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Optimal Control Points Problem in Domination Game on Large Scale Multi-Agent Systems

Tomomitsu Suzuki * **Koji Tsumura** **

* *The University of Tokyo, Bunkyo-ku 7-3-1, Tokyo, JAPAN (e-mail: tomomitsu_suzuki@ipc.i.u-tokyo.ac.jp)*
** *The University of Tokyo, Bunkyo-ku 7-3-1, Tokyo, JAPAN (e-mail: tsumura@i.u-tokyo.ac.jp), CREST, NEDO SIP*

Abstract: In this paper, we deal with large scale multi-agent consensus systems in which some agents are assumed to be weakly controlled with feedback and consider a domination game between several players. We assume that each player can choose a set of controlled agents and input signals to control the states of all the agents to its own desired reference state. The reference states of the players are assumed to be different from each other, therefore, a conflict occurs between the players. The problem for each player is to choose a set of controlled agents to dominate the whole system as possible and we call this as a domination game. To find the optimal set of the controlled agents is essentially a complex combination problem, however, in this paper, we show that the optimal set can be given by small calculations. This result provides a strategy to the players for the domination game and we discuss the relationship between the structure of networks and monopolistic/equally domination games.

Keywords: multi-agent systems, consensus, game, hierarchical control, passivity

1. INTRODUCTION

In the last several decades, based on the development of information and communication technology, large scale networked systems with many units in engineering and science have been actively dealt with and investigated. Examples of such systems include traffic systems, electric power networks, sensor networks. Moreover, behavior of non-engineering systems such as swarms of living things, social systems, or genetic networks have been analyzed mathematically (Reynolds (1987); Parrish (2002); Olfati-Saber et al. (2007); Gazi (2007)).

It is well-recognized that control theory has contributed substantially to these research fields, where each unit is modeled as an agent or a subsystem with input-output dynamics and the entire system is given as a group of connected agents through networks. Then, the main focus has been on the analysis of the stability or the equilibrium of the networked dynamical systems, and in particular, the consensus problem is a typical example (Fax & Murray (2004); Olfati-Saber et al. (2007)). The research interest on the consensus systems initially explored analysis of the dynamics of autonomous multi-agent systems. As an extension of the autonomous consensus systems, we can assume a case that the systems include external control inputs and the research interests turn to the external controller design problems (Pequito et al. (2013); Olshevsky (2014); Moothedath et al. (2018); Clark et al. (2012, 2017)).

From above background, our research group has investigated the external controller design problems, and specifically the optimal control points problem (Yamamoto & Tsumura (2011); Tsumura & Yamamoto (2013); Tsumura & Kawasaki (2014)). We dealt with a *feedback* controller design problem for large scale multi-agent systems where the number of agents is very large; however, the number of agents to be observed or directly controlled is limited. Then we considered *an optimal control/observation points problem* for a given control performance such as the convergence rate to a given reference signal or their quadratic error.

Motivated by those previous research, where we assumed that only one feedback controller is designed and connected by a designer or *a player*, in this paper, we deal with a case that several feedback controllers are designed and connected simultaneously to the systems by several competitive players in order to attain the different purposes; controlling the outputs of all the agents to their different reference signals. This implies a conflict occurs between the players and we call this case as *a domination game* on multi-agent systems. Then, we consider an optimal control points problem for each player to capture a greater share of the agents than the other players. An example of the applications of this domination game is the advertisement on the social networking service (SNS); several companies or players compete with each other for capturing a share of the agents on SNS.

As explained in later, this kind of optimal control points problems for multi-agent systems can be solved by a brute-force search with the comparison of the resultant control performances in every choice of the control points sets; however, when the number of agents is large and the number of the directly controlled agents is adequate, the number of the possible combinations of controlled agents easily become so large that it is not realistic to find the optimal control points in the sense of computation complexity. In Yamamoto & Tsumura (2011); Tsumura & Yamamoto (2013), which deal with a case of one player and there is no competition, we showed that the optimal control points for fast convergence rate of states in a consensus system can be given by the index called Alt-PageRank, a dual notion of the PageRank Page et al. (1999), of the network matrix, and the necessary computation complexity to find it is comparatively smaller than a brute-force search. Based on those previous results, in this paper, we show that the optimal control points problem for the domination game can be also solved by computing the Alt-PageRank and the order of the computation is $O(N^3)$ where N is the number of agents.

The following research efforts are related to this paper. In Pequito et al. (2013); Olshevsky (2014); Moothedath et al. (2018), the problem of finding the minimal set of control points to satisfy controllability of a total system and its computation complexity are discussed. In Clark et al. (2012, 2017), the problem of finding an optimal selection of the leader agent for a given control performance is discussed. The control structure is *a feed-forward* type and the authors show that the solution can be given with reasonable computation complexity using the submodularity property of the problem. In Ishii & Tempo (2010, 2014); Montijano et al. (2018), a distributed calculation of the importance index of subsystems in large scale networked systems is discussed.

This paper is organized as follows: In Section 2, we introduce the mathematical preparation and a motivative example of this research. In Section 3, we formulate the multi-agent systems, the domination game, and problem statements. In Section 4, we provide the main result of this paper. In Section 5, we show numerical simulations to verify the main result and discuss the strategy for the domination game. Finally, we summarize our research in Section 6.

2. PRELIMINARIES

In this paper, we will deal with networked multi-agent systems and discuss on an optimal selection of controlled agents with information of network structure for the domination game. In preparation, we briefly explain the overview of the domination game in networked multi-agent systems, and introduce fundamental notions and results on graph theory, eigenvectors, and eigenvalues of matrices in this section.

2.1 Overview of domination game in multi-agent systems

We deal with networked multi-agent systems, where each agent updates its own variable by a consensus algorithm according to its neighborhoods' variables. In the case of standard autonomous consensus systems, the variables of agents converge to an identical value, which is decided only by the initial values of the variables of the agents. On the other hand, in this paper, we suppose external feedback control signals to some of agents in order to control the consensus values. Moreover, we assume there are several players adding their own control signals to monopolize the agents such as to make agents' variables the player's own reference signal. We call this situation as a domination game between the competitive players in this paper. If each player can choose a set of agents that are directly input from the player's control signal, a problem of optimal choice of the controlled agents for the degree of monopoly arises. We call this as "an optimal control points problem."

This problem can be solved if the degree of the domination, which is explained in Section 3, can be calculated when sets of directly controlled agents are fixed and we apply a brute-force search among all the possible combinations of the controlled agents. However, the number of combinations becomes too large when the number of agents is large, the brute force search becomes impossible in practice. For example, when the number of the total agents is $N = 10^4$ and the number of the directly controlled agents is $\kappa = 5$, then, the order of the number of the combinations is approximated as

$$ {}_N\mathrm{C}_\kappa \sim 10^{20} \tag{1} $$

and the brute-force search is not realistic. From the above discussion, we consider solving the optimal control points problem with small calculation numbers in this paper.

2.2 Linear Algebraic Graph Theory

We employ the standard description of graphs to represent network structures of multi-agent systems. A directed graph $\mathcal{G}$ is defined by a pair $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, N\}$ represents a set of nodes and $\mathcal{E}$ a set of directed edges. An element $(j, i) \in \mathcal{E}$ represents a directed edge from node $j \in \mathcal{V}$ to node $i \in \mathcal{V}$. A neighbor set for node i is defined by $\mathcal{N}_i := \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ and it can be regarded as the collection of nodes $\{j\}$ that send information to i such that an edge $(j, i) \in \mathcal{E}$ represents a flow of signal from node j to node i. Also $\mathcal{B}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ represents the collection of nodes $\{j\}$ to which infromation is sent from i. The number of edges that are directed to node i is called *the in-degree* of i represented by $d_i := |\mathcal{N}_i|$.

A directed path from node j to node i is a set of edges

$$ \{(j, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_a, i) \mid i_1, i_2, \ldots, i_a \in \mathcal{V}\}. \tag{2} $$

A graph is called *strongly connected* if any node $j \in \mathcal{V}$ has a directed path to any other node $i \in \mathcal{V}$. On the other hand, graph $\mathcal{G}$ is said to have a *self-loop* when $\exists i \in \mathcal{V}$ such that $(i, i) \in \mathcal{E}$. Hereafter in this paper, we assume the following:

Assumption 2.1. Graph $\mathcal{G}$ is strongly connected and has no self-loop.

According to the standard formulation, we represent graph structures by matrices: the transition matrix and normalized graph Laplacian matrix (Fax & Murray (2004); Olfati-Saber et al. (2007)) as explained below.

The transition matrix Π of graph $\mathcal{G}$ is defined by

$$ \Pi_{ij} = \begin{cases} 1/d_i & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}. \tag{3} $$

Note that any row-sum of Π is equal to 1. *The normalized graph Laplacian matrix* L of graph $\mathcal{G}$ is defined by

$$ L := I - \Pi. \tag{4} $$

When graph $\mathcal{G}$ satisfies Assumption 2.1, we can define the corresponding normalized graph Laplacian matrix. An equivalent expression of a normalized graph Laplacian matrix is given as follows:

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -1/d_i & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

In general, *a spectral radius* of a square matrix is the largest absolute value of its eigenvalues and *the spectral circle* is a circle on the complex plane centered at the origin O, where its radius is equal to the spectral radius. A matrix is said to be *aperiodic* if it has only one eigenvalue on the spectral circle. Then, we also assume the following in this paper:

Assumption 2.2. Matrix Π of graph $\mathcal{G}$ is aperiodic.

2.3 Alt-PageRank

Alt-PageRank, which is proposed in Yamamoto & Tsumura (2011); Tsumura & Yamamoto (2013), is defined as an indicator of node importance in the control of networked systems. Let p_i be the Alt-PageRank of node i, and the column vector

$$\boldsymbol{p} = [p_1 \ p_2 \ \cdots \ p_N]^\top \tag{6}$$

is called as Alt-PageRank vector. The Alt-PageRank vector $\boldsymbol{p}$ of graph $\mathcal{G}$ whose transition matrix is Π, is defined by

$$\boldsymbol{p}^\top = \boldsymbol{p}^\top \Pi. \tag{7}$$

That is, the Alt-PageRank vector of graph $\mathcal{G}$ with Π is defined as the left eigenvector for eigenvalue 1 of Π. Note that from (4), (7) can be also represented as

$$\boldsymbol{p}^\top L = \boldsymbol{0}^\top. \tag{8}$$

The Alt-PageRank vector $\boldsymbol{p}$ is known to be unique except for an arbitrary scale factor and $p_i > 0$, $\forall i$ for a strongly connected graph whose transition matrix Π is aperiodic, then, hereafter in this paper, we assume that Alt-PageRank vector $\boldsymbol{p}$ is normalized as

$$\sum_{i=1}^{N} p_i = 1. \tag{9}$$

Moreover, each element is real (e.g., Saito (1996)) and it can be represented as

$$p_i = \sum_{j \in \mathcal{B}_i} \frac{p_j}{d_j}, \ \forall i. \tag{10}$$

3. DOMINATION GAME ON MULTI-AGENT SYSTEMS

In this section, we introduce the dynamics of consensus multi-agent systems with external players and formulate a domination game between the players.

3.1 Problem Formulation

At first, each agent at each node on a networked system is assumed to have its own state variable and a dynamics, and autonomously exchange the state variable with its neighborhood. The problem setting in Yamamoto & Tsumura (2011); Tsumura & Yamamoto (2013) is that there exists only one player (a control designer) who can input exogeneous control signals to some of the agents to make all the variables of the agents to a given reference signal. Then, the control objective is to make the convergence rate of the entire system the fastest. On the other hand, in this paper, we deal with a situation of competition, that is, there exist several competitive players each of whom can choose a set of controlled agents and input signals to control the states of all the agents to their own desired reference state as explained in the following.

Let the number of players be M. To represent the difference of effects from all the players' controls to agents, define the state of the i-th agent as

$$\boldsymbol{x}_i = \left[x_i^{(1)} \ x_i^{(2)} \ \cdots \ x_i^{(M)}\right]^\top, \ i = 1, 2, \ldots, N. \tag{11}$$

We also define the reference state vector $\boldsymbol{r}_m$ of the m-th player to $\boldsymbol{x}_i$, $\forall i$ as

$$\boldsymbol{r}_m = \boldsymbol{e}_m \in \mathbb{R}^M, \tag{12}$$

where $\boldsymbol{e}_m$ represents a unit vector such that its m-th element is 1 and the others are 0, then, the purpose of the m-th player to the i-th agent is $\boldsymbol{x}_i \to \boldsymbol{e}_m$, $\forall i$, that is,

$$\forall i \quad x_i^{(k)} \to 1, \ k = m \tag{13}$$

$$x_i^{(k)} \to 0, \ k \neq m \tag{14}$$

$$k = 1, 2, \ldots, M.$$

Therefore, for example, if $x_i^{(m)}$ is the maximum among $\{x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(M)}\}$, the m-th player has the largest effect on the i-th agent among the other players and this is a scenario of competition between the players.

We define a state vector of the total system as

$$\boldsymbol{x} = \left[\boldsymbol{x}_1^\top \ \boldsymbol{x}_2^\top \ \cdots \ \boldsymbol{x}_N^\top\right]^\top \in \mathbb{R}^{M \cdot N} \tag{15}$$

and also a vector

$$\boldsymbol{x}^{(k)} = \left[x_1^{(k)} \ x_2^{(k)} \ \cdots \ x_N^{(k)}\right]^\top \in \mathbb{R}^N, \tag{16}$$

which collects $x_i^{(k)}$ from agents $i = 1, 2, \ldots, N$.

Then, we get the following consensus system with the competitive external control inputs:

$$\text{if } i \notin \bigcup_{m=1}^{M} \mathcal{I}_m$$

$$\dot{x}_i^{(k)} = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j^{(k)} - x_i^{(k)}), \ k = 1, 2, \ldots M \tag{17}$$

$$\text{if } i \in \bigcup_{m=1}^{M} \mathcal{I}_m$$

$$\dot{x}_i^{(k)} = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j^{(k)} - x_i^{(k)}) + \sum_{i \in \bigcup_{m=1}^{M} \mathcal{I}_m} \ \sum_{m=1,2,\ldots,M} \varepsilon u_i^{(k)(m)},$$

$$k = 1, 2, \ldots M \tag{18}$$

where $\mathcal{I}_m$ is the set of controlled points of the m-th player, $|\mathcal{I}_m| = \kappa$, $\forall m$, ε $(0 < \varepsilon \ll 1)$ is a control gain, and $u_i^{(k)(m)}$ is a control input of the m-th player to $x_i^{(k)}$.

To attain (13) and (14) for the m-th player, we define control inputs $u_i^{(k)(m)}$ as follows:

$$\forall i \quad u_i^{(k)(m)} = 1 - x_i^{(k)},\ k = m \tag{19}$$

$$u_i^{(k)(m)} = -x_i^{(k)},\ k \neq m \tag{20}$$

$$k = 1,\, 2,\, \ldots,\, M$$

The control inputs (19) and (20) correspond to the control objective (13) and (14) of the m-th player. That is, (19) is for closing $x_i^{(m)}$ to 1 and (20) is for closing $x_i^{(k)}$, $k \neq m$, to 0 and this control rule is consistent with the purpose that $\boldsymbol{x}_i \to \boldsymbol{e}_m$.

By collecting the dynamics of $x_i^{(m)}$, $i = 1, 2, \ldots, N$, for each m, the above feedback system can also be represented by

$$\text{if } i \notin \bigcup_{m=1}^{M} \mathcal{I}_m$$

$$\dot{x}_i^{(m)} = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j^{(m)} - x_i^{(m)}) \tag{21}$$

$$\text{if } i \in \mathcal{I}_m$$

$$\dot{x}_i^{(m)} = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j^{(m)} - x_i^{(m)}) + \varepsilon(1 - x_i^{(m)}) \tag{22}$$

$$\text{if } i \in \mathcal{I}_k,\ k \neq m$$

$$\dot{x}_i^{(m)} = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j^{(m)} - x_i^{(m)}) - \varepsilon x_i^{(m)} \tag{23}$$

$$m = 1,\, 2,\, \ldots,\, M$$

The corresponding vector form of the dynamics of $\boldsymbol{x}^{(m)}$ for the m-th player can be represented as

$$\dot{\boldsymbol{x}}^{(m)} = -L_{\text{in}} \boldsymbol{x}^{(m)} + \varepsilon \boldsymbol{r}^{(m)},\ m = 1,\, 2,\, \ldots,\, M \tag{24}$$

where

$$L_{\text{in}} := L + \sum_{m=1}^{M} \sum_{i \in \mathcal{I}_m} \varepsilon \boldsymbol{e}_i \boldsymbol{e}_i^\top = L + E_{\text{in}} \tag{25}$$

$$E_{\text{in}} := \sum_{m=1}^{M} \sum_{i \in \mathcal{I}_m} \varepsilon \boldsymbol{e}_i \boldsymbol{e}_i^\top \tag{26}$$

$$\boldsymbol{r}^{(m)} := \begin{bmatrix} r_1^{(m)} & r_2^{(m)} & \cdots & r_N^{(m)} \end{bmatrix}^\top \in \mathbb{R}^N \tag{27}$$

$$r_i^{(m)} := \begin{cases} 1 & \text{if } i \in \mathcal{I}_m \\ 0 & \text{otherwise} \end{cases}. \tag{28}$$

In Section 4, we show that $\text{Re}[\lambda_i(-L_{\text{in}})] < 0$, $\forall i$, then system (24) has a unique equilibrium

$$\boldsymbol{x}^{(m)*} = \varepsilon L_{\text{in}}^{-1} \boldsymbol{r}^{(m)} \tag{29}$$

and $\boldsymbol{x}^{(m)}(t)$ converges to $\boldsymbol{x}^{(m)*}$. Each m-th player is assumed to choose a set of controlled points $\mathcal{I}_m$ to control the states of all the agents as $\boldsymbol{x}_i \to \boldsymbol{r}_m = \boldsymbol{e}_m$. Therefore, there exists a conflict between all the players and we consider the following problem in this paper:

Problem 3.1. For i and a given $\kappa > 0$ such as $\kappa = |\mathcal{I}_\ell|$, $\forall \ell$, find the following optimal choice $\mathcal{I}_{m,i}^o$:

$$\mathcal{I}_{m,i}^o := \arg\max_{\mathcal{I}_m} x_i^{(m)*} \tag{30}$$

$$\text{s.t. } x_i^{(m)*} \geq x_i^{(\ell)*},\ \forall \ell \neq m \tag{31}$$

Remark 3.1. We call the competition between the players to satisfy $x_i^{(m)*} \geq x_i^{(\ell)*}$, $\ell \neq m$, for many $\{i\}$ as *a domination game*. From the definition (30)–(31), it seems that $\mathcal{I}_{m,i}^o \neq \mathcal{I}_{m,j}^o$, $i \neq j$, in general, however, we can show that $\mathcal{I}_{m,i}^o = \mathcal{I}_{m,j}^o$, $\forall i, j$ in Theorem 4.1 and the domination is possible.

Similar to the one-player case, this optimal choice problem essentially can be solved by a brute-force search as mentioned before, thus, we select a combination of controlled points $\mathcal{I}_m$, $m = 1, 2, \ldots, M$, and calculate the equilibrium state, repeat this calculation for all the possible combinations, and finally compare all the equilibrium states to find the optimal one. However, as the number of possible choices of all the players' controlled points is too large when the number of agents is large, the brute-force search requires significantly high calculation costs and it is impossible in practice. Thus, a calculation method to find the optimal choice of controlled points at lower computation costs is needed.

4. MAIN RESULTS

Initially, we introduce the following lemma:

Lemma 4.1. (Hu & Hong (2007)). The real parts of all the eigenvalues of L_{in} are positive, that is, $-L_{\text{in}}$ is asymptotically stable.

Remark 4.1. From this lemma, it is obvious that the liner time invariant system (24) (equivalently, the system (17)–(18) or (21)–(23)) is asymptotically stable and it has a unique equilibrium (29).

By employing Lemma 4.1, we can derive the solution for Problem 3.1:

Theorem 4.1. On Problem 3.1, there exists $\bar{\varepsilon}(> 0)$ and when $\varepsilon \leq \bar{\varepsilon}$, the optimal choice $\mathcal{I}_{m,i}^o$ for the m-th player defined by (30)–(31) of the directly controlled agents of (24) is to choose κ agents by

$$\mathcal{I}_{m,i}^o = \{\iota(1),\, \iota(2),\, \ldots,\, \iota(\kappa)\} \tag{32}$$

where $\{\iota(\bullet)\}$ represent the indices of the elements of Alt-PageRank $\boldsymbol{p}$ of L and they are sorted as

$$p_{\iota(1)} \geq p_{\iota(2)} \geq \cdots \geq p_{\iota(\kappa)} \geq p_{\iota(\kappa+1)} \geq \cdots \geq p_{\iota(N)}. \tag{33}$$

Moreover,

$$\mathcal{I}_{m,i}^o = \mathcal{I}_{m,j}^o,\ \forall i, j. \tag{34}$$

Remark 4.2. In the proof, we show that

$$\forall m \quad \boldsymbol{x}^{(m)*} = K \tilde{\boldsymbol{x}}^{(m)*} \tag{35}$$

$$\tilde{\boldsymbol{x}}^{(m)*} = S_m \cdot \mathbf{1} + O(\varepsilon) \tag{36}$$

$$S_m := \sum_{i \in \mathcal{I}_m} p_i \tag{37}$$

where K is a constant for all m, and then the statement is derived straightforward. From above, we regard S_m as a score of the m-th player and we can regard the domination game as the score competition between the players. Players who have large scores can make the elements of their $\boldsymbol{x}^{(m)*}$ large compared to the others and, as a result, obtain a large part of the networked multi-agent system.

Remark 4.3. By employing Theorem 4.1, we can find the optimal choice of the directly controlled agents with less calculation costs compared with the brute-force search. The optimal choice of the controlled agents can be found only by calculating the Alt-PageRank $\boldsymbol{p}$, the left eigenvector of L corresponding to 0 eigenvalue, and choosing

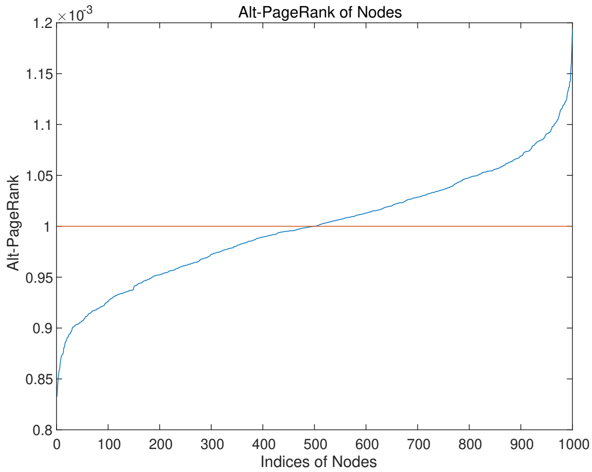

Fig. 1. Distribution of the elements $\{p_i\}$ of the Alt-PageRanks $\boldsymbol{p}$ in their ascending order of strongly connected networks including 1000 nodes (blue line: random network, red line: completely symmetric network)

the agents by (32)–(33). Then, in a case that choosing the controlled agents is exclusive between the players, the order of computational complexity to find the optimal set of controlled agents by brute-force is given by

$$_N\mathrm{C}_{\kappa M} \cdot \mathcal{O}(N^3), \tag{38}$$

where the first factor $_N\mathrm{C}_{\kappa M}$ is the number of all possible combinations of controlled agents' selections of all the players and the second factor $\mathcal{O}(N^3)$ is the order for solving (29) $\forall m$. However, the order of the necessary computational complexity by Theorem 4.1 is $\mathcal{O}(N^3)$, which is for calculating the Alt-PageRank $\boldsymbol{p}$ in (8). When the number of agents N is large, the first factor $_N\mathrm{C}_{\kappa M}$ in (38) becomes extremely large and our proposed method can substantially reduce the calculation costs compared to the brute-force search.

5. NUMERICAL SIMULATION

In this section, we verify the result of Theorem 4.1 and demonstrate its effectiveness by numerical simulations.

Initially, we randomly generate a strongly connected network with 1000 nodes and calculate its Alt-PageRank $\boldsymbol{p}$. The blue line in Figure 1 shows the values of the elements of this Alt-PageRank $\boldsymbol{p}$, where we sort the elements $\{p_i\}$ in ascending order of their values. The distribution of the elements is typical and it shows an S-curve. This implies that there exist differences in the importance of the control agents and the players should choose the agents with larger Alt-PageRanks or higher agent indices in Figure 1 to win the domination game. We also generate a completely symmetric and strongly connected network with 1000 nodes and calculate its Alt-PageRank $\boldsymbol{p}$. In this case, it is known that $p_i = 1/N$, $\forall i$. The red line in Figure 1 shows the similar distribution of its elements $\{p_i\}$ and we can observe that the distribution is flat. This implies that there is no difference in importance for control between the agents and also there is no significant superiority between the players in the domination game.

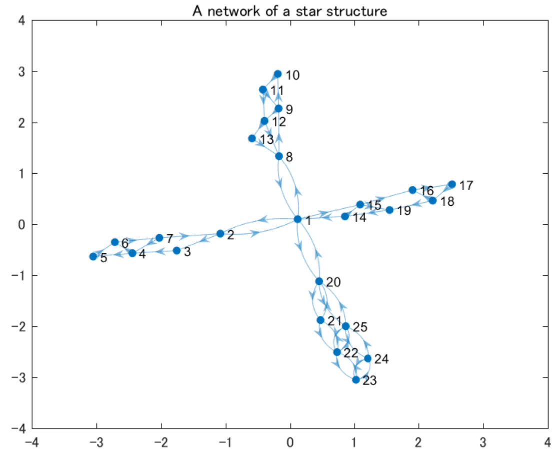

Fig. 2. A network of a star structure of 25 nodes (numbers on nodes represent the indices of agents)

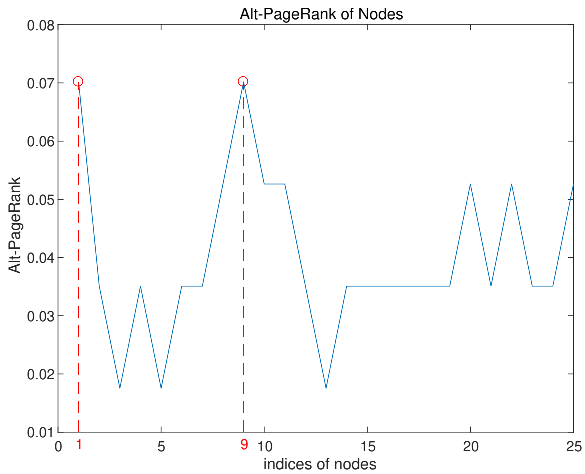

Fig. 3. The elements of the Alt-PageRank of the network in Figure 2.

We next generate a network of twenty-five agents, which has a star structure as shown in Figure 2 and calculate its Alt-PageRank $\boldsymbol{p}$. Figure 3 shows the Alt-PageRanks of all the agents and from Theorem 4.1 in the case $\kappa = 2$, it is known that the optimal choise of the directly controlled agents is $\mathcal{I}_\bullet^o = \{1, 9\}$. In a case of a 2-player domination game where $\kappa = 2$, suppse that player 1 chooses $\mathcal{I}_1^o = \{1, 9\}$ and player 2 chooses agents $\mathcal{I}_2 = \{3, 5\}$. The scores S_1 and S_2 of the two players are 0.1401 and 0.0351, respectively. Therefore, from Theorem 4.1, we can estimate that player 1 wins the domination game against player 2. Figure 4 shows that the equilibrium states $\tilde{x}_i^{(1)*}$ and $\tilde{x}_i^{(2)*}$ for $i = 1, 2, \ldots, 25$. It is observed that $\tilde{x}_i^{(1)*} > \tilde{x}_i^{(2)*}$, $\forall i$, that is, player 1 surpasses player 2 in all the agents and player 1 wins the domination game on this network.

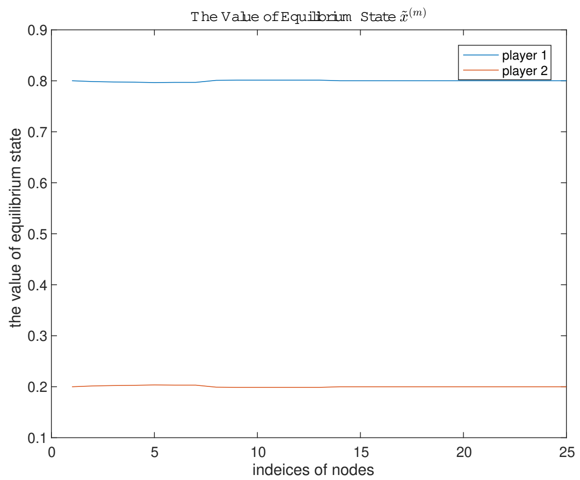

Fig. 4. Equillibrium states $\tilde{x}_i^{(1)*}$ and $\tilde{x}_i^{(2)*}$, $i = 1, 2, \ldots, 25$ in the case of Figure 2 where player 1 chooses agents $\mathcal{I}_1^o = \{1, 9\}$, and player 2 chooses agents $\mathcal{I}_2 = \{3, 5\}$ (blue line: $\tilde{x}_i^{(1)*}$, red line: $\tilde{x}_i^{(2)*}$, $i = 1, 2, \ldots, 25$)

6. CONCLUSION

In this paper, we dealt with large scale multi-agent consensus systems with external weak feedback control inputs. We assume that there exist a number of players where the purpose of each player is to control the agents' states to the players own reference signal by a feedback control and a domination game arises between the players. Then, we formulated an optimal control points problem for each player to obtain the share of the agents as much as possible and showed that the solutions can be found by using the Alt-PageRank in a reasonable computation complexity compared with a brute-force search. We then verified the theory in numerical simulations and demonstrated the efficiency of our method. We also discussed the strategy for the domination game with the network structure of the consensus systems.

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