Operator Inequality Approach for State-Feedback Stabilization of Infinite-Dimensional Systems: Synthesis via Dual of Input-to-State Operator

Izumi Masubuchi *

* *Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan (e-mail: msb@harbor.kobe-u.ac.jp).*

Abstract: This paper proposes a synthesis method of stabilizing state-feedback controllers for linear infinite-dimensional systems with possible unbounded input operators. A regularity condition is assumed in the sense of the existence of step responses for all initial conditions and constant inputs, under which the closed-loop system with any bounded state-feedback is well-posed with a filter on the input channels. Operator inequalities are provided on the dual space of the input-to-state operator, where any solution to the linear inequality provides a stabilizing filtered state feedback controller.

Keywords: Infinite-dimensional systems, stabilization, operator inequalities.

1. INTRODUCTION

Synthesis of feedback controllers for infinite-dimensional systems has been receiving a great deal of attention in recent years. Fruitful results have been presented for various specific classes of infinite-dimensional systems. Also some of papers are based on general frameworks for synthesis of controllers, particularly those based on linear or linearized operator inequalities; See e.g. Fridman and Orlov (2009a,b); Gahlawat and Peet (2017); Peet (2019) and references therein. A difficulty lies in that the partial differential equation derived as the result of feedback control might not be well-posed and for this problem case-by-case treatments are inevitably needed, if the plant has unbounded input and/or output operators. Detailed discussion on well-posedness of feedback systems can be found in Staffans (2005); Tucsnak and Weiss (2009, 2014).

This paper proposes a simple way of synthesis of stabilizing state-feedback controllers for linear infinite-dimensional systems with possible unbounded input operators. We assume a regularity condition in the sense of the existence of step responses for all initial conditions and constant inputs, under which the closed-loop system with any bounded state-feedback is well-posed via a filter on the input channels. Operator inequalities are provided on the dual space of the input-to-state operator Σ_{AB}, by which the plant variable $x \in X$ with input $u \in U$ is governed as

$$\dot{x} = \Sigma_{AB} \begin{bmatrix} x \\ u \end{bmatrix},$$

where X and U are Hilbert spaces and Σ_{AB} is a linear operator defined on the Cartesian product space fo X and U. This is adopted from Staffans (2005), where Σ_{AB} is written as $A\&B$. By this we can represent e.g. boundary control systems whose input appears in the boundary condition. We mention that in the proposed operator inequalities some of unknown variables can be eliminated in a similar fashion as the Finsler's lemma (Boyd et al. (1994)) in LMI-based synthesis for finite-dimensional systems.

2. DESCRIPTION OF THE SYSTEM

Let X and U be Hilbert spaces and consider a linear infinite-dimensional system represented as the following state equation on X:

$$\dot{x}(t) = \Sigma_{AB} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t > 0 \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where $x(t) \in X$ is the state and $u(t) \in U$ is the input of the system, with initial state x_0 at time $t = 0$. Let $Z = X \oplus U$ be the Cartesian product space of X and U and let Σ_{AB} be a linear operator from Z to X that defines the input-to-state characteristic of the system, where its domain is $D(\Sigma_{AB}) \subset Z$. The integral equation for (1)–(2) is

$$x(t) = x_0 + \Sigma_{AB} \int_0^t \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} ds, \qquad t \geq 0. \tag{3}$$

Define an operator $A : X \to X$ with its domain $D(A)$ as follows:

$$D(A) = \left\{ x \in X : \begin{bmatrix} x \\ 0 \end{bmatrix} \in D(\Sigma_{AB}) \right\}, \tag{4}$$

$$Ax = \Sigma_{AB} \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in D(A). \tag{5}$$

Assumption 1. Operators A and Σ_{AB} are closed.

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This assumption follows a part of those assumed in Staffans (2005) for system nodes.

We assume a simple regularity condition on the system in the sense of the existence of step response x for constant input u. Let $C(D,X)$ and $C^k(D,X)$ denote the spaces of continuous functions and k-times continuously differentiable functions from D to X, respectively.

Assumption 2. For every $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in X \oplus U$, there exists a unique $x \in C([0,\infty),X)$ that satisfies (3) for constant u with $u(t)=u_0$, $t \geq 0$.

If $u_0 = 0$, for every $x_0 \in X$, there exists a unique $x \in C([0,\infty),X)$ that satisfies

$$x(t) = x_0 + A\int_0^t x(s)ds, \qquad t \geq 0 \tag{6}$$

from Assumption 2. Lemma 3 below guarantees the existence of a solution to the following abstract Cauchy problem:

$$\dot{x}(t) = Ax(t), \qquad t > 0, \tag{7}$$

$$x(0) = x_0 \tag{8}$$

for $x_0 \in D(A)$ and that A generates a C_0 semigroup.

Lemma 3. The following three statements are equivalent:

(i) Operator A generates a C_0 semigroup on X.
(ii) Operator A is closed, $\rho(A) \neq 0$, and, for every $x_0 \in D(A)$, there exists a unique $x \in C^1([0,\infty),D(A))$ that satisfies (7)–(8), where $\rho(A)$ stands for the resolvent set of A.
(iii) Operator A is closed and, for every $x_0 \in X$, there exists a unique $x \in C([0,\infty),X)$ that satisfies (6).

Proof. See Theorem 20.2 of Miyadera (1996). See also Theorem 1.1.1 of Melnikova and Filinkov (2001).

From this lemma and Assumptions 1 and 2, A in (4)–(5) is densely defined and generates a C_0 semigroup $\mathbb{T}(t)$, $t \geq 0$ on X, with which solutions to (6) and (7)–(8) are given by

$$x(t) = \mathbb{T}(t)x_0, \qquad t \geq 0.$$

Moreover, Assumption 2 derives some useful properties of input-to-state operator Σ_{AB}. Define a densely defined operator $A_a : D(\Sigma_{AB}) \to Z$ by

$$A_a = \begin{bmatrix} \Sigma_{AB} \\ [\,0\ 0\,] \end{bmatrix}, \qquad D(A_a) = D(\Sigma_{AB}),$$

which is apparently closed since so is assumed for Σ_{AB}. From Assumption 2 and Lemma 3, we see the unique existence of $z \in C([0,\infty),Z)$ satisfying

$$z(t) = z_0 + A_a\int_0^t z(s)ds, \qquad t \geq 0$$

for every $z_0 \in Z$ and indeed $z(t)$ is given as

$$z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ u_0 \end{bmatrix}, \qquad z_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}. \tag{9}$$

Also there exists a unique $z \in C^1([0,\infty),Z)$ such that

$$\dot{z}(t) = A_a z(t), \qquad t > 0,$$
$$z(0) = z_0$$

for every $z_0 \in D(A_a) = D(\Sigma_{AB})$ from Lemma 3. Furthermore, A_a is densely defined and generates a C_0 semigroup $\mathbb{T}_a(t)$, $t \geq 0$ on X, with which the step responses are given by

$$x(t) = [\,I_X\ \ 0\,]\mathbb{T}_a(t)\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in X \oplus U,$$

where I_X stands for the identity operator on X. Clearly

$$\mathbb{T}(t) = [\,I_X\ \ 0\,]\mathbb{T}_a(t)\begin{bmatrix} I_X \\ 0 \end{bmatrix}.$$

Let $M_a > 0$, $\omega_a \in \mathbb{R}$ be constants for which $\|\mathbb{T}_a(t)\| \leq M_a e^{\omega_a t}$, $t \geq 0$.

Remark 4. The above introduction of an augmented system can be compared with that for boundary control systems in Section 3.3 of Curtain and Zwart (1995) with C^2 input. Our formulation here is rather simple and may be regarded as an explicit implementation of a filter to generate smoothen inputs from the measured state.

3. FILTERED STATE FEEDBACK

Let $\mathcal{L}(X,U)$ and $\mathcal{L}(U)$ denote the sets of bounded linear operators from X to U and U to U, respectively. For system (1), consider state feedback control with filtering:

$$\dot{u}(t) = Fx(t) + Gu(t), \tag{10}$$

where $F \in \mathcal{L}(X,U)$, $G \in \mathcal{L}(U)$. The transfer function of this controller is $(sI_U - G)^{-1}F$, which is a state feedback followed by a first-order low-pass filter. Combining (1) and (10), we have the closed-loop system equation as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{AB} \\ [\,F\ \ G\,] \end{bmatrix}\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \begin{cases} x(0) = x_0, \\ u(0) = u_0. \end{cases} \tag{11}$$

By using the notation A_a with

$$z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad z_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ I_U \end{bmatrix}$$

and

$$F_a = [\,F \quad G\,],$$

we can represent the closed-loop system (11) as

$$\dot{z}(t) = A_{cl}z(t), \qquad A_{cl} = A_a + B_aF_a.$$

Since A_a generates a C_0 semigroup and B_a and F_a are bounded, A_{cl} generates a C_0 semigroup $\mathbb{T}_{cl}(t)$, where we have $\|\mathbb{T}_{cl}(t)\| \leq M_a e^{(\omega_a + \|F_a\|)t}$, $t \geq 0$ and $D(A_{cl}) = D(A_a) = D(\Sigma_{AB})$.

Here we consider synthesis of F_a for which the closed-loop system is exponentially stable, i.e., we seek F_a such that $\mathbb{T}_{cl}$ satisfies

$$\|\mathbb{T}_{cl}(t)\| \leq M_a e^{-\alpha t}, \qquad t \geq 0 \tag{12}$$

for some $\alpha > 0$. Since Z is a Hilbert space, A_{cl}^*, the adjoint operator of A_{cl}, generates a C_0 semigroup $\mathbb{T}^*(t)$ (See Pazy

(1983), Corollary 10.6 of Chaper 1). Moreover, $\mathbb{T}_{cl}$ satisfies (12) if and only if $\mathbb{T}_{cl}^*$ satisfies

$$\|\mathbb{T}_{cl}^*(t)\| \le M_a e^{-\alpha t}, \qquad t \ge 0 \tag{13}$$

(See Pazy (1983), Lemma 10.1 of Chaper 1). We have $A_{cl}^* = A_a^* + F_a^* B_a^*$ and

$$A_a^* = \begin{bmatrix} \Sigma_{AB}^* & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix},$$
$$B_a^* = [\,0 \quad I_U\,],$$
$$F_a^* = \begin{bmatrix} F^* \\ G^* \end{bmatrix},$$

where $D(A_{cl}^*) = D(A_a^*) = D(\Sigma_{AB}^*) \oplus U$. Operators Σ_{AB}^*, A_a^* and A_{cl}^* are closed and $F^* \in \mathcal{L}(U, X)$, $G^* \in \mathcal{L}(U)$.

4. OPERATOR INEQUALITIES

Here we invoke Lyapunov methods (Datko (1970); Curtain and Zwart (1995)); The following Lyapunov inequality guarantees the exponential stability of $\mathbb{T}_{cl}^*$ and hence that of $\mathbb{T}_{cl}$:

$$\alpha_1\|w\|^2 \le \langle w,\ P_a w\rangle \le \alpha_2\|w\|^2 \qquad \forall w \in Z, \tag{14}$$
$$\langle P_a A_{cl}^* w,\ w\rangle + \langle w,\ P_a A_{cl}^* w\rangle \le -\gamma\|w\|^2 \qquad \forall w \in D(\Sigma_{AB}^*), \tag{15}$$

where $0 < \alpha_1 < \alpha_2$, $\gamma > 0$ and $P_a \in \mathcal{L}(Z)$ is self-adjoint. Let

$$P_a A_{cl}^* = P_a(A_a^* + F_a^* B_a^*) = P_a A_a^* + W_a^* B_a^*,$$

where

$$W_a = [\,W\ \ V\,] := F_a P_a \in \mathcal{L}(Z, U)$$

with $W \in \mathcal{L}(X, U)$, $V \in \mathcal{L}(U)$. Then

$$P_a A_{cl}^* = [\,P_a \Sigma_{AB}^* \quad W_a^*\,].$$

Thus we have obtained the following linearized operator inequality.

Proposition 5. Let $0 < \alpha_1 < \alpha_2$ and $\gamma > 0$. Suppose that there exists a self-adjoint operator $P_a \in \mathcal{L}(Z)$ and bounded operators $W \in \mathcal{L}(X, U)$, $V \in \mathcal{L}(U)$ such that

$$\alpha_1\|w\|^2 \le \langle w,\ P_a w\rangle \le \alpha_2\|w\|^2 \qquad \forall w \in Z, \tag{16}$$
$$\langle P_a \Sigma_{AB}^* f + W_a^* g,\ w\rangle + \langle w,\ P_a \Sigma_{AB}^* f + W_a^* g\rangle \le -\gamma\|w\|^2 \qquad \forall w \in D(\Sigma_{AB}^*), \tag{17}$$

where

$$w = \begin{bmatrix} f \\ g \end{bmatrix}, \quad f \in X, \quad g \in U$$

and

$$W_a = [\,W \quad V\,].$$

Then P_a is invertible in $\mathcal{L}(Z)$ and

$$F_a = [\,F \quad G\,] = W_a P_a^{-1}$$

is a pair of operators F and G by which $\mathbb{T}_{cl}$ is exponentially stable.

We can eliminate variable W_a in (17), which amounts to Finsler's lemma (Boyd et al. (1994)) for finite-dimensional LMIs. The first term of the left hand side of inequality (17) is divided as

$$\begin{aligned} &\langle P_a \Sigma_{AB}^* f + W_a^* g,\ w\rangle \\ &= \langle [\,I_X \quad 0\,] P_a \Sigma_{AB}^* f,\ g\rangle + \langle [\,0 \quad I_U\,] P_a \Sigma_{AB}^* f,\ g\rangle \\ &\quad + \langle g,\ Wf\rangle + \langle g,\ Vg\rangle. \end{aligned} \tag{18}$$

One may expect that setting

$$W = -[\,0 \quad I_U\,] P_a \Sigma_{AB}^*$$

will cancel out W and simplify the inequality. This is stated concretely as follows.

Corollary 6. Suppose that a self-adjoint operator $P_a \in \mathcal{L}(Z)$ satisfies (16) and

$$\langle [\,I_X \quad 0\,] P_a \Sigma_{AB}^* f,\ f\rangle + \langle f,\ [\,I_X \quad 0\,] P_a \Sigma_{AB}^* f\rangle \le -\gamma\|f\|^2 \tag{19}$$

for all $f \in [\,I_X \quad 0\,] D(\Sigma_{AB}^*)$, where $0 < \alpha_1 < \alpha_2$, $\gamma > 0$. Suppose also that operator

$$W_0 = -[\,0 \quad I_U\,] P_a \Sigma_{AB}^* : D(\Sigma_{AB}^*) \to U \tag{20}$$

is extended to an operator $W \in \mathcal{L}(X, U)$. Then (17) holds for $W = W_0$ with $V = -(\gamma/2) I_U$.

Proof. The proof is seen that under the assumption of the corollary,

$$\begin{aligned} &\text{Right hand side of (18)} \\ &\quad = \langle [\,I_X \quad 0\,] P_a \Sigma_{AB}^* f,\ f\rangle - \frac{\gamma}{2}\|g\|^2. \end{aligned}$$

Then (17) holds from this and (19).

Such an extension in Corollary 6 can be available if P_a is chosen an operator that 'integrates' $\Sigma_{AB}^* f$ when Σ_{AB}^* is an operator involving some differentiation of f. Another derivation of a controller is considered below.

Corollary 7. Suppose that (16) and (19) hold and

$$\|W_0 w\| \le k\|w\| \tag{21}$$

for all $w \in D(\Sigma_{AB}^*)$, where W_0 is defined as in (20). Then (17) is satisfied with $W = 0$, $V = -cI_U$ with $c > 0$ large enough, and γ replaced with $\gamma/2$.

Proof. From (19) and (21), setting $W = 0$ and $V = -cI_U$, we get

$$\begin{aligned} &\text{Left hand side of (17)} \\ &\quad \le -\gamma\|f\|^2 + 2k\|f\|\,\|g\| - c\|g\|^2 \\ &\quad \le -(\gamma/2)\|w\|^2, \end{aligned}$$

where the last inequality holds for some large $c > 0$.

5. CONCLUSION

In this paper, we showed a synthesis method of stabilizing state-feedback controllers for linear infinite-dimensional systems. We assumed a simple regularity condition of the existence of step responses. This suffices to guarantee the well-posedness of closed-loop systems with a filter on the input channel and any bounded feedback operators. Operator inequalities are provided on the dual space of the input-to-state operator, where any solution to the linear inequality provides a stabilizing filtered state-feedback controller.

Though the paper considered only stabilization, synthesis to satisfy control performance criteria can be considered by extending the proposed formulation via operator inequalities. The results of this paper can be extended in the spaces X_{-1}, X^d_{-1} (Staffans (2005); Tucsnak and Weiss (2009, 2014)) with more sophisticated regularity assumptions, where operator inequalities will provide bounded operators in the extended spaces.

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