_{1,1,j,l}(s_1) & \gamma_{1,2,j,l}(s_2) & \ldots & \gamma_{1,N,j,l}(s_N) \\ \gamma_{2,1,j,l}(s_1) & \gamma_{2,2,j,l}(s_2) & \ldots & \gamma_{2,N,j,l}(s_N) \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{N,1,j,l}(s_1) & \gamma_{N,2,j,l}(s_2) & \ldots & \gamma_{N,N,j,l}(s_N) \end{bmatrix}, \tag{12}
$$

$$
\rho_{j,l}(\eta) = \left[\rho_{1,j,l}(\eta) \; \rho_{2,j,l}(\eta) \; \cdots \; \rho_{N,j,l}(\eta) \right]^T \tag{13}
$$

We can rewrite compactly the inequalities (b) in Assumption 5, for $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $(j, l) \in E(S)$, as follows

$$
\begin{aligned}
& V_{vec,l}(f_j(x, u)) - V_{vec,j}(x) \leq \\
& (-A_{j,l} + \Gamma_{j,l})(V_{vec,j}(x)) + \rho_{j,l}(|u|)
\end{aligned} \tag{14}
$$

Assumption 6. For any $(j, l) \in E(S)$, for any $s \in \mathbb{R}_+^N$, $s > 0$, for any $k = 1, 2, \ldots, N$, $\Omega_{j,l}(s) e_k \neq 0$, where e_k, $k = 1, 2, \ldots, N$, is the canonical basis in $\mathbb{R}^N$. There exist a vector $\mu \in \mathbb{R}_+^N$, $\mu > 0$, functions $\beta_{i,j,l}$, $i = 1, 2, \ldots, N$, $(j, l) \in E(S)$, of class $\mathcal{K}_\infty$, such that, for any $s = [s_1 \; s_2 \; \ldots \; s_N]^T \in \mathbb{R}_{+,\neq 0}^N$, the following small-gain inequalities hold

$$
\mu^T \left(D_{j,l} \circ \Gamma_{j,l} \circ A_{j,l}^{-1}(s) - s \right) < 0, \; (j, l) \in E(S), \tag{15}
$$

where the function $D_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N$, $(j,l) \in E(S)$, is defined, for $s = [\, s_1 \ s_2 \ \ldots \ s_N \,]^T \in \mathbb{R}_+^N$, as follows

$$D_{j,l}(s) = s + \beta_{j,l}(s), \tag{16}$$

and $\beta_{j,l} : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is defined, for $s = [\, s_1 \ s_2 \ \ldots \ s_N \,]^T \in \mathbb{R}_+^N$, as follows

$$\beta_{j,l}(s) = [\, \beta_{1,j,l}(s_1) \ \beta_{2,j,l}(s_2) \ \cdots \ \beta_{N,j,l}(s_N) \,]^T \tag{17}$$

5. SMALL-GAIN THEOREM FOR ISS

The following theorem provides small-gain results for the ISS property of system (6).

Theorem 7. Let Assumption 5 and Assumption 6 hold. Then, system (6) is ISS. Moreover the p functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, defined, for $x \in \mathbb{R}^n$, as $V_i(x) = \mu^T V_{vec,i}(x)$, $i \in S$, satisfy conditions (b_1), (b_2) in Theorem 4, when applied to system (6).

Proof. The proof is obtained by the application of Theorem 4 to system (6), with the functions V_i, $i \in S$. As far as the conditions (b_1) in Theorem 4 are concerned, the following inequalities hold, for any $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \le V_i(x) \le \alpha_2(|x|) \tag{18}$$

with α_1, α_2 functions of class $\mathcal{K}_\infty$ defined, for $s \in \mathbb{R}^+$, as

$$\begin{aligned}
\alpha_1(s) = & \inf_{z=[\, z_1 \ z_2 \ \cdots \ z_N \,]^T \in \mathbb{R}_+^N, \ |z|=s} \\
& \left\{ \mu^T \left[\, \underline{\gamma}(z_1) \ \underline{\gamma}(z_2) \ \cdots \ \underline{\gamma}(z_N) \,\right]^T \right\}, \\
\alpha_2(s) = & \sup_{z=[\, z_1 \ z_2 \ \cdots \ z_N \,]^T \in \mathbb{R}_+^N, \ |z|=s} \\
& \left\{ \mu^T \left[\, \overline{\gamma}(z_1) \ \overline{\gamma}(z_2) \ \cdots \ \overline{\gamma}(z_N) \,\right]^T \right\}
\end{aligned} \tag{19}$$

As far as the conditions (b_2) in Theorem 4 are concerned, we have, from (14), for any $(j,l) \in E(S)$, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
& V_l(f_j(x,u)) - V_j(x,u) = \\
& \mu^T \left(V_{vec,l}(f_j(x,u)) - V_{vec,j}(x) \right) \le \\
& \mu^T(-A_{j,l} + \Gamma_{j,l})(V_{vec,j}(x)) + \mu^T \rho_{j,l}(|u|),
\end{aligned} \tag{20}$$

From (15), it follows that

$$\begin{aligned}
& \mu^T D_{j,l} \circ \Gamma_{j,l}(V_{vec,j}(x)) = \\
& \mu^T D_{j,l} \circ \Gamma_{j,l} \circ A_{j,l}^{-1} \circ A_{j,l}(V_{vec,j}(x)) \le \\
& \mu^T A_{j,l}(V_{vec,j}(x))
\end{aligned} \tag{21}$$

and thus the inequality follows

$$-\mu^T A_{j,l}(V_{vec,j}(x)) \le -\mu^T D_{j,l} \circ \Gamma_{j,l}(V_{vec,j}(x)) \tag{22}$$

From (20), (22) it follows that

$$\begin{aligned}
& V_l(f_j(x,u)) - V_j(x,u) \le -\mu^T D_{j,l} \circ \Gamma_{j,l}(V_{vec,j}(x)) \\
& + \mu^T \Gamma_{j,l}(V_{vec,j}(x)) + \mu^T \rho_{j,l}(|u|) = \\
& - \mu^T \Gamma_{j,l}(V_{vec,j}(x)) - \mu^T \beta_{j,l}(V_{vec,j}(x)) \\
& + \mu^T \Gamma_{j,l}(V_{vec,j}(x)) + \mu^T \rho_{j,l}(|u|) = \\
& - \mu^T \beta_{j,l}(V_{vec,j}(x)) + \mu^T \rho_{j,l}(|u|) \le \\
& - \widetilde{\alpha}_3(V_j(x)) + \widetilde{\alpha}_4(|u|),
\end{aligned} \tag{23}$$

where $\widetilde{\alpha}_3$, $\widetilde{\alpha}_4$ are the functions of class $\mathcal{K}_\infty$ and $\mathcal{K}$, respectively, defined, for $s \in \mathbb{R}_+$, as

$$\begin{aligned}
\widetilde{\alpha}_3(s) = & \inf_{(j,l) \in E(S), \ x \in \mathbb{R}^n} \{ \mu^T \beta_{j,l}(V_{vec,j}(x)) \\
& \mid \mu^T V_{vec,j}(x) = s \},
\end{aligned} \tag{24}$$

$$\widetilde{\alpha}_4(s) = \sup_{(j,l) \in E(S)} \{ \mu^T \rho_{j,l}(s) \} \tag{25}$$

Therefore, the inequalities (b_2) in Theorem 4 are satisfied with function α_3 of class $\mathcal{K}_\infty$ and function α_4 of class $\mathcal{K}$ defined, for $s \in \mathbb{R}_+$, as $\alpha_3(s) = \widetilde{\alpha}_3 \circ \alpha_1(s)$, $\alpha_4(s) = \widetilde{\alpha}_4(s)$. The proof of the theorem is complete.

Remark 8. The small-gain condition (15) is the switching discrete-time counterpart, with multiple Lyapunov functions and switches digraph, of the small-gain condition for continuous-time networked systems described by ODEs in (Dashkovskiy et al., 2011). In (Dashkovskiy et al., 2011) also linearly scaled gains were considered, leading to a nice condition on related coefficients, which was firstly used in the discrete-time case in (Pola et al., 2016). A same small-gain condition by linearly scaled gains as well as a small-gain condition by nonlinear gains as in (15) are used in (Battista & Pepe, 2018), for nonlinear networked discrete-time systems with bounded time-varying and uncertain time delays, by means of Lyapunov functions independent of the time delay, one for each subsystem. A key point in the proof of Theorem 7 is given by steps (21) and (22). Such simple mechanism was used in (Battista & Pepe, 2018) in the framework of networked discrete-time time-delay systems (see the proof of Theorem 8 in (Battista & Pepe, 2018)), and reveals to be helpful in the framework of networked switching systems with switches digraph considered in this paper, as well.

The following Corollary provides small-gain results for the 0-GAS property of system (6).

Corollary 9. Let, in (6), $u(\cdot) \equiv 0$. Let Assumption 5 hold with $u = 0$ and let Assumption 6 hold. Then, system (6) is 0-GAS. Moreover the p functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, defined, for $x \in \mathbb{R}^n$, as $V_i(x) = \mu^T V_{vec,i}(x)$, $i \in S$, satisfy conditions (b_1), (b_2) in Theorem 3, when applied to system (6).

Proof. Just consider subsystems described by functions $\widetilde{f}_{i,j} : \mathbb{R}^{n_i} \times \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$, $j \in S_i$, defined, for $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, as $\widetilde{f}_{i,j}(x,u) = f_{i,j}(x,0)$. Then, by Theorem 7, it follows that the new constructed networked system is ISS. As the solution of the new constructed system does not depend on the input, it follows that the input can be chosen identically zero. Thus the inequality (4) with $u(\cdot) = 0$ returns the inequality (3) for the system described by $\widetilde{f}_{i,j}$, which yields the 0-GAS property of the system described by (6), with $u(\cdot) \equiv 0$. As well, with the new constructed system, the Lyapunov functions V_i, $i \in S$, satisfy conditions (b_1), (b_2) in Theorem 7. Furthermore, since the left-hand side of the inequality in (b_2), in Theorem 7, does not depend on $u \in \mathbb{R}^m$, it follows that it holds in particular with $u = 0$, thus returning condition (b_2) in Theorem 3. The proof of the corollary is complete.

Remark 10. Notice that Assumption 6 requires the functions $\alpha_{i,j,l}$, $i = 1, 2, \ldots, N$, $(j, l) \in E(S)$, to be of class $\mathcal{K}_\infty$. Therefore, the weaker condition (b_2) in Theorem 3, which involves α_3 of class $\mathcal{K}$, is not exploited in Corollary 9. In Corollary 9, the constructed multiple Lyapunov functions satisfy condition (b_2) in Theorem 3 with α_3 of class $\mathcal{K}_\infty$.

6. EXAMPLE

In order to show how the methodology presented in the paper works, let us consider the following simple network of two scalar subsystems, with $S_1 = \{1\}$ (i.e., the first subsystem is not switching), and $S_2 = \{1, 2\}$ (i.e., the second subsystem is switching between two modes):

$$x_1(k+1) = 0.5x_1(k) + \delta_1 q_1(x_2(k)) + u(k)$$
$$x_2(k+1) = \begin{cases} 0.2x_2(k) + \delta_2 q_2(x_1(k)) + u(k) \\ 3x_2(k) + \delta_3 q_3(x_1(k)) \end{cases} \tag{26}$$

where $x_1(k), x_2(k), u(k) \in \mathbb{R}$, $k \in \mathbb{N}$, $\delta_i \in \mathbb{R}$, $i = 1, 2, 3$, $q_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, are arbitrary locally Lipschitz functions satisfying $|q_i(y)| \le |y|$, $y \in \mathbb{R}$, $i = 1, 2, 3$. With the notation of the paper, we have, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $u \in \mathbb{R}$,

$$f_{1,1}(x, u) = 0.5x_1 + \delta_1 q_1(x_2) + u$$
$$f_{2,1}(x, u) = 0.2x_2 + \delta_2 q_2(x_1) + u$$
$$f_{2,2}(x, u) = 3x_2 + \delta_3 q_3(x_1) \tag{27}$$

The second subsystem has an unstable mode (i.e., the mode 2 characterized by $f_{2,2}$). We have $S_1 = \{1\}$, $E_1(S_1) = \{(1, 1)\}$, $S_2 = \{1, 2\}$. Let

$$E_2(S_2) = \{(1, 1), (1, 2), (2, 1)\} \tag{28}$$

Notice that the switches digraph of the second subsystem does not allow a self loop $(2, 2)$, that is, whenever the second subsystem, at time k, is on unstable mode 2, then in $k+1$ it must switch to mode 1. In this case we have $p = 2$ and $S = \{1, 2\}$. Let the bijective function $L : S \to S_1 \times S_2$ be defined as follows: $L(1) = [\,1\ 1\,]$, $L(2) = [\,1\ 2\,]$. We obtain $E(S) = \{(1, 1), (1, 2), (2, 1)\}$. Let, for $x_i \in \mathbb{R}$, $i = 1, 2$, $V_{1,1}(x_1) = r_1|x_1|$, $V_{2,1}(x_2) = r_2|x_2|$, $V_{2,2}(x_2) = r_3|x_2|$, with r_i positive reals to be chosen, $i = 1, 2, 3$. As far as Assumption 5 is concerned, by exploiting the sector-boundedness property of the functions q_i, $i = 1, 2, 3$, we have, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $u \in \mathbb{R}$,

$$V_{1,L_1(1)}(f_{1,L_1(1)}(x, u)) - V_{1,L_1(1)}(x_1) \le$$
$$-0.5V_{1,L_1(1)}(x_1) + \frac{r_1|\delta_1|}{r_2} V_{2,L_2(1)}(x_2) + r_1|u|$$
$$V_{2,L_2(1)}(f_{2,L_2(1)}(x, u)) - V_{2,L_2(1)}(x_2) \le$$
$$-0.8V_{2,L_2(1)}(x_2) + \frac{r_2|\delta_2|}{r_1} V_{1,L_1(1)}(x_1) + r_2|u|$$
$$V_{1,L_1(2)}(f_{1,L_1(1)}(x, u)) - V_{1,L_1(1)}(x_1) \le$$
$$-0.5V_{1,L_1(1)}(x_1) + \frac{r_1|\delta_1|}{r_2} V_{2,L_2(1)}(x_2) + r_1|u|$$
$$V_{2,L_2(2)}(f_{2,L_2(1)}(x, u)) - V_{2,L_2(1)}(x_2) \le$$
$$-\frac{r_2 - 0.2r_3}{r_2} V_{2,L_2(1)}(x_2) + \frac{r_3|\delta_2|}{r_1} V_{1,L_1(1)}(x_1) + r_3|u|$$
$$V_{1,L_1(1)}(f_{1,L_1(2)}(x, u)) - V_{1,L_1(2)}(x_1) \le$$
$$-0.5V_{1,L_1(2)}(x_1) + \frac{r_1|\delta_1|}{r_3} V_{2,L_2(2)}(x_2) + r_1|u|$$
$$V_{2,L_2(1)}(f_{2,L_2(2)}(x, u)) - V_{2,L_2(2)}(x_2) \le$$
$$-\frac{r_3 - 3r_2}{r_3} V_{2,L_2(2)}(x_2) + \frac{r_2|\delta_3|}{r_1} V_{1,L_1(2)}(x_1) \tag{29}$$

Let us choose $r_1 = r_3 = 1$, $r_2 = 0.25$. As far as Assumption 6 is concerned, we have, for $s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \in \mathbb{R}^2_+$,

$$A_{1,1}(s) = \begin{bmatrix} 0.5s_1 \\ 0.8s_2 \end{bmatrix},\ A_{1,2}(s) = \begin{bmatrix} 0.5s_1 \\ 0.2s_2 \end{bmatrix},$$
$$A_{2,1}(s) = \begin{bmatrix} 0.5s_1 \\ 0.25s_2 \end{bmatrix},\ \Gamma_{1,1}(s) = \begin{bmatrix} 4|\delta_1|s_2 \\ 0.25|\delta_2|s_1 \end{bmatrix},$$
$$\Gamma_{1,2}(s) = \begin{bmatrix} 4|\delta_1|s_2 \\ |\delta_2|s_1 \end{bmatrix},\ \Gamma_{2,1}(s) = \begin{bmatrix} |\delta_1|s_2 \\ 0.25|\delta_3|s_1 \end{bmatrix} \tag{30}$$

Let us choose $\beta_{i,j,l}(s) = \omega s$, $s \in \mathbb{R}_+$, with ω suitable positive real, $i = 1, 2$, $(j, l) \in E(S)$. The application of the small-gain condition (15) leads to the following equivalent conditions (successful sufficiently small ω can be consequently chosen)

$$\max\{2|\delta_2|, 0.5|\delta_3|\} < \frac{\mu_1}{\mu_2} < \frac{1}{20|\delta_1|}, \tag{31}$$

where $\mu = [\,\mu_1\ \mu_2\,]^T$ is the vector invoked in the small-gain condition (6). Therefore, if the following condition on parameters δ_i, $i = 1, 2, 3$,

$$20|\delta_1| \max\{2|\delta_2|, 0.5|\delta_3|\} < 1 \tag{32}$$

holds, then the system described by (26) is ISS. Multiple Lyapunov functions for the entire system, by which the ISS property is proved according to Theorem 4, are given by $V_i : \mathbb{R}^2 \to \mathbb{R}^+$, $i = 1, 2$, defined, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, as $V_i(x) = \mu^T \begin{bmatrix} V_{1,1}(x) \\ V_{2,i}(x_2) \end{bmatrix}$, with μ_1 and μ_2 positive reals satisfying inequalities (31), and with above choice of r_i, $i = 1, 2, 3$. A solution (in the unknown variable μ), of inequalities (31), is guaranteed to exist by condition (32) on parameters δ_i, $i = 1, 2, 3$.

7. CONCLUSION

A small-gain theorem for the input-to-stability of a network of switching subsystems, equipped with switches digraphs, is here provided. The switches digraphs may accommodate for unstable subsystems in the family, as long as no consecutive steps on those subsystems are allowed. The more information they take, the smaller the number of required small-gain inequalities (with arbitrary switching equal to p^2, the square of the number of modes of the entire system). The provided small-gain theorem yields the explicit construction of multiple Lyapunov functions, one for each mode, for the entire switching system, on the basis of multiple Lyapunov functions for each subsystem. Future developments will concern the maximal allowed permanence on unstable subsystems such that the input-to-state stability is still preserved. A further development is the application of the results by multiple Lyapunov functions, here provided, to discrete-time systems with unknown and time-varying time delays (see Battista & Pepe, 2018, Hetel et al., 2008, Pepe et al., 2018), which naturally arise in networked remote control. Stabilization issues (see Deaecto et al., 2015, Fan et al., 2012) will be also topic of forthcoming investigations.

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