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A geometric characterization of the slow space of the Hamiltonian system arising from the singular LQR problem

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Abstract: In this paper we first characterize the *slow space* of a given state-space system. We provide this characterization in terms of an eigenspace of the corresponding Rosenbrock matrix pair. We also characterize the "good" slow space in terms of a stable eigenspace of the Rosenbrock matrix pair. Moreover, we show how the dimensions of these subspaces can be calculated from the determinant of the Rosenbrock matrix pencil. Then, we apply these results to the Hamiltonian system arising from the singular linear quadratic regulator (LQR) problem and explore a few interesting properties of the good slow space of this Hamiltonian system. Finally, we provide a feedback law to achieve the smooth optimal solutions.

Keywords: Generalized eigenvalues and eigenspaces, slow space, Hamiltonian system.

1. INTRODUCTION

Singular LQR problem is one of the classical problems in systems and control theory (Hautus and Silverman (1983), Willems et al. (1986)). This problem is still an area of active research (Reis et al. (2015), Ferrante and Ntogramatzidis (2018),Bhawal and Pal (2019)). The following is the formal statement of the infinite-horizon LQR problem:

Problem 1. Consider the stabilizable system defined by $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition $x(0) = x_0$, find an input $u(t)$ that minimizes the functional

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \qquad (1)$$

with $\lim_{t\to\infty} x(t) = 0$, where $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, S \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, such that $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] \geqslant 0$.

Problem 1 is called a *regular* LQR problem if $R > 0$, and a *singular* LQR problem if $R \geqslant 0$ with R being singular. It is well-known in the literature that the regular LQR problem admits an algebraic Riccati equation (ARE) given as:

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0. \qquad (2)$$

If $K_{\mathtt{max}}$ is the maximal solution of eq. (2), that is, $K_{\mathtt{max}} - K \geqslant 0$ for any other solution K of the ARE, then the LQR Problem 1 can be solved by using the feedback law $u = Fx$, where $F := -R^{-1}(B^T K_{\mathtt{max}} + S^T)$. Notice from eq. (2) that the existence of an ARE crucially depends on the invertibility of R. Naturally, a singular LQR problem does not admit an ARE and consequently can not be solved using the feedback law as mentioned before. Hautus and Silverman (1983) deals with the solution of the singular LQR problem, but it does not provide a feedback solution for the problem. In Reis et al. (2015), the notion of deflating subspaces has been used to provide a linear implicit control law which, unfortunately, often turns out to be not feedback implementable. The theory presented in Reis et al. (2015) assumes that the state and the input of the system are from the space of locally square-integrable functions. This assumption prevents the presence of impulses in the input and the states. Reis et al. (2015) also imposes a restriction on the initial condition of the system. Such a restriction on the initial condition is not desirable, because an initial condition of a system should ideally be free. For single-input systems, Bhawal and Pal (2019) provides a solution for any arbitrary initial condition. They also provide a PD feedback law for the optimal solution. Since the initial condition is free, the optimal trajectories corresponding to certain initial conditions are impulsive in nature. Hence, the function space assumed in Bhawal and Pal (2019) allows impulses in the input and the states. This solution is based on the notion of the *slow subspace* and the *fast subspace* of the *Hamiltonian system* arising from the singular LQR problem (Bhawal et al. (2019b)). We wish to extend the result based on the notion of slow and fast subspaces of the Hamiltonian system to the case of multi-input systems in our future research. Therefore, in this paper we characterize the slow space of the Hamiltonian system in terms of an eigenspace of the Hamiltonian matrix pencil. We also provide a static feedback law for the smooth optimal solutions.

2. NOTATION AND PRELIMINARIES

2.1 Notation

The symbols $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ are used for the sets of real numbers, complex numbers, and natural numbers, respectively. $\mathbb{R}_+$ denotes the set of non-negative real numbers. We use the symbols $\overline{\mathbb{C}}_+$ and $\mathbb{C}_-$ for the closed right-half and the open left-half of the complex plane, respectively. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of $\mathtt{n} \times \mathtt{p}$ matrices with elements from $\mathbb{R}$. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. We use the symbol $I_{\mathtt{n}}$ for an $\mathtt{n}\times\mathtt{n}$ identity matrix and the symbol $0_{\mathtt{n},\mathtt{m}}$ for an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. Symbol $\mathtt{col}(B_1, B_2, \ldots, B_{\mathtt{n}})$ represents a matrix of the form $[B_1^T \; B_2^T \; \cdots \; B_{\mathtt{n}}^T]^T$. The symbol $\mathtt{det}(A)$ represents the determinant of a square matrix A. Symbol $\mathtt{rank}\, A$ denotes the rank of a matrix A. We use the symbol $\mathtt{roots}(p(s))$ to denote the set of roots (over $\mathbb{C}$) of a polynomial $p(s)$ with real or complex coefficients (counted with multiplicity). The symbol $\mathtt{deg}(p(s))$ is used to denote the degree of the polynomial $p(s)$. The symbol $\sigma(\Gamma)$ denotes the set of eigenvalues of a square matrix Γ (counted with multiplicity). The symbol $|\Gamma|$ denotes the cardinality of a set Γ (counted with multiplicity). We use the symbols $A|_{\mathcal{S}}$ to denote the restriction of a matrix A to a subspace $\mathcal{S}$ (with respect to a suitable basis) and $\sigma(A|_{\mathcal{S}})$ to represent the set of eigenvalues of A restricted to the subspace $\mathcal{S}$. We use the symbol $\mathtt{dim}(\mathcal{S})$ to denote the dimension of a space $\mathcal{S}$. The symbol $\mathtt{img}\,A$ and $\mathtt{ker}\,A$ denote the image and nullspace of a matrix A, respectively. The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is represented by the symbol $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})$. The symbol $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})|_{\mathbb{R}_+}$ represents the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$.

2.2 Regular matrix pencils and their canonical form

Linear matrix pencils and their eigenvectors are crucially used throughout this paper. Hence, we define eigenvalues and eigenvectors of a linear matrix pencil next.

Definition 2. Consider a regular matrix pencil $(sU_1-U_2)\in\mathbb{R}[s]^{\mathtt{n}\times\mathtt{n}}$, i.e., $\mathtt{det}(sU_1-U_2)\neq 0$. Let $\lambda\in\mathtt{roots}\,(\mathtt{det}(sU_1-U_2))$. Then λ is called an *eigenvalue* of (U_1,U_2) and every nonzero vector $v\in\mathtt{ker}\,(\lambda U_1-U_2)$ is called an *eigenvector* of the matrix pair (U_1,U_2) corresponding to the eigenvalue λ.

We use the symbol $\sigma(U_1,U_2)$ to denote the set of eigenvalues of (U_1,U_2) (with $\lambda\in\sigma(U_1,U_2)$ included in the set as many times as its algebraic multiplicity). In this paper, we extensively use one of the canonical forms of a linear matrix pencil (see Dai (1989) for more on different canonical forms). We review the result that leads to such a canonical form next (Dai, 1989, Lemma 1-2.2).

Proposition 3. A matrix pair (U_1,U_2) is regular if and only if there exist nonsingular matrices Z_1 and Z_2 such that $Z_1U_1Z_2=\mathtt{diag}(I_{\mathtt{n}_1},N)$ and $Z_1U_2Z_2=\mathtt{diag}(U,I_{\mathtt{n}_2})$, where $\mathtt{n}_1+\mathtt{n}_2=\mathtt{n}$, $U\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, and $N\in\mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent.

The matrix pair $\left(\left[\begin{smallmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{smallmatrix}\right],\left[\begin{smallmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{smallmatrix}\right]\right)$ is said to be in a canonical form. Note that $\mathtt{det}(sU_1-U_2)=k\times\mathtt{det}(sI_{\mathtt{n}_1}-U)$, where $k\in\mathbb{R}\setminus\{0\}$. The following two lemmas provide some important properties related to the generalized eigenspaces.

Lemma 4. Consider the matrix pair $\left(\left[\begin{smallmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{smallmatrix}\right],\left[\begin{smallmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{smallmatrix}\right]\right)$, where $U\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, and $N\in\mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent. Let $\widetilde{W}_1\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{k}}$, $\widetilde{W}_2\in\mathbb{R}^{\mathtt{n}_2\times\mathtt{k}}$, and $\widetilde{\Gamma}\in\mathbb{R}^{\mathtt{k}\times\mathtt{k}}$ be such that

$$\begin{bmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{bmatrix}\begin{bmatrix}\widetilde{W}_1\\ \widetilde{W}_2\end{bmatrix}=\begin{bmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{bmatrix}\begin{bmatrix}\widetilde{W}_1\\ \widetilde{W}_2\end{bmatrix}\widetilde{\Gamma}. \tag{3}$$

Then, $\widetilde{W}_2=0$.

Proof. From eq. (3), we get $\widetilde{W}_2=N\widetilde{W}_2\widetilde{\Gamma}$. Now, if we keep substituting $\widetilde{W}_2=N\widetilde{W}_2\widetilde{\Gamma}$ on the right-hand side of the equation, then clearly we have $\widetilde{W}_2=N^i\widetilde{W}_2\widetilde{\Gamma}^i$ for all $i\in N$. But, N is a nilpotent matrix. Therefore. $\widetilde{W}_2=0$. □

Lemma 5. Let the matrix pair (U_1,U_2) with $U_1,U_2\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be such that $\mathtt{degdet}(sU_1-U_2)=:\mathtt{n}_1\neq 0$. Then,

(1) There exist a full column-rank matrix $W\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}_1}$ and $\Gamma\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ with $\mathtt{det}(sI_{\mathtt{n}_1}-\Gamma)=\mathtt{det}(sU_1-U_2)$ such that $U_2W=U_1W\Gamma$.
(2) There exist $T_1,T_2\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ non-singular such that $T_1U_1T_2=\mathtt{diag}(I_{\mathtt{n}_1},N)$ and $T_1U_2T_2=\mathtt{diag}(\Gamma,I_{\mathtt{n}_2})$, where $\mathtt{n}_1+\mathtt{n}_2=\mathtt{n}$ and $N\in\mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent.

Proof. (1): According to Proposition 3, there exist $Z_1,Z_2\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ non-singular such that $Z_1U_1Z_2=\mathtt{diag}(I_{\mathtt{n}_1},N)$ and $Z_1U_2Z_2=\mathtt{diag}(U,I_{\mathtt{n}_2})$, where $U\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$, $\mathtt{det}(sI_{\mathtt{n}_1}-U)=\mathtt{det}(sU_1-U_2)$, and $N\in\mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ is nilpotent. Evidently, if a matrix $\Gamma\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ is similar to the matrix U, then there exists $W_1\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ non-singular such that $UW_1=W_1\Gamma$. Then, clearly, $\left[\begin{smallmatrix} U & 0\\ 0 & I_{\mathtt{n}_2}\end{smallmatrix}\right]\left[\begin{smallmatrix} W_1\\ 0\end{smallmatrix}\right]=\left[\begin{smallmatrix} I_{\mathtt{n}_1} & 0\\ 0 & N\end{smallmatrix}\right]\left[\begin{smallmatrix} W_1\\ 0\end{smallmatrix}\right]\Gamma$ holds. Consequently, the equation $U_2W=U_1W\Gamma$ is satisfied, where $W:=Z_2^{-1}\left[\begin{smallmatrix} W_1\\ 0\end{smallmatrix}\right]$. Since $W_1\in\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ is nonsingular, we must have that $W\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}_1}$ is full column-rank. Also, $\mathtt{det}(sI_{\mathtt{n}_1}-\Gamma)=\mathtt{det}(sI_{\mathtt{n}_1}-U)=\mathtt{det}(sU_1-U_2)$.
(2): Recall that $W_1^{-1}UW_1=\Gamma$. Define $\widetilde{T}:=\mathtt{diag}(W_1,I_{\mathtt{n}_2})$, $T_1:=\widetilde{T}^{-1}Z_1$, and $T_2:=Z_2\widetilde{T}$. It is evident that T_1 and T_2 are non-singular. Further, it is easy to verify that $T_1U_1T_2=\mathtt{diag}(I_{\mathtt{n}_1},N)$ and $T_1U_2T_2=\mathtt{diag}(\Gamma,I_{\mathtt{n}_2})$. □

2.3 The slow subspace

Definition 6. Consider the system Σ given by

$$\frac{d}{dt}x(t)=Ax(t)+Bu(t),\ \ y(t)=Cx(t)+Du(t), \tag{4}$$

where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B\in\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C\in\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, and $D\in\mathbb{R}^{\mathtt{m}\times\mathtt{m}}$. A state $x_0\in\mathbb{R}^{\mathtt{n}}$ is called *weakly unobservable* if there exists an input $u(t)\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ such that $y(t;x_0,u)\equiv 0$ for all $t\geqslant 0$, where $y(t;x_0,u)$ is the output of the system corresponding to the initial condition x_0 and the input $u(t)$. The collection of all such weakly unobservable states is called the *weakly unobservable subspace* or the *slow space* of the state-space and is denoted by $\mathcal{O}_w$.

The following property of the slow space is crucially used in this paper ((Hautus and Silverman, 1983, Theorem 3.10)).

Proposition 7. The slow space $\mathcal{O}_w$ is the largest subspace $\mathcal{V}$ of the state-space for which there exists a feedback $F\in\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A+BF)\mathcal{V}\subseteq\mathcal{V}$ and $(C+DF)\mathcal{V}=0$.

An important subspace of the slow space, $\mathcal{O}_w$ is the "good" slow space $\mathcal{O}_{wg}$. We formally define this subspace next.

Definition 8. The good slow space $\mathcal{O}_{wg}$ is the largest subspace $\mathcal{V}$ of the state-space for which there exists a feedback $F\in\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that

$$(A+BF)\mathcal{V}\subseteq\mathcal{V},\ (C+DF)\mathcal{V}=0,\ \text{and}\ \sigma((A+BF)|_{\mathcal{V}})\subseteq\mathbb{C}_-.$$

3. CHARACTERIZATION OF THE SLOW SPACE

Corresponding to the system Σ (see eq. (4)), we define

$$U_1:=\left[\begin{smallmatrix} I_{\mathtt{n}} & 0\\ 0 & 0\end{smallmatrix}\right]\in\mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}\ \text{and}\ U_2:=\left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right]. \tag{5}$$

The pair (U_1,U_2) is called the Rosenbrock matrix pair and the matrix (sU_1-U_2) is the Rosenbrock matrix pencil corresponding to the system Σ. Throughout this paper we assume that the matrix pencil is regular. We present this section in two parts. In the first part, we characterize the slow space of Σ, and in the second part we characterize the good slow space of Σ.

3.1 Characterization of the slow space in terms of an eigenspace of the Rosenbrock matrix pair

In the following theorem we characterize the slow space, $\mathcal{O}_w$, of the system Σ. This theorem also provides us with the dimension of the subspace $\mathcal{O}_w$.

Theorem 9. Consider the system Σ defined in eq. (4) and the corresponding Rosenbrock matrix pair (U_1,U_2) as defined in eq. (5). Assume that $\mathtt{det}(sU_1-U_2)\neq 0$ and $\mathtt{degdet}(sU_1-U_2)=:\mathtt{n_s}$. Let $V_1\in\mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_2\in\mathbb{R}^{\mathtt{m}\times\mathtt{n_s}}$ be such that $\mathtt{col}(V_1,V_2)$ is full column-rank and

$$\underbrace{\begin{bmatrix} A & B\\ C & D\end{bmatrix}}_{U_2}\begin{bmatrix} V_1\\ V_2\end{bmatrix}=\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0\\ 0 & 0\end{bmatrix}}_{U_1}\begin{bmatrix} V_1\\ V_2\end{bmatrix}J, \tag{6}$$

where $J\in\mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\mathtt{det}(sI_{\mathtt{n_s}}-J)=\mathtt{det}(sU_1-U_2)$. (Such V_1,V_2, and J exist due to Lemma 5.) Let $\mathcal{O}_w$ be the slow space of Σ. Then, the following statements hold:

(1) V_1 is full column-rank.
(2) $\mathcal{O}_w=\mathtt{img}V_1$.
(3) $\mathtt{dim}(\mathcal{O}_w)=\mathtt{n_s}$.

Proof. (1): To the contrary, assume that V_1 is not full column-rank. Then, there exists $T \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$, non-singular such that $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] T = \left[\begin{smallmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{smallmatrix}\right]$, where $V_{11} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_1}}$, $\mathtt{n_1} := \mathtt{rank}V_1$, and $V_{22} \in \mathbb{R}^{\mathtt{m}\times(\mathtt{n_s}-\mathtt{n_1})}$. Define $\widehat{J} := T^{-1}JT =: \begin{bmatrix} \widehat{J}_{11} & \widehat{J}_{12} \\ \widehat{J}_{21} & \widehat{J}_{22} \end{bmatrix}$, where $\widehat{J}_{11} \in \mathbb{R}^{\mathtt{n_1}\times\mathtt{n_1}}$ and $\widehat{J}_{22} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{n_1})\times(\mathtt{n_s}-\mathtt{n_1})}$. So, from eq. (6) it follows that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \widehat{J}, \tag{7}$$

Thus, $AV_{11} + BV_{21} = V_{11}\widehat{J}_{11},\ BV_{22} = V_{11}\widehat{J}_{12},$

$$CV_{11} + DV_{21} = 0, \text{ and } DV_{22} = 0. \tag{8}$$

From eq. (8) it is clear that the $\widehat{J}_{21}$ and $\widehat{J}_{22}$ blocks in $\widehat{J}$ are free; in particular, choose $\widehat{J}_{21} = 0$ and $\widehat{J}_{22}$ such that $\sigma(\widehat{J}_{22}) \subsetneq \mathtt{roots}(\mathtt{det}(sU_1 - U_2))$ also satisfy eq. (7). In that case, clearly $\mathtt{det}(sI_{\mathtt{n_s}} - \widehat{J}) \neq \mathtt{det}(sU_1 - U_2)$. This is a contradiction, because $\widehat{J} = T^{-1}JT$ and thus $\mathtt{det}(sI_{\mathtt{n_s}} - \widehat{J}) = \mathtt{det}(sI_{\mathtt{n_s}} - J) = \mathtt{det}(sU_1 - U_2)$. So, our initial assumption cannot be true. Hence, V_1 is full column-rank.

(2): Since V_1 is full column-rank, there exists $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $V_2 = FV_1$. So, eq. (6) can also be written as

$$(A + BF)V_1 = V_1 J \text{ and } (C + DF)V_1 = 0. \tag{9}$$

From eq. (9) and Proposition 7 it follows that $\mathtt{img}V_1 \subseteq \mathcal{O}_w$. To the contrary assume that $\mathtt{img}V_1 \neq \mathcal{O}_w$. Thus, there exists a non-trivial subspace $\mathcal{V}_{\mathtt{e}}$ such that $\mathtt{img}V_1 \oplus \mathcal{V}_{\mathtt{e}} = \mathcal{O}_w$. Define $\ell := \mathtt{dim}\mathcal{V}_{\mathtt{e}}$ and let $V_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times\ell}$ be such that $\mathtt{img}V_{\mathtt{e}} = \mathcal{V}_{\mathtt{e}}$. Now, from (Wonham, 1985, Lemma 5.7) and Proposition 7 it is evident that there exists $F_{\mathtt{e}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $F_{\mathtt{e}}|_{\mathtt{img}V_1} = F|_{\mathtt{img}V_1}$ and

$$(A + BF_{\mathtt{e}})\mathcal{O}_w \subseteq \mathcal{O}_w \text{ and } (C + DF_{\mathtt{e}})\mathcal{O}_w = 0. \tag{10}$$

Thus, from eq. (9) it follows that

$$(A + BF_{\mathtt{e}})V_1 = V_1 J \text{ and } (C + DF_{\mathtt{e}})V_1 = 0. \tag{11}$$

Also, since $\mathtt{img}V_{\mathtt{e}} \subseteq \mathcal{O}_w$, from eq. (10) it is clear that there exist $T_1 \in \mathbb{R}^{\mathtt{n_s}\times\ell}$ and $T_{\mathtt{e}} \in \mathbb{R}^{\ell\times\ell}$ such that

$$(A + BF_{\mathtt{e}})V_{\mathtt{e}} = [V_1\ \ V_{\mathtt{e}}] \begin{bmatrix} T_1 \\ T_{\mathtt{e}} \end{bmatrix} \text{ and } (C + DF_{\mathtt{e}})V_{\mathtt{e}} = 0. \tag{12}$$

Recall that $F_{\mathtt{e}}|_{\mathtt{img}V_1} = F|_{\mathtt{img}V_1}$. Thus, $F_{\mathtt{e}}V_1 = FV_1 = V_2$. So, combining eq. (11) and eq. (12) together, we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} \begin{bmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{bmatrix}, \tag{13}$$

where $V_{2\mathtt{e}} := F_{\mathtt{e}}V_{\mathtt{e}}$. Due to Statement (2) of Lemma 5, there exist nonsingular $Y, Z \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$ such that

$$U_1 = Y \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} Z \text{ and } U_2 = Y \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} Z, \tag{14}$$

where $N \in \mathbb{R}^{(\mathtt{n}+\mathtt{m}-\mathtt{n_s})\times(\mathtt{n}+\mathtt{m}-\mathtt{n_s})}$ is nilpotent. Thus, using eq. (14) in eq. (13) and using invertibility of Y we get that

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} Z \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} Z \begin{bmatrix} V_1 & V_{\mathtt{e}} \\ V_2 & V_{2\mathtt{e}} \end{bmatrix} \begin{bmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{bmatrix}. \tag{15}$$

Define $Z\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} =: \begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix}$ and $Z\begin{bmatrix} V_{\mathtt{e}} \\ V_{2\mathtt{e}} \end{bmatrix} =: \begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ \widehat{V}_{2\mathtt{e}} \end{bmatrix}$, where $\widehat{V}_1 \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ and $\widehat{V}_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n_s}\times\ell}$. We rewrite eq. (15) as

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{V}_1 & \widehat{V}_{\mathtt{e}} \\ \widehat{V}_2 & \widehat{V}_{2\mathtt{e}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \widehat{V}_1 & \widehat{V}_{\mathtt{e}} \\ \widehat{V}_2 & \widehat{V}_{2\mathtt{e}} \end{bmatrix} \begin{bmatrix} J & T_1 \\ 0 & T_{\mathtt{e}} \end{bmatrix}. \tag{16}$$

Thus, from Lemma 4, we have $[\widehat{V}_2\ \ \widehat{V}_{2\mathtt{e}}] = 0$. Since Z is non-singular, $\mathtt{rank}\begin{bmatrix} \widehat{V}_1 \\ \widehat{V}_2 \end{bmatrix} = \mathtt{rank}\begin{bmatrix} \widehat{V}_1 \\ 0 \end{bmatrix} = \mathtt{rank}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathtt{n_s}$. Thus, $\widehat{V}_1$ is non-singular. Hence, $\mathtt{img}\widehat{V}_{\mathtt{e}} \subseteq \mathtt{img}\widehat{V}_1$. So,

$$\mathtt{img}\begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ \widehat{V}_{2\mathtt{e}} \end{bmatrix} = \mathtt{img}\begin{bmatrix} \widehat{V}_{\mathtt{e}} \\ 0 \end{bmatrix} \subseteq \mathtt{img}\begin{bmatrix} \widehat{V}_1 \\ 0 \end{bmatrix} \Rightarrow \mathtt{img}\begin{bmatrix} V_{\mathtt{e}} \\ V_{2\mathtt{e}} \end{bmatrix} \subseteq \mathtt{img}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

Therefore $\mathtt{img}V_{\mathtt{e}} \subseteq \mathtt{img}V_1$. This is a contradiction. Thus, there does not exist any nontrivial subspace $\mathcal{V}_{\mathtt{e}}$ such that $\mathtt{img}V_1 \oplus \mathcal{V}_{\mathtt{e}} = \mathcal{O}_w$. This, again, is a contradiction to the assumption that $\mathtt{img}V_1 \neq \mathcal{O}_w$. Hence, $\mathcal{O}_w = \mathtt{img}V_1$.

(3): $\mathtt{rank}V_1 = \mathtt{n_s}$ and $\mathcal{O}_w = \mathtt{img}V_1$. Hence, $\mathtt{dim}(\mathcal{O}_w) = \mathtt{n_s}$. □

3.2 Characterization of the good slow space in terms of a stable eigenspace of the Rosenbrock matrix pair

Notice that, we can partition $\sigma(J)$ as $\sigma(J) = \sigma_{\mathtt{g}}(J) \cup \sigma_{\mathtt{b}}(J)$, where $\sigma_{\mathtt{g}}(J) \subseteq \mathbb{C}_-$ and $\sigma_{\mathtt{b}}(J) \subseteq \overline{\mathbb{C}}_+$. Define $\mathtt{n_g} := |\sigma_{\mathtt{g}}(J)|$. Clearly, there exists a non-singular matrix $T \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}$ such that $T^{-1}JT = \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}$, where $J_{\mathtt{g}} \in \mathbb{R}^{\mathtt{n_g}\times\mathtt{n_g}}$, $J_{\mathtt{b}} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{n_g})\times(\mathtt{n_s}-\mathtt{n_g})}$, $\sigma(J_{\mathtt{g}}) = \sigma_{\mathtt{g}}(J)$, and $\sigma(J_{\mathtt{b}}) = \sigma_{\mathtt{b}}(J)$. Define

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} T =: \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix}, \tag{17}$$

where $V_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_g}}$ and $V_{2\mathtt{b}} \in \mathbb{R}^{\mathtt{m}\times(\mathtt{n_s}-\mathtt{n_g})}$. From eq. (6) it follows that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} T = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} TT^{-1}JT$. So, by eq. (17)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{1\mathtt{b}} \\ V_{2\mathtt{g}} & V_{2\mathtt{b}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & 0 \\ 0 & J_{\mathtt{b}} \end{bmatrix}. \tag{18}$$

Now we characterize the good slow space, $\mathcal{O}_{wg}$ of Σ.

Lemma 10. Consider the system Σ and the corresponding Rosenbrock matrix pair (U_1, U_2) as defined in eq. (4) and eq. (5), respectively. Assume that $\mathtt{det}(sU_1 - U_2) \neq 0$. Consider the matrix $V_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_g}}$ as defined in eq. (17) and $\mathtt{n_g} = |\sigma_{\mathtt{g}}(J)|$. Then, the following statements hold:

(1) $V_{1\mathtt{g}}$ is full column-rank.
(2) $\mathcal{O}_{wg} = \mathtt{img}V_{1\mathtt{g}}$.
(3) $\mathtt{dim}(\mathcal{O}_{wg}) = \mathtt{n_g}$.

Proof. (1): Since V_1 is full column-rank (see Theorem 9) and T is non-singular, it is evident that $V_1T = [V_{1\mathtt{g}}\ V_{1\mathtt{b}}]$ is full column-rank. Consequently, $V_{1\mathtt{g}}$ is full column-rank.

(2): Since $V_{1\mathtt{g}}$ is full column-rank, there exists $F_{\mathtt{g}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $V_{2\mathtt{g}} = F_{\mathtt{g}}V_{1\mathtt{g}}$. Thus, from eq. (18) it follows that $(A + BF_{\mathtt{g}})V_{1\mathtt{g}} = V_{1\mathtt{g}}J_{\mathtt{g}}$ and $(C + DF_{\mathtt{g}})V_{1\mathtt{g}} = 0$. Recall that $\sigma(J_{\mathtt{g}}) \subseteq \mathbb{C}_-$. Thus, $\mathtt{img}V_{1\mathtt{g}} \subseteq \mathcal{O}_{wg}$. Now, to the contrary, we assume that $\mathtt{img}V_{1\mathtt{g}} \neq \mathcal{O}_{wg}$. So, there exists a non-trivial subspace $\mathcal{V}_{\mathtt{eg}}$ such that $\mathtt{img}V_{1\mathtt{g}} \oplus \mathcal{V}_{\mathtt{eg}} = \mathcal{O}_{wg}$. Define $\mathtt{n_{eg}} := \mathtt{dim}\mathcal{V}_{\mathtt{eg}}$ and let $V_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_{eg}}}$ be such that $\mathtt{img}V_{\mathtt{eg}} = \mathcal{V}_{\mathtt{eg}}$. Next, by Definition 8, there exists $F_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF_{\mathtt{eg}})\mathcal{O}_{wg} \subseteq \mathcal{O}_{wg}$, $(C + DF_{\mathtt{eg}}) = 0$, $\sigma((A + BF_{\mathtt{eg}})|_{\mathcal{O}_{wg}}) \subseteq \mathbb{C}_-$, and $F_{\mathtt{eg}}|_{\mathtt{img}V_{1\mathtt{g}}} = F_{\mathtt{g}}|_{\mathtt{img}V_{1\mathtt{g}}}$. Thus, there exist $T_1 \in \mathbb{R}^{\mathtt{n_g}\times\mathtt{n_{eg}}}$ and $T_2 \in \mathbb{R}^{\mathtt{n_{eg}}\times\mathtt{n_{eg}}}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}, \tag{19}$$

where $\sigma(T_2) \subseteq \mathbb{C}_-$, $V_{2\mathtt{eg}} := F_{\mathtt{eg}}V_{\mathtt{eg}}$, and $V_{2\mathtt{g}} = F_{\mathtt{g}}V_{1\mathtt{g}} = F_{\mathtt{eg}}V_{1\mathtt{g}}$ ($\because F_{\mathtt{eg}}|_{\mathtt{img}V_{1\mathtt{g}}} = F_{\mathtt{g}}|_{\mathtt{img}V_{1\mathtt{g}}}$). Similar to the proof of Statement (2) of Theorem 9, we use eq. (14) in eq. (19) to obtain

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} Z \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} Z \begin{bmatrix} V_{1\mathtt{g}} & V_{\mathtt{eg}} \\ V_{2\mathtt{g}} & V_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \tag{20}$$

Define $Z\begin{bmatrix} V_{1\mathtt{g}} \\ V_{2\mathtt{g}} \end{bmatrix} =: \begin{bmatrix} \widehat{V}_{1\mathtt{g}} \\ \widehat{V}_{2\mathtt{g}} \end{bmatrix}$ and $Z\begin{bmatrix} V_{\mathtt{eg}} \\ V_{2\mathtt{eg}} \end{bmatrix} =: \begin{bmatrix} \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{eg}} \end{bmatrix}$, where $\widehat{V}_{1\mathtt{g}} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_g}}$ and $\widehat{V}_{\mathtt{eg}} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_{eg}}}$. Thus, from eq. (20), we get

$$\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{eg}} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n_s}} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \widehat{V}_{1\mathtt{g}} & \widehat{V}_{\mathtt{eg}} \\ \widehat{V}_{2\mathtt{g}} & \widehat{V}_{2\mathtt{eg}} \end{bmatrix} \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \tag{21}$$

Due to Lemma 4, $[\widehat{V}_{2\mathtt{g}}\ \ \widehat{V}_{2\mathtt{eg}}] = 0$. Thus, eq. (21) reduces to

$$J\,[\widehat{V}_{1\mathtt{g}}\ \ \widehat{V}_{\mathtt{eg}}] = [\widehat{V}_{1\mathtt{g}}\ \ \widehat{V}_{\mathtt{eg}}] \begin{bmatrix} J_{\mathtt{g}} & T_1 \\ 0 & T_2 \end{bmatrix}. \tag{22}$$

From eq. (22), it is evident that $\sigma(J_{\mathtt{g}}) \cup \sigma(T_2) \subseteq \sigma(J)$. But, we have assumed that $\sigma(J) \cap \mathbb{C}_- = \sigma(J_{\mathtt{g}})$. Hence, $\sigma(T_2) \subseteq \overline{\mathbb{C}}_+$. This is a contradiction. Accordingly, there does not exist any non-trivial subspace $\mathcal{V}_{\mathtt{eg}}$ such that $\mathtt{img}V_{1\mathtt{g}} \oplus \mathcal{V}_{\mathtt{eg}} = \mathcal{O}_{wg}$. Hence, $\mathtt{img}V_{1\mathtt{g}} = \mathcal{O}_{wg}$.

(3): Directly follows from Statement (1) and Statement (2).

4. APPLICATION TO THE HAMILTONIAN SYSTEM ARISING FROM THE SINGULAR LQR PROBLEM

In this section, we apply the results developed in Section 3 to a special system, namely the Hamiltonian System arising from the singular LQR problem (Problem 1). Since, $R \geqslant 0$, there exists an orthogonal $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ such that $U^T R U = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \widehat{R}\end{smallmatrix}\right]$, where $\widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ and $\mathtt{r} := \mathtt{rank}R$. If we define $BU =: [\,B_1 \; B_2\,]$ and $SU =: [\,S_1 \; S_2\,]$, where $B_2, S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$, then we have that $S_1 = 0$ (see (Bhawal et al., 2019a, Lemma 1)). Hence, without loss of generality, any singular LQR problem can be written as:

Problem 11. Consider the stabilizable system given by

$$\frac{d}{dt}x(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \tag{23}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B_1 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{m}-\mathtt{r})}$, and $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. Then, for every initial condition $x(0) = x_0$, find an input $u(t) := \mathtt{col}(u_1(t), u_2(t))$ that minimizes the functional

$$J(x_o, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t)\end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix} \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t)\end{bmatrix} dt, \tag{24}$$

with $\lim_{t\to\infty} x(t) = 0$, where $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}, \widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$, $\left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right] \geqslant 0$, and $\widehat{R} > 0$.

Next, we define the *primal* for the LQR Problem 11.

Definition 12. Consider the LQR Problem 11. Let

$$\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix} = \begin{bmatrix} C^T \\ 0 \\ D_2^T\end{bmatrix} [\,C \; 0 \; D_2\,], \tag{25}$$

where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, and $\mathtt{p} := \mathtt{rank}\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix}$.

Define the system $\Sigma_{pr} : \frac{d}{dt}x(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t)$ and $y(t) = Cx(t) + D_2 u_2(t)$. We call the system Σ_{pr}, the *primal* for the LQR Problem 11.

Another important system arising from an LQR problem is the Hamiltonian system. We obtain this system using Pontryagin's maximum principle (PMP) to Problem 11:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1 \\ u_2\end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2 \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R}\end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u_1 \\ u_2\end{bmatrix}, \tag{26}$$

where $E \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$ is partitioned conforming to the partition in H. $\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right]$ is called the state-costate pair. It follows from Pontryagin's maximum principle that (x^*, u^*) is an optimal trajectory of the primal Σ_{pr} if and only if there exists z^* such that (x^*, z^*, u^*) belongs to the Hamiltonian system. Hence, the trajectories of the Hamiltonian system are of special interest. Recall that, $\widehat{R}$ is non-singular, and hence u_2 can be eliminated from eq. (26) to obtain

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0\end{bmatrix}}_{E_{\mathtt{r}}} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1\end{bmatrix} = \underbrace{\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0\end{bmatrix}}_{H_{\mathtt{r}}} \begin{bmatrix} x \\ z \\ u_1\end{bmatrix}, \tag{27}$$

where $\widetilde{A} := A - B_2\widehat{R}^{-1}S_2^T$, $A_z := B_2\widehat{R}^{-1}B_2^T$, $\widetilde{B} := B_1$, and $\widetilde{Q} := Q - S_2\widehat{R}^{-1}S_2^T \geqslant 0$ (by using the notion of Schur complement on the cost matrix). This system is called the *reduced Hamiltonian system.* Throughout this paper, we assume that $\mathtt{det}(sE_{\mathtt{r}} - H_{\mathtt{r}}) \neq 0$. Notice that the reduced Hamiltonian system admits an output-nulling representation:

$$\Sigma_{\mathtt{Ham}} : \; \frac{d}{dt}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] = \underbrace{\begin{bmatrix} \widetilde{A} & -A_z \\ -\widetilde{Q} & -\widetilde{A}^T\end{bmatrix}}_{A_{\mathtt{r}}}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] + \underbrace{\begin{bmatrix} \widetilde{B} \\ 0\end{bmatrix}}_{B_{\mathtt{r}}} u_1 \text{ and } 0 = \underbrace{[\,0 \; \widetilde{B}^T\,]}_{C_{\mathtt{r}}}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right]. \tag{28}$$

Clearly, $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as defined in eq. (27) is the Rosenbrock matrix pair of $\Sigma_{\mathtt{Ham}}$. Say, $\Lambda := \sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-$, $\mathtt{n_s} := |\Lambda|$, $V_1, V_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$, and $V_3 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n_s}}$ be such that the columns of the matrix $V_{\mathtt{e}} := \mathtt{col}(V_1, V_2, V_3)$ form a basis for the $\mathtt{n_s}$-dimensional stable eigenspace of $(E_{\mathtt{r}}, H_{\mathtt{r}})$, i.e.,

$$\underbrace{\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0\end{bmatrix}}_{H_{\mathtt{r}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3\end{bmatrix}}_{V_{\mathtt{e}}} = \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0\end{bmatrix}}_{E_{\mathtt{r}}} \begin{bmatrix} V_1 \\ V_2 \\ V_3\end{bmatrix} J, \tag{29}$$

where $J \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_s}}, \sigma(J) = \Lambda$. Thus, we can directly apply Lemma 10 to infer that the good slow space $\mathcal{O}_{wg}$ of $\Sigma_{\mathtt{Ham}}$ is given by $\mathcal{O}_{wg} = \mathtt{img}\left[\begin{smallmatrix} V_1 \\ V_2\end{smallmatrix}\right]$. How the subspace $\mathcal{O}_{wg}$ can be used to solve the regular LQR problem is well-known in the literature (see (Ionescu et al., 1999, Chapter 5)). In Bhawal and Pal (2019) $\mathcal{O}_{wg}$ has been used to solve the singular LQR problem for the single-input case. How to use this subspace to solve the singular LQR problem for the multi-input case is a matter of our future research.

Next, we divide this section in two parts: we first show a relation between the good slow space ($\mathcal{V}_g$) of the primal Σ_{pr} and the subspace $\mathcal{O}_{wg}$. In the second part, we show that the subspace $\mathtt{img}V_{\mathtt{e}}$ is *disconjugate* (see Definition 15).

4.1 Relation between the spaces $\mathcal{V}_g$ and $\mathcal{O}_{wg}$

The following lemma is crucially used to establish a relation between $\mathcal{V}_g$ and $\mathcal{O}_{wg}$.

Lemma 13. Consider the LQR Problem 11 and the corresponding primal Σ_{pr} as defined in Definition 12. Further define the system $\Sigma_{aux} : \frac{d}{dt}x(t) = \widetilde{A}x(t) + \widetilde{B}u(t)$, $y(t) = \widetilde{C}x(t)$, where $\widetilde{A}, \widetilde{B}$ are as defined in eq. (27) and $\widetilde{C} := C - D_2\widehat{R}^{-1}S_2^T$. Let $\mathcal{V}_g$ and $\mathcal{W}_g$ be the good slow spaces of Σ_{pr} and Σ_{aux}, respectively. Then, $\mathcal{V}_g = \mathcal{W}_g$.

Proof. ($\mathcal{W}_g \subseteq \mathcal{V}_g$): Say $\mathtt{dim}(\mathcal{W}_g) =: \mathtt{g_1}$ and $W_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g_1}}$ be such that $\mathcal{W}_g = \mathtt{img}W_g$. Clearly, there exist $F \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ and $J_1 \in \mathbb{R}^{\mathtt{g_1}\times\mathtt{g_1}}$ such that $(\widetilde{A} + \widetilde{B}F)W_g = W_g J_1$ and $\widetilde{C}W_g = 0$, where $\sigma(J_1) \subseteq \mathbb{C}_-$. Hence, from definition of $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$, it immediately follows that $(A + B_1F - B_2\widehat{R}^{-1}S_2^T)W_g = W_g J_1 \Rightarrow (A + [\,B_1 \; B_2\,]\left[\begin{smallmatrix} F \\ -\widehat{R}^{-1}S_2^T\end{smallmatrix}\right])W_g = W_g J_1$. Also, $(C - D_2\widehat{R}^{-1}S_2^T)W_g = 0 \Rightarrow (C + [\,0 \; D_2\,]\left[\begin{smallmatrix} F \\ -\widehat{R}^{-1}S_2^T\end{smallmatrix}\right])W_g = 0$. Consequently, $\mathcal{W}_g \subseteq \mathcal{V}_g$.

($\mathcal{V}_g \subseteq \mathcal{W}_g$): Say, $\mathtt{dim}(\mathcal{V}_g) =: \mathtt{g_2}$ and $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g_2}}$ be such that $\mathcal{V}_g = \mathtt{img}V_g$. Thus, there exist $F_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}, F_2 \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$, and $J_2 \in \mathbb{R}^{\mathtt{g_2}\times\mathtt{g_2}}$ such that

$$(A + [\,B_1 \; B_2\,]\left[\begin{smallmatrix} F_1 \\ F_2\end{smallmatrix}\right])V_g = V_g J_2 \text{ and } (C + [\,0 \; D_2\,]\left[\begin{smallmatrix} F_1 \\ F_2\end{smallmatrix}\right])V_g = 0, \tag{30}$$

where $\sigma(J_2) \subseteq \mathbb{C}_-$. Now, $(C + [\,0 \; D_2\,]\left[\begin{smallmatrix} F_1 \\ F_2\end{smallmatrix}\right])V_g = 0 \Rightarrow D_2F_2V_g = -CV_g \Rightarrow D_2^TD_2F_2V_g = -D_2^TCV_g$. Notice, from eq. (25), that $D_2^TD_2 = \widehat{R}$ and $C^TD_2 = S_2$. Thus, we have

$$F_2V_g = -\widehat{R}^{-1}S_2^TV_g. \tag{31}$$

Next, using eq. (31) in eq. (30), we get $(\widetilde{A} + \widetilde{B}F_1)V_g = V_gJ_2$ and $\widetilde{C}V_g = 0$. Thus, $\mathcal{V}_g \subseteq \mathcal{W}_g$. Consequently, $\mathcal{V}_g = \mathcal{W}_g$. □

The next lemma shows that $\mathcal{V}_g$ is embedded into $\mathcal{O}_{wg}$.

Lemma 14. Let $\mathcal{V}_g$ and $\mathcal{O}_{wg}$ be the good slow spaces of the primal Σ_{pr} (Definition 12) and the Hamiltonian system $\Sigma_{\mathtt{Ham}}$ (eq. (28)), respectively. Define the subspace $\mathcal{V}_{\mathtt{gHam}} := \{\left[\begin{smallmatrix} v \\ 0\end{smallmatrix}\right] \in \mathbb{R}^{2\mathtt{n}} \mid v \in \mathcal{V}_g\}$. Then, $\mathcal{V}_{\mathtt{gHam}} \subseteq \mathcal{O}_{wg}$.

Proof. Let $\mathtt{g} := \mathtt{dim}\mathcal{V}_{\mathtt{g}}$ and $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$ be such that $\mathcal{V}_g = \mathtt{img}V_g$. Thus, using Lemma 13, we infer that there exists $F \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ such that

$$(\widetilde{A} + \widetilde{B}F)V_g = V_gJ_g \text{ and } \widetilde{C}V_g = 0, \tag{32}$$

where $\sigma(J_g) \subseteq \mathbb{C}_-$. Also, since $\widetilde{C} = C - D_2\widehat{R}^{-1}S_2^T$, it is easy to verify that $\widetilde{C}^T\widetilde{C} = \widetilde{Q}$. Hence, defining $V_{3g} := FV_g$, we get:

$$\begin{bmatrix} A_{\mathtt{r}} & B_{\mathtt{r}} \\ C_{\mathtt{r}} & 0 \end{bmatrix} \begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g} \end{bmatrix} = \begin{bmatrix} I_{2\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \\ V_{3g} \end{bmatrix} J_g, \tag{33}$$

where $A_{\mathtt{r}}, B_{\mathtt{r}}$, and $C_{\mathtt{r}}$ are as defined in eq. (28). From eq. (33), it is clear that $A_{\mathtt{r}} \left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] + B_{\mathtt{r}} V_{3g} = (A_{\mathtt{r}} + B_{\mathtt{r}} \left[\begin{smallmatrix} F & 0_{(\mathtt{m}-\mathtt{r}),\mathtt{n}} \end{smallmatrix}\right]) \left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] = \left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] J_g$; and $C_{\mathtt{r}} \left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] = 0$. Thus, from Definition 8, it is evident that $\mathtt{img}\left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] \subseteq \mathcal{O}_{wg}$. But, notice that $\mathtt{img}\left[\begin{smallmatrix} V_g \\ 0_{\mathtt{n},\mathtt{g}} \end{smallmatrix}\right] = \mathcal{V}_{\mathtt{gHam}}$. Hence, $\mathcal{V}_{\mathtt{gHam}} \subseteq \mathcal{O}_{wg}$. □

4.2 Disconjugacy of $\mathtt{img}V_{\mathtt{e}}$

In this section, we show that the subspace $\mathtt{img}V_{\mathtt{e}}$ (eq. (29)) is disconjugate. Following is the definition of disconjugacy.

Definition 15. Let $\mathcal{W}$ be an eigenspace of the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as defined in eq. (27). Assume that the columns of the matrix W form a basis for the eigenspace $\mathcal{W}$. Further, conforming to the partition in $H_{\mathtt{r}}$, say W be partitioned as $\mathtt{col}(W_1, W_2, W_3)$. Then, $\mathcal{W}$ is said to be *disconjugate* if W_1 is full column-rank.

From this definition, it is clear that disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ is equivalent to V_1 (eq. (29)) being full column-rank. We show at the end of this section that if the system starts from an initial condition from $\mathtt{img}V_1$, then the singular LQR problem can be solved using a smooth input. Thus, disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ provides us with the basis and dimension of the subspace of the state-space for which the problem can be solved using smooth input if the system starts from that subspace. Disconjugacy of $\mathtt{img}V_{\mathtt{e}}$ also enables us to provide a feedback law if the initial condition is from $\mathtt{img}V_1$ (Theorem 19). To prove that V_1 is full column-rank, we need two auxiliary results. We present these results one by one. The first auxiliary result is a well-known result about the left- and right-eigenspaces of the Hamiltonian matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ (Ionescu et al. (1999)). For the sake of easy referencing, we present this as a proposition next.

Proposition 16. Let the columns of the matrix $V_{\mathtt{e}} = \mathtt{col}(V_1, V_2, V_3)$ form a basis for the eigenspace of the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$ corresponding to the eigenvalues in Λ, where $(E_{\mathtt{r}}, H_{\mathtt{r}})$ is as defined in eq. (27), and $\Lambda := \sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-$. Then, $V_1^T V_2 = V_2^T V_1$.

Next, recall from Lemma 14 that $\mathtt{img}\left[\begin{smallmatrix} V_g \\ 0 \end{smallmatrix}\right] \subseteq \mathcal{O}_{wg}$. Thus, there exist $V_{12}, V_{22} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$, where $\mathtt{n_s} := \mathtt{dim}(\mathcal{O}_{wg})$, such that $\mathcal{O}_{wg} = \mathtt{img}\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right]$. We use this fact in the following crucial lemma.

Lemma 17. Let $V_{12}, V_{22} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ be such that $\mathcal{O}_{wg} = \mathtt{img}\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right]$, where $\mathtt{n_s} := \mathtt{dim}(\mathcal{O}_{wg}), \mathtt{g} := \mathtt{dim}(\mathcal{V}_{\mathtt{g}})$, $V_g \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$, and $\mathtt{img}V_g = \mathcal{V}_g$. Then, the following are true:

(1) V_{22} is full column-rank.
(2) $V_{22}^T V_{12} > 0$.
(3) $[V_g \; V_{12}]$ is full column-rank.

Proof. (1): Since $\mathtt{img}\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right] = \mathcal{O}_{wg}$, by Definition 8, there exist $F_{\mathtt{e}} \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times 2\mathtt{n}}$, $\Gamma_{12} \in \mathbb{R}^{\mathtt{g}\times(\mathtt{n_s}-\mathtt{g})}$, and $\Gamma_{22} \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times(\mathtt{n_s}-\mathtt{g})}$ such that

$$(A_{\mathtt{r}} + B_{\mathtt{r}} F_{\mathtt{e}}) \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} V_g & V_{12} \\ 0 & V_{22} \end{bmatrix} \begin{bmatrix} \Gamma_{12} \\ \Gamma_{22} \end{bmatrix}, \text{ and } C_{\mathtt{r}} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 0, \tag{34}$$

where $A_{\mathtt{r}}, B_{\mathtt{r}}$, and $C_{\mathtt{r}}$ are as defined in eq. (28). Define $V_{32} := F_{\mathtt{e}} \left[\begin{smallmatrix} V_{12} \\ V_{22} \end{smallmatrix}\right]$. Combining eq. (33) and eq. (34), we get

$$\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0 \end{bmatrix} \begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32} \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32} \end{bmatrix} \underbrace{\begin{bmatrix} J_g & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix}}_{J_s}. \tag{35}$$

Now, since $\mathtt{img}\left[\begin{smallmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \end{smallmatrix}\right] = \mathcal{O}_{wg}$, it is evident that $\sigma(J_s) \subseteq \mathbb{C}_- \Rightarrow \sigma(\Gamma_{22}) \subseteq \mathbb{C}_-$. Next, from eq. (35), we get

$$\widetilde{A} V_{12} - A_z V_{22} + \widetilde{B} V_{32} = V_g \Gamma_{12} + V_{12} \Gamma_{22}, \tag{36}$$

$$-\widetilde{Q} V_{12} - \widetilde{A}^T V_{22} = V_{22} \Gamma_{22}, \tag{37}$$

$$\widetilde{B}^T V_{22} = 0. \tag{38}$$

Clearly, $\mathtt{img}\left[\begin{smallmatrix} V_g & V_{12} \\ 0_{\mathtt{n},\mathtt{g}} & V_{22} \\ V_{3g} & V_{32} \end{smallmatrix}\right]$ is an eigenspace of the matrix pair $(E_{\mathtt{r}}, H_{\mathtt{r}})$. So, due to Proposition 16, $\left[\begin{smallmatrix} V_g & V_{12} \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 0_{\mathtt{n},\mathtt{g}} & V_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_{\mathtt{n},\mathtt{g}} & V_{22} \end{smallmatrix}\right]^T \left[\begin{smallmatrix} V_g & V_{12} \end{smallmatrix}\right]$; which further implies that

$$V_{22}^T V_g = 0 \text{ and } V_{22}^T V_{12} = V_{12}^T V_{22}. \tag{39}$$

Next, we pre-multiply eq. (36) by V_{22}^T and eq. (37) by $-V_{12}^T$, and then add them together to get

$$V_{22}^T \widetilde{A} V_{12} - V_{22}^T A_z V_{22} + V_{22}^T \widetilde{B} V_{32} + V_{12}^T \widetilde{Q} V_{12} + V_{12}^T \widetilde{A}^T V_{22}$$
$$= V_{22}^T V_g \Gamma_{12} + V_{22}^T V_{12} \Gamma_{22} - V_{12}^T V_{22} \Gamma_{22}. \tag{40}$$

By using eq. (38) and eq. (39), eq. (40) reduces to:

$$V_{22}^T \widetilde{A} V_{12} - V_{22}^T A_z V_{22} + V_{12}^T \widetilde{Q} V_{12} + V_{12}^T \widetilde{A}^T V_{22} = 0. \tag{41}$$

Now, to the contrary, we assume that V_{22} is not full column-rank. So, there exists $w \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times 1}, w \neq 0$ such that $V_{22}w = 0$. Thus, on pre- and post-multiplication of eq. (41) by w^T and w, respectively, we get $w^T V_{12}^T \widetilde{Q} V_{12} w = 0$. But, since $\widetilde{Q} \geqslant 0$, we have $\widetilde{Q} V_{12} w = 0$. Hence,

$$\mathtt{ker}V_{22} \subseteq \mathtt{ker}\widetilde{Q}V_{12}. \tag{42}$$

Post-multiplying eq. (37) by w, we get $-\widetilde{Q}V_{12}w - \widetilde{A}^T V_{22} w = V_{22}\Gamma_{22}w$. But, recall that $V_{22}w = 0$ and $\widetilde{Q}V_{12}w = 0$. Consequently, $V_{22}\Gamma_{22}w = 0$. Hence, $\mathtt{ker}V_{22}$ is Γ_{22}-invariant.

So, there exists a full column-rank matrix $T \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times\bullet}$ such that $V_{22}T = 0$ and $\Gamma_{22}T = T\Gamma, \sigma(\Gamma) \subseteq \sigma(\Gamma_{22}) \subseteq \mathbb{C}_-$. Moreover, from eq. (42), we have $\widetilde{Q}V_{12}T = 0$. Now, post-multiplying eq. (36) by T and using the fact that $V_{22}T = 0$ and $\Gamma_{22}T = T\Gamma$, we have

$$\widetilde{A} V_{12} T + \widetilde{B} V_{32} T = V_{\mathtt{g}} \Gamma_{12} T + V_{12} T \Gamma. \tag{43}$$

Since $\widetilde{C}^T \widetilde{C} = \widetilde{Q}$, from $\widetilde{Q} V_{12} T = 0$, it is clear that

$$\widetilde{C} V_{12} T = 0. \tag{44}$$

Thus, combining eq. (32), eq. (43), and eq. (44), we get

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix} \begin{bmatrix} V_{\mathtt{g}} & V_{12}T \\ V_{3g} & V_{32}T \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{\mathtt{g}} & V_{12}T \\ V_{3g} & V_{32}T \end{bmatrix} \begin{bmatrix} J_g & \Gamma_{12} \\ 0 & \Gamma \end{bmatrix}.$$

Since $\sigma(J_g), \sigma(\Gamma) \subseteq \mathbb{C}_-$, from Lemma 10, it follows that $\mathtt{img}\left[\begin{smallmatrix} V_{\mathtt{g}} & V_{12}T \end{smallmatrix}\right]$ is contained in the largest good slow space, $\mathcal{W}_g$, of the system $\Sigma_{aux} : \frac{d}{dt}x = \widetilde{A}x + \widetilde{B}u, y = \widetilde{C}x$. But, from Lemma 13 we know that $\mathcal{W}_g = \mathcal{V}_g = \mathtt{img}V_g$. So, $\mathtt{img}\left[\begin{smallmatrix} V_{\mathtt{g}} & V_{12}T \end{smallmatrix}\right] = \mathtt{img}V_g$. Thus, there exist $\alpha_1 \in \mathbb{R}^{\mathtt{g}\times 1}$ and a non-zero $\alpha_2 \in \mathbb{R}^{\bullet\times 1}$ such that

$$V_g \alpha_1 + V_{12} T \alpha_2 = 0. \tag{45}$$

Recall that, $V_{22}T = 0$. Thus, $V_{22}T\alpha_2 = 0$. Combining this with eq. (45), we have $\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} \alpha_1 \\ T\alpha_2 \end{smallmatrix}\right] = 0$. But, T being full column-rank and $\alpha_2 \neq 0$ implies that $T\alpha_2 \neq 0$. Consequently, we have a non-zero vector $\left[\begin{smallmatrix} \alpha_1 \\ T\alpha_2 \end{smallmatrix}\right]$ inside $\mathtt{ker}\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right]$. This is contradiction, because $\left[\begin{smallmatrix} V_g & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right]$ is full column-rank. Therefore, V_{22} must be full column-rank.

(2): Due to eq. (39), $V_{22}^T V_{12}$ is symmetric. We prove that $V_{22}^T V_{12} > 0$ in two steps: first, we show that $V_{22}^T V_{12} \geqslant 0$, and then we show that $V_{22}^T V_{12}$ is non-singular. Pre-multiplying eq. (36) by V_{22}^T and using eq. (38) and eq. (39), we have

$$V_{22}^T \widetilde{A} V_{12} - V_{22}^T A_z V_{22} = V_{22}^T V_{12} \Gamma_{22}. \tag{46}$$

Also, by taking transpose of eq. (37), and then post-multiplying by V_{12}, we obtain

$$-V_{12}^T\widetilde{Q}V_{12} - V_{22}^T\widetilde{A}V_{12} = \Gamma_{22}^T V_{22}^T V_{12}. \qquad (47)$$

Adding eq. (46) and eq. (47) together, we get

$$-V_{12}^T\widetilde{Q}V_{12} - V_{22}^T A_z V_{22} = V_{22}^T V_{12}\Gamma_{22} + \Gamma_{22}^T V_{22}^T V_{12}. \qquad (48)$$

Since, $\widetilde{Q} \geqslant 0$ and $A_z = B_2\widehat{R}^{-1}B_2^T \geqslant 0$, we have $V_{12}^T\widetilde{Q}V_{12} + V_{22}^T A_z V_{22} \geqslant 0$. Now, notice that eq. (48) is a Lyapunov equation. Recall that Γ_{22} is Hurwitz. Thus, by Lyapunov's theorem, we conclude that $V_{22}^T V_{12} \geqslant 0$.

Next, to the contrary, assume that $V_{22}^T V_{12}$ is singular. Then, we must have that $(\left[\begin{smallmatrix}\widetilde{Q} & 0\\ 0 & A_z\end{smallmatrix}\right]\left[\begin{smallmatrix}V_{12}\\ V_{22}\end{smallmatrix}\right], \Gamma_{22})$ is unobservable (see (Wonham, 1985, Lemma 12.2)). Thus, there exists a nonzero $v \in \mathbb{C}^{(\mathtt{n_s}-\mathtt{g})\times 1}$ such that

$$\Gamma_{22}v = \lambda v, \text{ for some } \lambda \in \sigma(\Gamma_{22}) \subseteq \mathbb{C}_-, \text{ and}$$

$$\left[\begin{smallmatrix}\widetilde{Q} & 0\\ 0 & A_z\end{smallmatrix}\right]\left[\begin{smallmatrix}V_{12}\\ V_{22}\end{smallmatrix}\right]v = 0 \Leftrightarrow \widetilde{Q}V_{12}v = 0 \ \& \ A_z V_{22}v = 0. \qquad (49)$$

Now, right-multiplying eq. (37) with v and using $\widetilde{Q}V_{12}v = 0$ from eq. (49), we have

$$\widetilde{A}^T V_{22}v = (A - B_2\widehat{R}^{-1}S_2^T)^T V_{22}v = -\lambda V_{22}v. \qquad (50)$$

Also, $A_z V_{22}v = B_2\widehat{R}^{-1}B_2^T V_{22}v = 0 \Leftrightarrow \widehat{R}^{-1}B_2^T V_{22}v = 0 \Leftrightarrow B_2^T V_{22}v = 0$. Combining this with eq. (38), we get $v^T V_{22}^T [\, B_1 \ B_2 \,] = 0$, because $\widetilde{B} = B_1$. Further, using $B_2^T V_{22}v = 0$ in eq. (50), we get that $A^T V_{22}v = -\lambda V_{22}v$. From Statement (1) of this lemma, we know that V_{22} is full column-rank. So, v being a non-zero vector implies that $V_{22}v \neq 0$. Thus, $V_{22}v$ is an eigenvector of A^T corresponding to the eigenvalue $-\lambda$. But, $v^T V_{22}^T [\, B_1 \ B_2 \,] = 0$ and $\lambda \in \mathbb{C}_- \Rightarrow -\lambda \in \overline{\mathbb{C}}_+$. This contradicts the assumption that $(A, [\, B_1 \ B_2 \,])$ is stabilizable (see Problem 11). Hence, $V_{22}^T V_{12}$ is non-singular. We also showed that $V_{22}^T V_{12} \geqslant 0$. Therefore, $V_{22}^T V_{12} > 0$.

(3): Say, $\beta_1 \in \mathbb{R}^{\mathtt{g}\times 1}$ and $\beta_2 \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})\times 1}$ be such that $[\, V_g \ V_{12} \,]\left[\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right] = 0$. Pre-multiplying this equation with V_{22}^T and using eq. (39), we have $V_{22}^T V_{12}\beta_2 = 0$. But, from Statement (2) of this lemma, we know that $V_{22}^T V_{12}$ is non-singular. So, $\beta_2 = 0$. This further implies that $V_g\beta_1 = 0$, which, in turn, implies that $\beta_1 = 0$, because V_g is full column-rank. Thus, $[\, V_g \ V_{12} \,]\left[\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right] = 0 \Rightarrow \left[\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right] = 0$. Hence, $[\, V_g \ V_{12} \,]$ is full column-rank. □

Now, we are in a position to show that the subspace $\mathtt{img}V_{\mathtt{e}}$ is disconjugate. We present this result as a theorem next.

Theorem 18. Let $(E_{\mathtt{r}}, H_{\mathtt{r}})$ be the Hamiltonian matrix pair as defined in eq. (27). Also, consider the eigenspace, $\mathtt{img}V_{\mathtt{e}}$, of $(E_{\mathtt{r}}, H_{\mathtt{r}})$ as define in eq. (29) . Then, $\mathtt{img}V_{\mathtt{e}}$ is disconjugate.

Proof. Recall that $\mathtt{img}\left[\begin{smallmatrix}V_1\\ V_2\end{smallmatrix}\right] = \mathcal{O}_{wg}$. But, from the statement of Lemma 17, we have that $\mathtt{img}\left[\begin{smallmatrix}V_g & V_{12}\\ 0 & V_{22}\end{smallmatrix}\right] = \mathcal{O}_{wg}$. Thus, $\mathtt{img}\left[\begin{smallmatrix}V_1\\ V_2\end{smallmatrix}\right] = \mathtt{img}\left[\begin{smallmatrix}V_g & V_{12}\\ 0 & V_{22}\end{smallmatrix}\right] \Rightarrow \mathtt{img}V_1 = \mathtt{img}[\, V_g \ V_{12} \,]$. Now, V_1 and $[\, V_g \ V_{12} \,]$ both have $\mathtt{n_s}$ number of columns. Furthermore, from Statement (3) of Lemma 17, we get that $[\, V_g \ V_{12} \,]$ is full column-rank. Hence, we must have that V_1 is full column-rank. Therefore, $\mathtt{img}V_{\mathtt{e}}$ is disconjugate. □

The following theorem renders the optimal trajectories and a feedback law to solve Problem 11, if the initial condition is from $\mathtt{img}V_1$.

Theorem 19. Consider the singular LQR Problem 11, V_1, and J as defined in eq. (29). Suppose $x_0 = V_1\alpha$, $\alpha \in \mathbb{R}^{\mathtt{n_s}\times 1}$, is an arbitrary initial condition from $\mathtt{img}V_1$. Then,

(1) (x_s, u_{s_1}, u_{s_2}) is the optimal trajectory, where $x_s := V_1 e^{Jt}\alpha$, $u_{s_1} := V_3 e^{Jt}\alpha$, and $u_{s_2} := -\widehat{R}^{-1}(S_2^T V_1 + B_2^T V_2)e^{Jt}\alpha$.
(2) There exist feedbacks $F_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ and $F_2 \in \mathbb{R}^{\mathtt{r}\times\mathtt{n}}$ such that $u_{s_1} = F_1 x_s$ and $u_{s_2} = F_2 x_s$.

Proof. (1): Define $z_0 := V_2\alpha$ and $z_s := V_2 e^{Jt}\alpha$. Then, using eq. (29), it is easy to verify that $(x_s, z_s, u_{s_1}, u_{s_2})$ is a trajectory for the Hamiltonian system defined by eq. (26) corresponding to the initial condition (x_0, z_0). It can also be verified that (x_s, u_{s_1}, u_{s_2}) is a trajectory for the system $\frac{d}{dt}x = Ax + B_1u_1 + B_2u_2$ corresponding to the initial condition x_0. Hence, from Pontryagin's maximum principle it follows that (x_s, u_{s_1}, u_{s_2}) is the optimal trajectory corresponding to the initial condition x_0.

(2): From Theorem 18, it follows that V_1 is full column-rank. Thus, there exist $K_1 \in \mathbb{R}^{(\mathtt{m}-\mathtt{r})\times\mathtt{n}}$ and $K_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $V_3 = K_1V_1$ and $V_2 = K_2V_1$. Define $F_1 := K_1$ and $F_2 := -\widehat{R}^{-1}(S_2^T + B_2^T K_2)$. Then, it is evident that $u_{s_1} = F_1x_s$ and $u_{s_2} = F_2x_s$. This completes the proof. □

5. CONCLUSION

In this paper we have provided a characterization of the slow and the good slow spaces. This characterization automatically gives a method to compute these subspaces from an eigenspace of the corresponding Rosenbrock system matrix. Furthermore, we have shown how to obtain the dimensions of these subspaces from the degree of the determinant of the Rosenbrock matrix pencil. Then, we have applied these results to the Hamiltonian system obtained from the singular LQR problem to explore some interesting properties. We have used the good slow space of the Hamiltonian to provide a feedback which solves the singular LQR problem when the initial condition of the system belongs to a certain subspace. This space has been used in Bhawal and Pal (2019) to solve the singular LQR problem for any arbitrary initial condition for the single-input case. We wish to use the results developed in this paper to solve the problem for the multi-input case.

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