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Structural Characterization of Controllability in Timed Continuous Petri Nets using Invariant Subspaces ⋆

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Abstract: This work deals with the *controllability* analysis in Timed Continuous Petri Nets ($TCPN$s) under infinite server semantics, a fluid relaxation that can model highly populated Discrete Event Systems. Here, the *full rank-controllability* property is defined, ensuring that the $TCPN$ is controllable over the equilibrium markings in each of the *regions* of its reachability space. This allows forcing the $TCPN$ systems to work at interesting operation points such as maximum production states, safety regions, to mention a few. Herein two structural conditions for full rank-controllability, one necessary and the other sufficient, are introduced, avoiding the enumeration of all the *configurations* required in other approaches. Finally, based on this, a polynomial algorithm to test the full rank-controllability is provided.

Keywords: Continuous Petri nets, Controllability, Piecewise-linear systems, Hybrid Systems.

1. INTRODUCTION

There exist several formalisms to model Discrete Event Systems (DES), among them, the *Petri nets* (PNs) are a formal tool that is widely used for modeling, analysis and control of DES since they capture their main characteristics, such as causal relationships, concurrence, and mutual exclusion. Unfortunately, as other DES formalisms, they suffer from the *state explosion* problem, particularly when they are heavily marked, leading to computationally intractable analysis. To cope with this problem, *continuous Petri nets* ($CPNs$) were introduced allowing to over-approximate the PN reachability set by a convex one. Later, the notion of time was introduced to CPNs, leading to the *timed continuous PNs* ($TCPN$s) to deal with the performance analysis of the system (David and Alla, 2010; Silva et al., 2011). In this paper we study $TCPN$s under *infinite server semantics* (ISS). Its evolution can be described by piecewise affine systems with *polyhedral regions*. It has been shown that, for highly marked systems, $TCPN$ systems provide a good approximation of the performance of timed PNs (Fraca et al., 2014). It has also been shown that $TCPN$s are appropriate to model different systems such as manufacturing systems (Silva et al., 2014), health management systems (Dotoli et al., 2009), traffic systems (Tolba et al., 2005; Júlvez and Boel, 2010), and epidemiological models (Beccuti et al., 2013), among others.

Controllability in $TCPN$s is a fundamental property that has been widely studied. If a $TCPN$ system is controllable, its *marking can be driven* to a required value by reducing the firing speed at the *controllable* transitions (Mahulea et al., 2008). In Silva et al. (2011), it was highlighted that P-flows induce uncontrollable invariants, constraining the controllability of $TCPN$s. In Vázquez et al. (2014) the controllability property was defined as the possibility to *drive the $TCPN$ system between its potential equilibrium markings*. In that work, the study of controllability was divided in two cases: a) *when all the transitions are controllable*, and b) *when there are uncontrollable transitions.* In the former, if *the support of all the P-semiflows is initially marked*, the consistency of the net is enough to guarantee controllability (a structural condition). In the latter, since $TCPN$ systems can be seen as piecewise-linear systems, the controllability at each given marking region, where the system behaves linearly, is characterized by using controllability matrices. The main drawback of this kind of result is that the number of regions *grows exponentially* with respect to the number of synchronizations in the net, hence the complexity of the controllability analysis also grows exponentially.

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This work focuses on finding structural conditions (i.e., explained from the graph structure) that guarantee the *rank-controllability* of a $TCPN$, a property that implies controllability as proposed in Vázquez et al. (2014). The analysis is performed from three related perspectives: the *net graph exploration*, the *geometrical controllability analysis* of each linear mode (corresponding to the different regions), and the *controllability matrices* of the linear modes. It is shown that some *invariant subspaces of the linear modes* are related to structural objects, being computable without enumerating regions. For this purpose, the *flow dynamic equation* and its controllability matrix are introduced. It is proved that *the flow controllability matrix* loses rank when there exist *uncontrollable flow invariants* due to P-flows and choice places. After that, it is proved that the existence of uncontrollable flow invariants (that are characterized from the net structure) may lead to a loss of rank-controllability. Finally, these conditions are exploited by an algorithm to determine if a $TCPN$ is *full rank-controllable*, i.e., rank-controllable in every linear mode.

This work is organized as follows: Section 2 gives an introduction to $TCPN$s. Section 3 presents the rank-controllability property. Next, Section 4 introduces the concept of *influeced nodes by the control actions* and states a structural necessary condition for full rank controllability. In Section 5, the net based structural characterization of *flow invariants* is presented. Moreover, the relation between flow invariants and the rank-controllability is established. Section 6 introduces an algorithm to test the full rank-controllability of the $TCPN$, avoiding the enumeration of all the linear modes, as required by other approaches. An example is presented in Section 7. Finally, some conclusions are presented in Section 8.

2. BASIC CONCEPTS

The reader should consult David and Alla (2010) and Silva et al. (2011) for a deeper insight on $TCPN$s. Basic controllability results for linear systems can be consulted in Wonham (1979) and Chen (1998). A vector that lies in the null space (resp. the left null space) of a matrix $\mathbf{A}$ is said to be a *right annuler of* $\mathbf{A}$ (resp. *left annuler of* $\mathbf{A}$).

2.1 Continuous Petri nets

Definition 1. A *continuous Petri net* (CPN) system is a pair $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ is a P/T net and $\mathbf{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$ is the initial marking. $P = \{p_1, p_2, \ldots, p_n\}$ is a finite set of nodes called places; $T = \{t_1, t_2, \ldots, t_m\}$ is a finite set of nodes called transitions; $P \cap T = \emptyset$. $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ matrices representing the *weighted arcs* going from places to transitions and from transitions to places, respectively. The *enabling degree* of a transition t_i is given by $enab(t_i, \mathbf{m}) = \min_{p_j \in {}^\bullet t_i} \{\mathbf{m}[p_j]/\mathbf{Pre}[p_j, t_i]\}$; a transition t_i is *enabled* at $\mathbf{m}$ iff $\forall p_j \in {}^\bullet t_i$, $\mathbf{m}[p_j] > 0$. Hence, any enabled transition t_i can be fired in any certain positive amount $\alpha \leq enab(t_i, \mathbf{m})$; the firing of t_i leads to a new marking according to the fundamental equation of the continuous Petri net:

$$\mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \qquad (1)$$

where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the *token-flow* matrix, $\boldsymbol{\sigma} = \left[\alpha_1 \cdots \alpha_{|T|}\right]^T$ and α_i is the amount of firing of t_i.

For pre- and postsets we use the conventional dot notation, e.g., ${}^\bullet t = \{p \in P | \mathbf{Pre}[p, t] \neq 0\}$. In this work we will consider *strongly connected* (s.c.) nets (i.e., for every pair of nodes x and y, there is a path leading from x to y).

If $\mathbf{x} \neq \mathbf{0}$ (resp. $\mathbf{y} \neq \mathbf{0}$) is a solution of $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ (resp. $\mathbf{y}^T \cdot \mathbf{C} = \mathbf{0}$) then it is named *T-flow* (resp. *P-flow*). Matrix $\mathbf{B}_y$ denotes a basis for the P-flows of $\mathcal{N}$. Nonnegative T-flows (resp. P-flows) are called *T-semiflows* (resp. *P-semiflows*). $\mathcal{N}$ is said to be *consistent* (resp. *conservative*) if there exists a T-semiflow $\mathbf{x} > \mathbf{0}$ (resp. P-semiflow $\mathbf{y} > \mathbf{0}$). The *support* of a vector $\mathbf{x} \in \mathbb{R}^{|T|}$ (resp. $\mathbf{y} \in \mathbb{R}^{|P|}$), denoted by $||\mathbf{x}||$ (resp. $||\mathbf{y}||$), is the set $||\mathbf{x}|| = \{t_i \in T | \mathbf{x}[t_i] \neq 0\}$ (resp. $||\mathbf{y}|| = \{p_j \in P | \mathbf{y}[p_j] \neq 0\}$).

Places and transitions of a PN can be classified based on their input and output nodes: *Join transitions* are transitions with more than one input place; *Choice places* are places with more than one output transition. The set of choice places is denoted as $P_C = \{p_i \in P || p_i^\bullet| > 1\}$.

2.2 Timed continuous Petri nets

Definition 2. A timed continuous Petri net ($TCPN$) system is a time-driven continuous-state system described by the tuple $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$, where $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ is a CPN system and the vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ is a function that assigns to each transition a positive value representing its *firing rate* per enabled server. The marking is time-dependent and its derivative is defined from (1) as:

$$\dot{\mathbf{m}} = \mathbf{C}\dot{\boldsymbol{\sigma}}(\tau), \quad \mathbf{m}(0) = \mathbf{m}_0 \qquad (2)$$

where $\mathbf{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ is named the *flow* of the transitions.

This work deals with the study of $TCPN$ systems under *infinite server semantics*, where the flow of a transition t_i is defined as $f_i(\mathbf{m}) = \lambda_i enab(t_i, \mathbf{m})$. The *min* operator in $enab(t_i, \mathbf{m})$ leads to the concept of configurations:

Definition 3. A *configuration* $\mathcal{C}$ is a set of arcs (p, t_k) of $\mathcal{N}$, one per transition, s.t. $p \in {}^\bullet t_k$. Here, the notation used to represent a configuration is: $\mathcal{C}_i = \{(p_j, t_1), \cdots, (p_k, t_{|T|})\}$. The $|T| \times |P|$ *configuration matrix* $\mathbf{\Pi}_i$, associated to the configuration $\mathcal{C}_i$, is a matrix s.t. its entries are defined as:

$$\mathbf{\Pi}_i[j,k] = \begin{cases} \frac{1}{\mathbf{Pre}[k,j]} & \text{if } (p_k, t_j) \in \mathcal{C}_i \\ 0 & \text{otherwise} \end{cases}$$

Definition 4. The *T-coverture* of a configuration $\mathcal{C}_i$ is the set of places $\mathcal{TC}_i = \{p \in P | (p, t_j) \in \mathcal{C}_i,\ t_j \in T\}$.

At a given marking $\mathbf{m}$, the flow of a transition t_k is defined by the marking of the place p that provides the minimum ratio in $\min_{p \in {}^\bullet t_k} \{\mathbf{m}[p]/\mathbf{Pre}[p, t_k]\}$. In that case, it is said that p *constrains* the flow of t_k. A configuration $\mathcal{C}_i$ is *active* at marking $\mathbf{m}$ if $\mathbf{enab}(\mathbf{m}) = \mathbf{\Pi}_i \mathbf{m}$. Then, the set of reachable markings can be *partitioned* (except on the borders) into *regions*, convex subsets, one per configuration. A reachable marking $\mathbf{m}$ belongs to region $\mathcal{R}_i$ if $\mathcal{C}_i$ is active at $\mathbf{m}$.

The marking evolution in region $\mathcal{R}_i$ is described by the *i-th TCPN mode*, or dynamic equation, $\dot{\mathbf{m}} = \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{m}$ where $\mathbf{\Lambda} = diag(\boldsymbol{\lambda})$ and $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i$ is its *dynamic matrix.* In region $\mathcal{R}_i$ the flow through the transitions is:

$$\mathbf{f}(\tau) = \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{m} \qquad (3)$$

Moreover, from (3) and (1), the transitions flow for the *i-th* configuration can also be expressed as:

$$\mathbf{f}(\tau) = \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{m}(0) + \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{C}\sigma(\tau) \quad (4)$$

This representation will be useful in the sequel.

2.3 Controllable transitions in $TCPNs$

Control actions in $TCPN$ systems can only reduce the flow through the transitions. That is, a transition cannot work faster than its nominal speed characterized by λ_i.

Definition 5. The *control vector* $\mathbf{u} \in \mathbb{R}^{|T|}$ is defined s.t. u_i represents the control action on t_i and $0 \leq u_i \leq \lambda_i \cdot enab(t_i, \mathbf{m})$. The effective flow through a controlled transition t_i is given by $w_i = f_i - u_i$.
Transitions in which control actions can be applied are named *controllable*. The set of all controllable transitions is denoted by T_c and the set of uncontrollable transitions is $T_{nc} = T \setminus T_c$. If $t_i \in T_{nc}$ then u_i must be null.

The behaviour of a *controlled* $TCPN$ system is described by the state equation and constraints:

$$\begin{aligned} \dot{\mathbf{m}} = \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{C}[T_c]\mathbf{u}[T_c], \\ \mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m} \end{aligned} \quad (5)$$

where $\mathbf{\Pi}(\mathbf{m}) = \mathbf{\Pi}_i$ whenever $\mathbf{m} \in \mathcal{R}_i$, $\mathbf{C}[T_c]$ is the *input matrix* (it only contains the columns of $\mathbf{C}$ related to transitions in T_c) and $\mathbf{u}[T_c]$ is the restriction of $\mathbf{u}$ to the controllable transitions. A control action that fulfills the required constraints is called *suitably bounded* (s.b.). The following definition will be useful later:

Definition 6. A $|T| \times |T_c|$ selector matrix, S_{T_c}, is named the *selector of controllable transitions*, i.e., $\mathbf{C}S_{T_c} = \mathbf{C}[T_c]$.

In this work, it will be useful to consider the evolution of the flow as a state equation. Then, by the derivative of (3) and Eq. (5), the flow system in $\mathcal{R}_i$ is described as:

$$\dot{\mathbf{f}} = \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{C}\mathbf{f} - \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{C}S_{T_c}\mathbf{u}[T_c] \quad (6)$$

2.4 State invariants and Equilibrium markings in $TCPNs$

Whenever a P-flow, $\mathbf{y}$, is present in a $TCPN$ system, every reachable marking $\mathbf{m}$ satisfy: $\mathbf{y}^T\mathbf{m} = \mathbf{y}^T\mathbf{m}_0$. In other words, linear dependencies between the markings appear (called *token conservation laws*) and its evolution is restricted to an invariant formally described as follows:

Definition 7. Given $\langle \mathcal{N}, \mathbf{m}_0 \rangle$, $Class(\mathbf{m}_0)$ represents the set of markings that agree with the P-flows: $Class(\mathbf{m}_0) = \{\mathbf{m} \in \mathbb{R}^{|P|}_{\geq 0} | \mathbf{B}_y^T\mathbf{m} = \mathbf{B}_y^T\mathbf{m}_0\}$.

If $\mathcal{N}$ is consistent and conservative, $Class(\mathbf{m}_0)$ corresponds to the *reachable markings* of the untimed system. For the case where $T_c \subsetneq T$, the $TCPN$ systems are not controllable over $Class(\mathbf{m}_0)$. In Vázquez et al. (2014), it was proposed to study the *controllability property over sets of equilibrium markings*, which are the "potential steady states" of the system.

Definition 8. Any marking $\mathbf{m}^q$ for which $\exists \mathbf{u}^q$ s.b. such that $\mathbf{C}(\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m}^q)\mathbf{m}^q - \mathbf{u}^q) = \mathbf{0}$ is said to be an *equilibrium marking*. The set of all equilibrium markings is denoted as $\mathbb{E}$. The set of equilibrium markings in $\mathcal{R}_i$ is $E_i = \{\mathbf{m} | \mathbf{m} \in \mathbb{E} \cap \mathcal{R}_i\}$.

Definition 9. Let $\mathbf{m}^q \in \mathbb{E}$. The set of *fully controlled transitions* at $\mathbf{m}^q$, T_f^q, is defined as $T_f^q = \{t_j \in T_c | 0 < u_j^q < \lambda_j enab(t_j, \mathbf{m}^q)\}$. The set $T_p^q = T_c \setminus T_f^q$ is defined as the set of the *partially controlled transitions*.

Next, an interesting subset of E_i that is composed of all the equilibrium markings at which $T_c = T_f^q$ is introduced.

Definition 10. The set $E_i^* = \{\mathbf{m}^q \in E_i | \forall t_i \text{ in } T_c, 0 < u_i^q < \lambda_i enab(t_i, \mathbf{m}^q)\}$ is defined as the set of *fully controllable equilibrium markings* (at which all the inputs, related to controllable transitions, can be arbitrarily controlled).

2.5 Controllability in linear systems

Throughout this work the Popov-Belevitch-Hautus controllability test, for linear systems, is used.

Proposition 11. (PBH Test) A linear time-invariant dynamic system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *is controllable iff the left eigenvectors of* $\mathbf{A}$ *are not orthogonal to* $\mathbf{B}$.

In detail, a left eigenvector $\boldsymbol{v}$ of $\mathbf{A}$ fulfills $\boldsymbol{v}^T\mathbf{A} = \beta\boldsymbol{v}^T$. Thus, given the controllability matrix of the system

$$\mathbb{C} = \left[\mathbf{B} \ \mathbf{A}\mathbf{B} \cdots \ \mathbf{A}^{n-1}\mathbf{B} \right], \quad (7)$$

if a left eigenvector $\boldsymbol{v}$ of $\mathbf{A}$ premultiplies $\mathbb{C}$ then:

$$\boldsymbol{v}^T\mathbb{C} = \left[\boldsymbol{v}^T\mathbf{B} \ \beta\boldsymbol{v}^T\mathbf{B} \cdots \ \beta^{n-1}\boldsymbol{v}^T\mathbf{B} \right] \quad (8)$$

Hence, if $\boldsymbol{v}^T$ is orthogonal to $\mathbf{B}$, then $\boldsymbol{v}^T$ is a left annuler of the controllability matrix, thus the system is uncontrollable. In other words, there exists an *invariant subspace* of $\mathbf{A}$, characterized by $\boldsymbol{v}$, that the control cannot affect (i.e., an *uncontrollable invariant*).

3. CONTROLLABILITY IN $TCPN$ SYSTEMS

This section introduces the rank-controllability notion used in this work. First, let us recall the classical definition of the controllability matrix for each configuration.

Definition 12. (Vázquez et al., 2014) Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a $TCPN$ system that is evolving in configuration $\mathcal{C}_i$. The controllability matrix of the system in configuration $\mathcal{C}_i$ is:

$$\mathbb{C}_i = \left[\mathbf{C}[T_c] \ \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{C}[T_c] \ ... \ (\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i)^{|P|-1}\mathbf{C}[T_c] \right] \quad (9)$$

Notice that the rank of this controllability matrix never exceeds the rank of $\mathbf{C}$. Since $rank(\mathbf{C}) = |P| - rank(\mathbf{B}_y^T)$, then it is the maximum possible rank of any $\mathbb{C}_i$.

Definition 13. A $TCPN$ is said to be *rank-controllable at configuration* $\mathcal{C}_i$ if $rank(\mathbb{C}_i) = |P| - rank(\mathbf{B}_y^T)$. Moreover, it is said to be *full rank-controllable* if it is rank-controllable in every configuration.

Next proposition states that, under generic conditions, rank-controllability implies controllability as proposed in Vázquez et al. (2014).

Proposition 14. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a $TCPN$ system. If $\mathbf{m}_0 > \mathbf{0}$, $T_f^q = T_c$ and $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is rank-controllable in $\mathcal{C}_i$, then, it is controllable over E_i^*.

Proof. Considering the hypothesis $\mathbf{m}_0 > \mathbf{0}$ and $T_f^q = T_c$, Theorem 5.6 in Vázquez et al. (2014) states that a $TCPN$ is controllable over E_i^* if, for any pair of equilibrium markings $\mathbf{m}_1, \mathbf{m}_2 \in E_i^*$, it is true that $\mathbf{m}_2 - \mathbf{m}_1 \in \text{Img}(\mathbb{C}_i)$.

Now, consider any $\mathbf{m}_0, \mathbf{m}_1 \in E_i^*$. From (1), any reachable marking $\mathbf{m}_1 - \mathbf{m}_0 \in \text{Img}(\mathbf{C})$. Assuming that the $TCPN$ is rank-controllable at $\mathcal{C}_i$, then $\text{Img}(\mathbb{C}_i) = \text{Img}(\mathbf{C})$, therefore $\mathbf{m}_1 - \mathbf{m}_0 \in \text{Img}(\mathbb{C}_i)$, then controllability follows. □

Previous proposition states that the controllability of a $TCPN$ system, in a configuration $\mathcal{C}_i$, can be guaranteed if it is rank-controllable at $\mathcal{C}_i$. Through this work, we will show that the full rank-controllability property can be analyzed avoiding the enumeration of all the configurations. In fact, in Section 5, the loss of rank-controllability will be related to the presence of *flow invariants* (equality relations involving transition flows). By considering the dynamics of the flow, as in (6), the uncontrollable invariant subspaces of the system can be studied from its controllability matrix:

$$\mathbb{M}_i = \left[\mathbf{\Lambda\Pi}_i\mathbf{C}S_{T_c} \ (\mathbf{\Lambda\Pi}_i\mathbf{C})^2 S_{T_c} \ \ldots \ (\mathbf{\Lambda\Pi}_i\mathbf{C})^{|T|} S_{T_c}\right] \quad (10)$$

Notice that the matrix $\mathbb{M}_i$ has a similar structure than the controllability matrix $\mathbb{C}_i$, in fact, $\mathbb{C}_i = \mathbf{C}\Psi_i$ where:

$$\Psi_i = \left[S_{T_c} \quad \mathbf{\Lambda\Pi}_i\mathbf{C}S_{T_c} \quad \ldots \quad (\mathbf{\Lambda\Pi}_i\mathbf{C})^{|P|-1} S_{T_c}\right] \quad (11)$$

The difference between $\mathbb{M}_i$ and Ψ_i appears in the first term S_{T_c} and the length of the expansion. This fact will be used to related the loss of rank controllability with the presence of flow invariants (Theorem 27).

4. INFLUENCE OF THE CONTROLLABLE TRANSITIONS

This section is devoted to the study of how the activity on a controllable transition *influence* the marking of the places. It will be shown that a necessary condition for full rank-controllability is that all the places in the net are *influeced by the controllable transitions*. Consider the place transition sequence (Fig. 1) with $T_c = \{t_1\}$. Then, the $TCPN$ represented by this sequence becomes rank-controllable. In detail, for $\boldsymbol{\lambda} = [\lambda_1, ..., \lambda_n]^T$, its dynamic equation is described by:

$$\dot{\mathbf{m}} = \begin{bmatrix} -\lambda_1 & 0 & \ldots & 0 \\ \lambda_1 & -\lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \lambda_{n-1} & -\lambda_n \end{bmatrix} \mathbf{m} + \begin{bmatrix} -1 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \mathbf{u} \quad (12)$$

and it can be easily verified that the rank of its controllability matrix is equal to the rank of the token-flow matrix.

This observation suggests that if a transition is controllable, then its control action *influences* over the flow of transitions down-stream in the sequence, which is true for Join-Free (JF) systems. However, join transitions may stop the influence propagation at some configurations. See, for instance, the net depicted in Fig. 2.a). In this system, if $T_c = \{t_3\}$, then the markings in p_3, p_4, p_5, p_{12} and p_{13} can be influenced. For instance, the marking at p_4 is described by $\dot{m}_4 = w_3 - \lambda_4 m_4$, thus m_4 can be controlled by means of w_3, moreover, the flow at t_4 is given by $f_4 = \lambda_4 m_4$, thus f_4 can be controlled by means of controlling m_4. The same occurs for p_3, p_5, p_{12} and p_{13}. Now, the flow at transition t_5 and the marking in place p_7 cannot be influenced if p_6 constrains the flow of t_5 since, in such case, the flow at t_5 is given by $f_5 = \lambda_5 m_6$. A similar reasoning is applied to every join transition. In other words, a place (transition) is said to be influenced by the control actions in all the configurations if it is always possible to state its marking evolution (its flow) in terms of some control action.

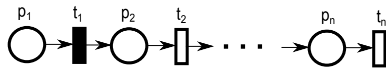

Fig. 1. Place transition sequence.

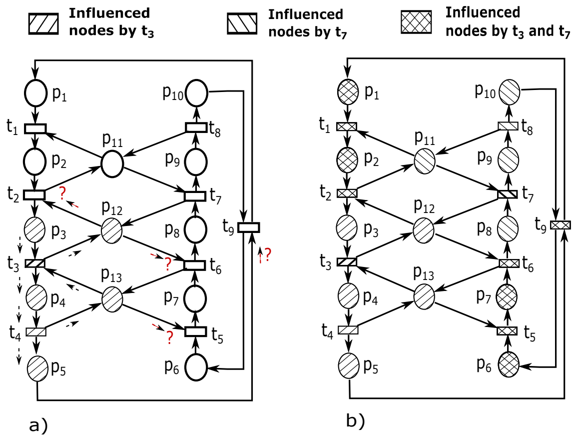

Fig. 2. On influence in $TCPN$s: a) Influenced nodes by the control action of t_3. For some configurations, it is blocked at join transitions. b) Influenced nodes when $T_c = \{t_3, t_7\}$. In this case $P_I = P$ and $T_I = T$.

Definition 15. Let $\mathcal{N}$ be a CPN where T_c is the set of controllable transitions. A place p_s is said to be *influenced by the control actions of* T_c if, for any configuration $\mathcal{C}_i$, there is a directed path from a transition $t_j \in T_c$ to p_s such that for any transition $t_k \neq t_j$ in the path, it is constrained by its previous place node in the path. A transition t_s is said to be *influenced by* T_c if $\forall p \in {}^\bullet t_s$, p is being influenced. The *set of influenced places* and *transitions* are denoted as P_I and T_I, respectively.

Consider the net depicted in Fig. 2.b). If both, t_3 and t_7 are controlled, then the markings in p_3, p_4, p_5, p_8, p_9, p_{10}, p_{11}, p_{12} and p_{13} are influenced. Moreover, since the markings in p_5 and p_{10} are influenced by the control actions, then the join transition t_9 is influenced as well in any configuration. Consequently, p_1 and p_6 are also influenced by the control actions. Following this reasoning, $P_I = P$.

It is worth noticing that *influence* and *controllability* are two different concepts. As explained above, if the $TCPN$ is a place-transition chain, then the control has influence over all the nodes, thus it is controllable. However, if circuits and other structural objects are found, then marking and flow invariants are introduced. This may affect the controllability. In such case, the influence is not *independent* on each place, thus it does not imply rank-controllability.

Example 16. Consider the net of Fig. 2.b). In this case, $P_I = P$ and $T_I = T$. However, this does not imply full rank-controllability. For example, with $\boldsymbol{\lambda} = [1, ..., 1]^T$, $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is not rank-controllable at some configurations. For instance, $\mathcal{C}_a = \{(p_{11}, t_1), (p_2, t_2), (p_{13}, t_3), (p_4, t_4), (p_{13}, t_5), (p_7, t_6), (p_{11}, t_7), (p_9, t_8), (p_5, t_9)\}$, where $rank(\mathbb{C}_a) = |P| - rank(\mathbf{B}_y^T) - 1$, i.e., $\exists \boldsymbol{v}$ s.t. $\boldsymbol{v}^T \mathbb{C}_a = \mathbf{0}$ and $\boldsymbol{v} \notin \text{Img}(\mathbf{B}_y)$.

On the contrary, *influence on all the places is a necessary condition for full rank-controllability.*

Proposition 17. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$ and T_c be the set of controllable transitions. If the $TCPN$ is full rank-controllable, the corresponding $P_I = P$.

It is worth noticing that influence can be verified in polynomial time, while the complexity of computing the influenced places by enumerating configurations is exponential.

Proposition 18. Let $\mathcal{N}$ be a CPN and T_c the set of controllable transitions. Then, Algorithm 1 computes, in polynomial time, the sets of influenced nodes, P_I and T_I, in all the configurations.

Proof. Notice that the loop of Alg. 1 ends, at most, after $|P|-1$ iterations. Moreover, the operations from the loop are performed in polynomial time. Then, the previous algorithm computes P_I and T_I in polynomial time. □

Algorithm 1: Sets of influenced nodes by T_c, $\forall \mathcal{C}_i$.

Initialize: $T_I := T_c, P_I := {}^\bullet T_c \cup T_c^\bullet$.
repeat
  $T_A := T_I; P_A := P_I;$
  $T_I := T_A \cup \{t \in P_A^\bullet \mid t \notin T_A \wedge {}^\bullet t \subseteq P_A\}$
  $P_I := P_A \cup {}^\bullet(T_I \setminus T_A) \cup (T_I \setminus T_A)^\bullet$
until $P_I = P_A$;
return P_I *and* T_I

Algorithm 1 can be used as a preliminary step in the full rank-controllability analysis: if influence over all the places is not satisfied, T_c is not adequate to ensure the full rank-controllability of the $TCPN$. On the contrary, if the condition is met, further analysis is necessary to conclude over the full rank-controllability of the system. This will be addressed in the following sections.

5. STRUCTURAL CHARACTERIZATION OF FLOW INVARIANTS AND RANK-CONTROLLABILITY.

This section is devoted to show the relation between flow invariants and the full rank-controllability in $TCPN$s, as stated in Theorem 27. The main idea of this section is to look for flow invariants by using structural net objects such as P-flows, choice places, or from the image of matrix $\mathbf{C}$. First, the definition of flow invariant is introduced.

Definition 19. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a $TCPN$ system at a configuration $\mathcal{C}_i$. Consider the flow vector at $\mathcal{C}_i$, $\mathbf{f}(\tau) = \boldsymbol{\Lambda}\boldsymbol{\Pi}_i\mathbf{m}(\tau)$ and its entries $f_1(\tau), ..., f_m(\tau)$. A *flow invariant* ($FI$) is an equality

$$\boldsymbol{v}^T\mathbf{f} = v_1 f_1(\tau) + ... + v_m f_m(\tau) = c \tag{13}$$

that holds in a subset of markings of $\mathcal{R}_i$, $\forall \tau$ when $\mathcal{C}_i$ is active, where $c \in \mathbb{R}$ and $v_1, ..., v_m \in \mathbb{R}$, not all zero.

Now consider the flow controllability matrix (Eq. (10)):

$$\mathbb{M}_i = \left[\boldsymbol{\Lambda\Pi}_i\mathbf{C}S_{T_c} \ (\boldsymbol{\Lambda\Pi}_i\mathbf{C})^2 S_{T_c} \ ... \ (\boldsymbol{\Lambda\Pi}_i\mathbf{C})^{|T|} S_{T_c}\right]$$

In accordance to the PBH Test, if the flow system is not controllable in region $\mathcal{R}_i$, then, there exist vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and/or $\boldsymbol{\delta}$ such that:

(1) $\boldsymbol{\alpha}^T\boldsymbol{\Lambda\Pi}_i = \mathbf{0}$
(2) $\boldsymbol{\beta}^T\boldsymbol{\Lambda\Pi}_i\mathbf{C} = \mathbf{0} \wedge \boldsymbol{\beta}^T\boldsymbol{\Lambda\Pi}_i \neq \mathbf{0}$
(3) $\boldsymbol{\delta}^T\boldsymbol{\Lambda\Pi}_i\mathbf{C} = \gamma\boldsymbol{\delta}^T \wedge \boldsymbol{\delta}^T S_{T_c} = \mathbf{0}$, where $\gamma \neq 0$

These vectors correspond to invariant subspaces of matrix $\boldsymbol{\Lambda\Pi}_i\mathbf{C}$; as will be seen in the following subsections, they correspond to flow invariants.

Since $\boldsymbol{\Lambda}$ is a full rank matrix, the existence of flow invariants characterized by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, in cases (1) and (2), depends only on the kernels of matrices $\boldsymbol{\Pi}_i$ and $\boldsymbol{\Pi}_i\mathbf{C}$, respectively, i.e. a merely structural condition. On the contrary, in case (3) the existence of a left annuler depends also on the value of matrix $\boldsymbol{\Lambda}$, i.e., given a particular structure, they may not appear for any timing.

Consequently, flow invariants corresponding to cases (1) and (2) are named *structural flow invariants* (SFIs). Flow invariants corresponding to case (3) are named *timed flow invarints* (TFIs). In particular, an invariant of case (1) is a *flow invariant induced by a choice place* (Choice-SFI) since they are related to the existence of choice places; an invariant of case (2) is a *flow invariant induced by a token conservation law* (Conservation-SFI) since they are related to the existence of P-flows.

5.1 Flow invariants induced by choice places

This section shows that choice places may lead to Choice-SFIs in particular configurations.

Example 20. Consider any $\mathcal{C}_i$ of $\mathcal{N}$, depicted in Fig. 2, s.t. $(p_{12}, t_2), (p_{12}, t_6) \in \mathcal{C}_i$. Thus, $f_2 = \lambda_2 m_{12}$ and $f_6 = \lambda_6 m_{12}$. Hence, $f_2/\lambda_2 - f_6/\lambda_6 = 0$, i.e., a Choice-$SFI$ is present in $\mathcal{C}_i$. Next proposition formalizes this.

Proposition 21. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$ at a configuration $\mathcal{C}_i$. Let p_c be a choice place s.t. $p_c \in \mathcal{TC}_i$. If p_c constrains $k > 1$ of its output transitions at $\mathcal{C}_i$, then p_c introduces $k-1$ left annulers in $\boldsymbol{\Lambda\Pi}_i$, leading to the corresponding Choice-SFIs.

Proof. Rename transitions and places in such a way that $\{t_1, \ldots, t_k\}$ are the output transitions of p_c and this is the first place appearing in the token-flow matrix. Hence:

$$\boldsymbol{\Pi}_i = \left[\begin{array}{c|c} 1/\mathbf{Pre}[1,1] & 0 \ \ldots \ 0 \\ \vdots & \vdots \ \ddots \ \vdots \\ 1/\mathbf{Pre}[1,k] & 0 \ \ldots \ 0 \\ \hline \mathbf{0} & \boldsymbol{\Pi}' \end{array}\right] \tag{14}$$

Then, the vectors $\mathbf{a}_1 = [\mathbf{Pre}[1,1] \ -[\mathbf{Pre}[1,2] \ 0 \ ... \ 0]^T$, ..., $\mathbf{a}_{k-1} = [\mathbf{Pre}[1,1] \ 0 ... \ 0 \ -[\mathbf{Pre}[1,k] \ 0 \ ... \ 0]^T$ are left annulers of $\boldsymbol{\Pi}_i$. Since $\boldsymbol{\Lambda}$ is a full rank matrix, the system $\boldsymbol{\alpha}_j^T\boldsymbol{\Lambda} = \mathbf{a}_j^T$, $j \in \{1, ..., k-1\}$, has solution. Then $\boldsymbol{\alpha}_j^T\boldsymbol{\Lambda\Pi}_i = 0$, i.e. $\alpha_j[1]f_1 + ... + \alpha_j[m]f_m = 0$. Thus, it follows that Choice-SFIs appear. □

Notice that if two different choice places p_{c_1}, p_{c_2} are contained in $\mathcal{TC}_i$, the left annulers (i.e., the Choice-SFIs) introduced by p_{c_1} are linearly independent from those introduced by p_{c_2}.

5.2 Flow invariants induced by token conservation laws

This section shows that the existence of P-flows may lead to Conservation-SFIs in particular configurations.

Example 22. Consider the net in Fig. 2 where $||\mathbf{y}_1|| = \{p_1, p_2, p_3, p_4, p_5\}$ is the support of a P-semiflow of $\mathcal{N}$ (i.e., the token conservation law $m_1 + m_2 + m_3 + m_4 + m_5 = c$ exists). If $||\mathbf{y}_1|| \subseteq \mathcal{TC}_i$, then, the flows of the transitions in $||\mathbf{y}_1||^\bullet$ are given by $f_1 = \lambda_1 m_1$,$f_2 = \lambda_2 m_2$, $f_3 = \lambda_3 m_3$, $f_4 = \lambda_4 m_4$ and $f_9 = \lambda_9 m_5$. Thus, the marking invariant can be written as $f_1/\lambda_1 + f_2/\lambda_2 + f_3/\lambda_3 + f_4/\lambda_4 + f_9/\lambda_9 = c$, i.e., a Conservation-$SFI$ that will be present in $\mathcal{C}_i$. The following proposition formalizes this kind of flow invariants.

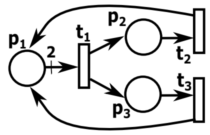

Fig. 3. $TCPN$ system with timing $\boldsymbol{\lambda} = [\lambda_1 \;\; \lambda_2 \;\; \lambda_3]^T$.

Proposition 23. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$ at a configuration $\mathcal{C}_i$. Let $\mathbf{y}$ be a P-flow of $\mathcal{N}$ such that $||\mathbf{y}|| \subseteq \mathcal{TC}_i$. Then $\mathbf{y}$ introduces left annulers in $\boldsymbol{\Lambda\Pi}_i\mathbf{C}$, leading to Conservation-SFIs.

Proof. Suppose that a P-flow $\mathbf{y}_j = [y_1 \cdots y_n]^T$ is s.t. $||\mathbf{y}_j|| \subseteq \mathcal{TC}_i$. If $y_k \neq 0$ then the place p_k is constraining the flow of at least one transition in $\mathcal{C}_i$. Without loss of generality name these transitions as $t_a, t_b, ..., t_l$. Thus $\pi_{a,k}, \pi_{b,k}, ...\pi_{l,k} \neq 0$ (the entries of matrix $\boldsymbol{\Pi}_i$). Moreover, at $\mathcal{C}_i$, each transition is constrained by only one place, thus rows of $\boldsymbol{\Pi}_i$ have only one non-null entry. Hence, there exists scalars $b_1, ..., b_m$ such that

$$[b_1 \;\; ... \;\; b_m]\boldsymbol{\Pi}_i = \mathbf{y}_j{}^T \tag{15}$$

Since $\mathbf{y}_j{}^T\mathbf{C} = \mathbf{0}$, then $\mathbf{b}_j = [b_1 \;\; ... \;\; b_m]$ is a left annuler of $\boldsymbol{\Pi}_i\mathbf{C}$. Moreover, matrix $\boldsymbol{\Lambda}$ has full rank, then

$$\boldsymbol{\beta}_j^T = \mathbf{b}_j^T\boldsymbol{\Lambda}^{-1} \tag{16}$$

is a left annuler of the matrix $\boldsymbol{\Lambda\Pi}_i\mathbf{C}$ generated by the P-flow $\mathbf{y}_j$. Now, from (4) and (16) it follows:

$$\boldsymbol{\beta}_j{}^T\mathbf{f}(\tau) = \boldsymbol{\beta}_j{}^T\boldsymbol{\Lambda\Pi}_i\mathbf{C}\sigma + \boldsymbol{\beta}_j{}^T\boldsymbol{\Lambda\Pi}_i\mathbf{m}(0)$$

Since $\boldsymbol{\beta}_j{}^T\boldsymbol{\Lambda\Pi}_i\mathbf{C}\sigma = 0$, then:

$$\boldsymbol{\beta}_j{}^T\mathbf{f}(\tau) = \mathbf{y}^T\mathbf{m}(0) = constant.$$

Thus, a Conservation-SFI exists. □

5.3 Timed flow invariants

This section deals with timed flow invariants. Next example illustrates a particular case of these invariants.

Example 24. Consider the $TCPN$ in Fig. 3 with $T_c = \{t_1\}$. It is a Join-Free net, so it has a unique configuration. According to Eq. (10), its flow controllability matrix is:

$$\mathbb{M} = \left[\begin{array}{c|c|c} -\lambda_1 & \lambda_1(2\lambda_1+\lambda_2+\lambda_3)/2 & -\lambda_1(2\lambda_1(\lambda_1+\lambda_2+\lambda_3)+\lambda_2^2+\lambda_3^2)/2 \\ \lambda_2 & -\lambda_2(\lambda_1+\lambda_2) & \lambda_2(\lambda_1(2\lambda_1+3\lambda_2+\lambda_3)+2\lambda_2^2)/2 \\ \lambda_3 & -\lambda_3(\lambda_1+\lambda_3) & \lambda_3(\lambda_1(2\lambda_1+3\lambda_3+\lambda_2)+2\lambda_3^2)/2 \end{array}\right]$$

Clearly, if $\lambda_2 = \lambda_3$, the second and third rows of $\mathbb{M}$ are equal. In other words, $\boldsymbol{\delta}^T = [0 \;\; 1 \;\; -1]$ is a left annuler of $\mathbb{M}$ iff $\lambda_2 = \lambda_3$. Let us assume that $\lambda_2 = \lambda_3$. Then, $\boldsymbol{\delta}$ characterizes an invariant of the flow system (it can be easily verified that it corresponds to the case (3)). For simplicity, consider any $\mathbf{m}_0$ s.t. $\mathbf{m}_0[p_2] = \mathbf{m}_0[p_3]$. Since $f_2 = \lambda_2 m_2$ and $f_3 = \lambda_3 m_3$, then $\forall \mathbf{m} \in Class(\mathbf{m}_0)$ s.t. $m_2 = m_3$ it holds that $\boldsymbol{\delta}^T\mathbf{f} = 0$, i.e., the flow invariant $f_2 - f_3 = 0$ exists over such marking set. Moreover, since $\dot{\mathbf{m}}_2 = f_1 - f_2 - u_1$ and $\dot{\mathbf{m}}_3 = f_1 - f_3 - u_1$, then $\dot{\mathbf{m}}_2 = \dot{\mathbf{m}}_3$ $\forall\tau$ and the evolution of the system will be restricted to such marking set.

Timed flow invariants cannot be characterized only from the net structure, and their computation is a hard task since all the configurations must be analyzed. This section introduces a necessary condition for the existence of these flow invariants.

Proposition 25. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$. If there exist a configuration $\mathcal{C}_i$ in which a timed flow invariant (TFI) appears, then, $\mathbf{C}S_{T_c}$ has left annulers $\boldsymbol{v}$ s.t. $\boldsymbol{v}^T\mathbf{C} \neq \mathbf{0}$.

Proof. Assume that there exist a configuration $\mathcal{C}_i$ in which a TFI appear. Then, $\exists\boldsymbol{\delta}$ s.t. $\boldsymbol{\delta}^T\boldsymbol{\Lambda\Pi}_i\mathbf{C} = \gamma\boldsymbol{\delta}^T$, with $\gamma \neq 0$, and $\boldsymbol{\delta}^T S_{T_c} = \mathbf{0}$. Thus, $\boldsymbol{\delta} = \mathbf{C}^T(\boldsymbol{\Pi}_i^T\boldsymbol{\Lambda\delta})(1/\gamma)$, i.e. $\boldsymbol{\delta} \in \text{Img}(\mathbf{C}^T)$. Hence, $\exists\boldsymbol{v}$ s.t. $\mathbf{C}^T\boldsymbol{v} = \boldsymbol{\delta}$, and $S_{T_c}{}^T\mathbf{C}^T\boldsymbol{v} = 0$. Thus $\boldsymbol{v}^T\mathbf{C}S_{T_c} = \mathbf{0}$, i.e., $\boldsymbol{v}$ is a left annuler. □

5.4 Uncontrollable flow invariants and rank-controllability

This section relates the full rank-controllability property with the existence of *uncontrollable flow invariants*, i.e., FIs involving the flow of solely uncontrollable transitions.

Definition 26. Let $\boldsymbol{v}^T\mathbf{f} = c$ be a flow invariant. If $||\boldsymbol{v}|| \subseteq T_{nc}$, it is said to be an *uncontrollable flow invariant* (UFI).

As it will be shown, if there are no UFIs in any configuration, then the $TCPN$ will be full rank-controllable. The structural flow invariants can be easily computed and tested to check if some of them are UFIs. The timed flow invariants are, by definition (see case (3)), UFIs since the invariant $\boldsymbol{\delta}$ is orthogonal to the selector of controllable transitions. Thus, according to proposition 25, their non existence can be guaranteed if $\mathbf{C}S_{T_c}$ has no left annulers, $\boldsymbol{v}$, s.t. $\boldsymbol{v}^T\mathbf{C} \neq \mathbf{0}$. The following theorem formalizes these ideas.

Theorem 27. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$. If there are no UFIs in all the configurations, then the $TCPN$ is full rank-controllable.

Proof. Part I: Consider any $\mathcal{C}_i$ of $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$. Let us demonstrate that if there are no UFIs in $\mathcal{C}_i$ then $rank([S_{T_c}, \mathbb{M}_i]) = |T|$. Proceeding by contradiction, suppose that there are no UFIs but $rank([S_{T_c}, \mathbb{M}_i]) < |T|$, then there exists $\boldsymbol{v}$ s.t. $\boldsymbol{v}^T S_{T_c} = \mathbf{0}$ and $\boldsymbol{v}^T\mathbb{M}_i = \mathbf{0}$. This last equality means that the flow state equation (6) is uncontrollable. Thus, by the PBH test (Proposition 11) it follows that $\boldsymbol{v}$ is a left eigenvector of $\boldsymbol{\Lambda\Pi}_i\mathbf{C}$. Moreover, $\boldsymbol{v}^T S_{T_c} = \mathbf{0}$ means that $\boldsymbol{v}$ is orthogonal to the selector of controllable transitions, which implies $||\boldsymbol{v}|| \subset T_{nc}$, then $\boldsymbol{v}$ is an UFI, which is a contradiction.

Part II: Now, let us show that $\text{rank}(\Psi_i) = \text{rank}([S_{T_c}, \mathbb{M}_i])$, despite the difference in the length of the expansions. First, if $|P| > |T|$ then Ψ_i has more columns than $[S_{T_c}, \mathbb{M}_i]$, but, by the Caley-Hamilton theorem, the columns of Ψ_i associated to the terms with exponents $k > |T|$ are linearly dependent of those columns of $[S_{T_c}, \mathbb{M}_i]$, thus both matrices have the same rank. On the other hand, if $|T| > |P|$, $[S_{T_c}, \mathbb{M}_i]$ has more columns than Ψ_i, however, by construction $rank(\boldsymbol{\Lambda\Pi}_i\mathbf{C}) \leq |P|$, then the columns of $[S_{T_c}, \mathbb{M}_i]$ associated to the terms with exponents $k > |P|$ are linearly dependent of those columns in Ψ_i, thus both matrices have the same rank.

Part III: Finally, if there are no $UFIs$ in $\mathcal{C}_i$, from the above, $|T| = rank([S_{T_c}, \mathbb{M}_i]) = rank(\Psi_i)$. Thus, since $\mathbb{C}_i = \mathbf{C}\Psi_i$ and Ψ_i has full row rank, then $\text{rank}(\mathbb{C}_i) = \text{rank}(\mathbf{C})$, i.e., the $TCPN$ is rank-controllable in $\mathcal{C}_i$. Moreover, since the previous is valid $\forall\mathcal{C}_i$, then the $TCPN$ is full rank-controllable □

6. TEST FOR FULL RANK-CONTROLLABILITY

In this Section, the previous results are integrated in an algorithm to polynomially test sufficient conditions for full rank-controllability.

According to theorem 27, full rank-controllability can be guaranteed if there are no UFIs in any $\mathcal{C}_i$. In order to test if a $TCPN$ fulfills this condition, a matrix that contains its potential SFIs is firstly introduced. Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a $TCPN$ where $j = |P_C|$ is the number of choice places in $\mathcal{N}$ and $k = dim(\mathbf{B}_y)$ is the dimension of the P-flow basis. Let $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_j$ be a basis for the Choice-SFIs induced by $p_{c1}, ..., p_{cj} \in P_C$ (as in Prop. 21). Let $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_k$ be a basis for the Conservation-SFIs induced by the P-semiflows $\mathbf{y}_1, ..., \mathbf{y}_k$ of the basis $\mathbf{B}_y$ (as in Prop. 23). The matrix of potential SFIs, for all the configurations, is:

$$F^s = [\boldsymbol{\alpha}_1 \ ... \ \boldsymbol{\alpha}_j | \ \boldsymbol{\beta}_1 \ ... \ \boldsymbol{\beta}_k]^T \tag{17}$$

It can be seen that the dimension of each of the bases for the SFIs, and thus the number of rows in F^s, are linear functions of the number of choice places, its output transitions and the dimension of $\mathbf{B}_y$.

Example 28. Consider the $TCPN$ in figure 5. The net has 2 choice places, p_5 and p_6. The support of the P-semiflows of $\mathcal{N}$ that form a basis $\mathbf{B}_y$ are: $||\mathbf{y}_1|| = \{p_2, p_3, p_4, p_{17}\}$, $||\mathbf{y}_2|| = \{p_8, p_9, p_{10}, p_{18}\}$, $||\mathbf{y}_3|| = \{p_2, p_5, p_{10}\}$, $||\mathbf{y}_4|| = \{p_4, p_6, p_8\}$, $||\mathbf{y}_5|| = \{p_{13}, p_{14}\}$, $||\mathbf{y}_6|| = \{p_{15}, p_{16}\}$, $||\mathbf{y}_7|| = \{p_1, p_2, p_3, p_4, p_{11}, p_{14}, p_{16}\}$, $||\mathbf{y}_8|| = \{p_7, p_8, p_9, p_{10}, p_{12}, p_{14}, p_{16}\}$. Let us compute its matrix F^s.

See, for instance, the place p_6 and its output transitions t_3 and t_5. According to Prop. 21, $\forall \mathcal{C}_i$ s.t. $(p_6, t_3), (p_6, t_5) \in \mathcal{C}_i$, the flow invariant $f_3/\lambda_3 - f_5/\lambda_5 = 0$ exist. The corresponding Choice-SFI is characterized by the vector:

$$\boldsymbol{\alpha}_6 = [0 \ 0 \ 1/\lambda_3 \ 0 \ -1/\lambda_5 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

For example, let us compute the flow invariants associated to the P-semiflow $\mathbf{y}_3$. For any configuration in which each of the places in $||\mathbf{y}_3||$ constrains, at least, one of its output transitions, t_1, t_2, t_7, and t_8, a flow invariant involving the flow of those transitions will be present. For instance, consider any configuration $\mathcal{C}_j$ s.t. $(p_5, t_1), (p_2, t_2), (p_5, t_7), (p_{10}, t_8) \in \mathcal{C}_j$, i.e., any of the 36 configurations that include all the possible flow invariants that $\mathbf{y}_3$ may introduce. Then, as in proof of proposition 23, a basis for the possible flow invariants is:

$$\boldsymbol{\beta}_3 = \begin{bmatrix} 1/\lambda_1 & 1/\lambda_2 & 0 & 0 & 0 & 0 & 0 & 1/\lambda_8 & 0 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 & 0 & 0 & 0 & 1/\lambda_7 & 1/\lambda_8 & 0 & 0 & 0 \end{bmatrix}^T$$

By following the same reasoning, the matrix of potential structural flow invariants, in all the configurations, is:

$$F^s = \left[\begin{array}{ccccccccccc} 0 & 0 & 1/\lambda_3 & 0 & -1/\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\lambda_1 & 0 & 0 & 0 & 0 & 0 & -1/\lambda_7 & 0 & 0 & 0 & 0 \\ \hline 1/\lambda_1 & 1/\lambda_2 & 1/\lambda_3 & 1/\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\lambda_5 & 1/\lambda_6 & 1/\lambda_7 & 1/\lambda_8 & 0 & 0 & 0 \\ 0 & 0 & 1/\lambda_3 & 1/\lambda_4 & 0 & 1/\lambda_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\lambda_4 & 1/\lambda_5 & 1/\lambda_6 & 0 & 0 & 0 & 0 & 0 \\ 1/\lambda_1 & 1/\lambda_2 & 0 & 0 & 0 & 0 & 0 & 1/\lambda_8 & 0 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 & 0 & 0 & 0 & 1/\lambda_7 & 1/\lambda_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\lambda_9 & 1/\lambda_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\lambda_{10} & 1/\lambda_{11} \\ 1/\lambda_1 & 1/\lambda_2 & 1/\lambda_3 & 1/\lambda_4 & 0 & 0 & 0 & 0 & 1/\lambda_9 & 1/\lambda_{10} & 1/\lambda_{11} \\ 0 & 0 & 0 & 0 & 1/\lambda_5 & 1/\lambda_6 & 1/\lambda_7 & 1/\lambda_8 & 1/\lambda_9 & 1/\lambda_{10} & 1/\lambda_{11} \end{array}\right] \tag{18}$$

Finally, the algorithm 2 tests the aforementioned condition for full rank-controllability. First, according to Prop. 25, a necessary condition for the existence of $UTFI$s is that there exist a left annuler, $\boldsymbol{v}$, of matrix $\mathbf{C}S_{T_c}$, s.t., $\boldsymbol{v}^T\mathbf{C} \neq \mathbf{0}$. In other words, the nonexistence of $UTFI$s can be guaranteed if $ker(\mathbf{C}^T) = ker(S_{T_c}^T\mathbf{C}^T)$, i.e., $rank(\mathbf{C}) = rank(\mathbf{C}S_{T_c})$. If this condition is fulfilled, the repeat/until loop verifies that there are no $USFI$s by checking that each row of F^s is not orthogonal to S_{T_c}, i.e., that all the possible structural flow invariants involve the flow of a controllable transition.

Algorithm 2: Test for ensuring rank-controllability in all the configurations.

Input: **Pre**, **Post**, $\boldsymbol{\lambda}$ and the selector of T_c, S_{T_c}.
Output: A variable $Flag$ that tells if the $TCPN$ is full rank-controllable.
Initialize: $Flag := 1$, $l := 0$, $\mathbf{C} := \mathbf{Post} - \mathbf{Pre}$
if $rank(\mathbf{C}S_{T_c}) = rank(\mathbf{C})$ **then**
  compute F^s (Eq. (17))
  repeat
    $l := l + 1$
    if $F^s[l, \bullet] \cdot S_{T_c} = \mathbf{0}$ **then** $Flag := 0$ **end**
  until $Flag = 0$ *or* $l = \#$ *of rows in* F^s;
else
  $Flag := 0$
end
end algorithm

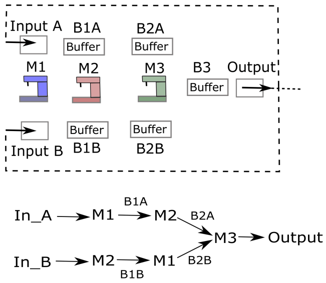

Fig. 4. Production process of a manufacturing system: Logical layout and process plan.

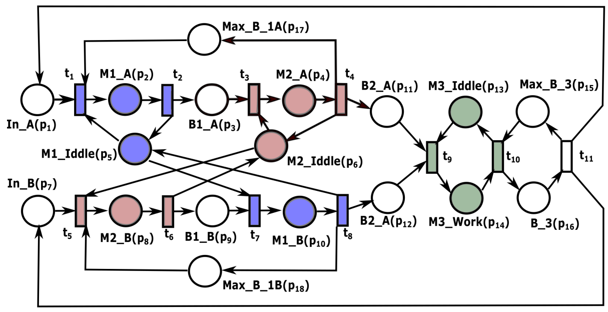

Fig. 5. $TCPN$ that models the flexible system in Fig. 4. It has 216 possible configurations.

The Algorithm 2 test the full-controllability property in polynomial time. If it gives a value $Flag = 1$, it means that *the $TCPN$ is full rank-controllable*. Otherwise, it is not possible to conclude if this property is fulfilled since the algorithm tests a sufficient condition.

7. ILLUSTRATIVE EXAMPLE

As an illustrative example, consider the flexible manufacturing system of Fig. 4 (Silva et al., 2014). A net structure dealing with the logical layout and process plan is presented in Fig. 5. It has 216 possible configurations. A rank-controllability analysis will be performed by using the algorithm presented in the previous section.

Assuming that machine $M3$ is always working at its nominal speed, the firing rate of the transition that model the discharge of material from that machine cannot be modified, i.e., $t_{10} \in T_{nc}$. On the contrary, it is assumed that $M1$ and $M2$ are not necessarily working at their nominal speeds, i.e., $t_2, t_4, t_6, t_8 \in T_c$. Furthermore, during the process it is always possible to decide the quantity of parts to process on an available machine and the output buffer can always be emptied leaving room in the system to process more parts, i.e. $t_1, t_3, t_5, t_7, t_9, t_{11} \in T_c$. Then, the set of controllable transitions is

$$T_c = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{11}\}.$$

Then, using the matrix F^s of the $TCPN$ (18) and Algorithm 2 we can verify, in polynomial time, that the timed net is full rank-controllable.

Notice that, even though the sets of equilibrium markings of a $TCPN$ system depend on the initial marking, $\mathbf{m}_0$ is not considered here. However, by using the presented approach we can conclude that, $\forall \mathbf{m}_0$ that marks the support of all the P-semiflows, the system is controllable $\forall \mathcal{C}_i$, over the corresponding sets E_i^* (Prop. 14). In contrast, the study of this property by enumerating configurations means that, for each different $\mathbf{m}_0$, a new analysis must be carried out by studying each of configuration of the system, which, in general $TCPN$s, may become intractable.

As a concluding remark, due to the lack of a complete characterization of the timed flow invariants, the presented results are clearly restrictive. However, since these invariants appear for specific values of the timing, we conjecture that full rank-controllability can be studied *generically*, where generic is understood as a property fullfilled for *almost all timing*, focusing only on the analysis of structural flow invariants. See for instance the case in example 24, where the timed flow invariant appears iff $\lambda_2 = \lambda_3$. In other words, $\forall \boldsymbol{\lambda}$ s.t. $\lambda_2 \neq \lambda_3$, there are no $UTFI$s in the $TCPN$. In fact, for any selection of $T_c \neq \emptyset$, the $TCPN$ does not exhibit $USFI$s, thus it is rank-controllable. Moreover, consider the example of this section. If the timing is chosen randomly, the system does not exhibit timed flow invariants. Then, the non existence of $USFI$s is a sufficient condition for full rank-controllability. For instance, if $T_c = \{t_1, t_3, t_5, t_7, t_9, t_{11}\}$ (i.e., the transitions related to the loading of all the machines and the output buffer emptying), by using the repeat/until loop of algorithm 2 it can be easily verified that there will be no $USFI$s, then, the $TCPN$ is full rank-controllable for almost any timing. This will be explored in future works, in order to state stronger results for full rank-controllability by using the presented approach.

8. CONCLUSIONS

It has been stated that controllability in $TCPN$ systems, under infinite server semantics, can be studied trough the structural property of *rank-controllability*. Then, two structural conditions for *full rank-controllability*, one necessary and the other sufficient, are provided. As a first necessary condition, the control actions must *influence* the marking at all the places. This can be verified from the net structure for all the configurations in *polynomial* time. As a second condition, the influence on the marking of each place is required to be *independent*. For this, uncontrollable flow invariants (UFIs) were defined and characterized by using structural objects of the net. Next, it has been shown that the existence of UFIs may lead to new marking invariants and it was proved that if *there are no UFIs in the $TCPN$, then, it is full rank-controllable.* Finally, a polynomial time algorithm to test sufficient conditions for full rank-controllability is provided.

A next step on this research is to extend the presented results to particular subclasses in order to obtain stronger results and determine supplementary information about the connection between the controllability property and the net structure. In addition, further investigation of the connexity of the sets of equilibrium markings is required since, together with full rank-controllability, it guarantees controllability over *the set of all equilibrium markings of the system.* Finally, a future direction will be to exploit the results herein presented for the synthesis of controllers for the case where there exist uncontrollable transitions.

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