Switched Fractional Order Model Reference Adaptive Control for Unknown Linear Time Invariant Systems ⋆

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Abstract: This paper presents a model reference adaptive controller for linear time invariant systems with unknown parameters. The adaptive laws used to estimate the controller parameters are based on fractional differential equations, whose orders are switched among a fractional value in the interval $(0, 1)$ and 1 at certain time instants. Boundedness of the signals in the resulting controlled system is proved using recent results. A simulation study is provided for a second order system to show how the proposed control strategy can improve the behavior and decrease the control energy used, compared to the classic model reference adaptive controllers and fractional order model reference adaptive controllers with non-switched adaptive laws.

Keywords: Control energy, Fractional control, Model Reference Adaptive Control, Switching theory.

1. INTRODUCTION

Adaptive systems refer to the identification or control of partially known systems, for which conventional techniques cannot achieve a satisfactory performance. Different adaptive algorithms for the stable identification and control of integer order systems with unknown parameters have been developed in the last forty years, such as the proposal of adaptive laws using the gradient technique approach (Narendra & Annaswamy, 2005).

Not a long time ago, fractional operators (integrals and derivatives of real orders (Kilbas et al., 2006)) were and keep being introduced in adaptive control schemes (Aguila-Camacho & Duarte-Mermoud, 2013; Tejado et al., 2014; Vinagre et al., 2002, 2008), allowing to increase the degrees of freedom in the design and adjustment of the controller, obtaining controlled systems with better performance as compared with integer order schemes.

One of the applications of fractional operators in adaptive control schemes is the use of fractional adaptive laws in model reference adaptive control, being the plant to be controlled described by a classic integer order differential equation (mixed order model reference adaptive control MO-MRAC). Some works have been published, reporting advantages for the MO-MRAC, such as better management of noise and disturbances, smoother control signals, etc., compared to the cases using classic integer order adaptive laws (Aguila-Camacho & Duarte-Mermoud, 2013, 2016). Analytical results for these mixed order cases took a longer time to appear, and even when we can already find some results in literature (Gallegos et al., 2019; Aguila-Camacho et al., 2019), still there are lots of cases without analytical support. Beyond the lack of analytical support for some MO-MRAC cases, regarding the behavior of fractional order control for integer order plants, published works usually report advantages compared to the case using classic adaptive laws, but usually a trade-off between system behavior and control energy used appears (Aguila-Camacho & Duarte-Mermoud, 2016, 2013; Aguila-Camacho & Ponce, 2018). That is, sometimes using fractional adaptive laws gives lower control energy but the convergence speed of the control error is lower or transients are worst, and some other times is the other way around. Thus, it looks like there is no way of achieving both results at the same time for most of the cases.

A recently presented work (Aguila-Camacho & Gallegos, 2019) introduced alternative fractional order adaptive laws to control a first order integer system in a MO-MRAC scheme, where the fractional order of the adaptive law is not fixed but switched between a real number in $(0, 1)$ and the classic case, which uses order 1, at some finite time T. This work reported that the proposed switched fractional order adaptive law (when carefully designed) allows obtaining better system performance and management of the control energy, compared to the classic adaptive law and fixed (non-switched) fractional order adaptive law. This paper extends the result in (Aguila-Camacho & Gallegos, 2019) to the case of systems not necessarily of first order, but higher order systems. Also, it extends the results in (Aguila-Camacho & Gallegos, 2019), allowing more than one switch, driven by an additional external signal.

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The paper is organized as follows. Section 2 presents some definitions and mathematical tools used along the paper. Section 3 presents the control problem, the proposed switched fractional order adaptive laws and the proof of boundedness of the signals in the resulting control scheme, together with the proof of the convergence to zero of the control error. Finally, Section 4 includes a simulation example for the control of a second order system, to show the advantages of the proposed switched fractional order adaptive laws compared to the classic adaptive laws and the non-switched fractional adaptive laws available in literature.

2. PRELIMINARIES

This section introduces some definitions, notation and mathematical tools used in this paper.

2.1 Definitions

Fractional operators are integrals and derivatives of orders that can be real or complex numbers (Kilbas et al., 2006).

The fractional integral of a function is defined by (Diethelm, 2004)

$$\left[{}_aI^\alpha f\right](t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (1)$$

where $\alpha > 0$, Γ is the gamma function and $m = \lceil \alpha \rceil$.

The Caputo fractional derivative is used in this paper, and is defined as

$$\left[{}_aD^\alpha f\right](t) := \left[{}_aI^{m-\alpha} D^m f\right](t). \qquad (2)$$

2.2 Continuity of solutions of fractional order systems

Consider the system of integral equations

$$y_i(t) = p_i(t) + I^{\alpha_i}\left[f_i(\cdot, y(\cdot))\right](t), \qquad (3)$$

where $\alpha_i > 0$, $y_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $f_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$. The following theorem state conditions for uniqueness and continuity of solutions to (3).

Theorem 1. (Gallegos et al., 2019) Consider system represented in (3) with $p : [0, T] \to \mathbb{R}^n$ a continuous function and $f_i(\cdot\cdot)$ continuous functions in their first variables and Lipschitz continuous functions in their second variables, for every $i = 1, \ldots, n$. Then:

i) There exists a unique continuous solution $y \in \mathcal{C}[0, T]$ to system (3).

ii) $y \in \mathcal{C}[0, T]$ is a solution to system (3) for

$$p_i(t) := \sum_{k=0}^{[\alpha_i]-1} \frac{t^k}{k!} y_{i_0}^{(k)} \qquad (4)$$

if and only if each of its components y_i is a solution to ${}_0D^\alpha y_i = f_i(t, y)$ with initial condition $y_i^{(k)}(0) = y_{i_0}^{(k)}$ for $k = 1, \ldots, [\alpha] - 1$ and $i = 1, \ldots, n$.

3. CONTROL PROBLEM AND PROPOSED CONTROL STRATEGY

Let us consider a single-input single-output linear time invariant plant described by

$$\dot{x}_p(t) = A_p x_p(t) + b_p u(t), \; x_p(0) = x_{p_0} \qquad (5)$$

where $A_p \in \mathbb{R}^{n \times n}$ is an unknown constant matrix, $b_p \in \mathbb{R}^n$ is a known constant vector, $u \in \mathbb{R}$ is the input of the system and $x_p \in \mathbb{R}^n$ is the state, which is assumed to be accessible.

An asymptotically stable reference model is specified by the linear time-invariant system described by

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t), \; x_m(0) = x_{m_0} \qquad (6)$$

where $r \in \mathbb{R}$ is a bounded $\mathcal{C}^1$ reference input, $A_m \in \mathbb{R}^{n \times n}, b_m \in \mathbb{R}^n$ are known and A_m is a Hurwitz matrix. It is assumed that $x_m(t)$, for all $t \geq 0$, represents the desired trajectory for $x_p(t)$.

3.1 Control strategy

To solve this problem, it is assumed that constant $k^* \in \mathbb{R}$ and $\theta^* \in \mathbb{R}^n$ exist such that

$$b_m = k^* b_p \qquad A_p + b_p {\theta^*}^T = A_m \qquad (7)$$

Classic model reference adaptive control can then be used (see Narendra & Annaswamy (2005)), where the control signal is given by

$$u(t) = \theta^T(t) x_p(t) + k^* r(t), \qquad (8)$$

with $\theta(t) \in \mathbb{R}^n$ an adjustable parameter vector.

Adaptive laws to estimate $\theta(t)$ in the classic case were proposed as (Narendra & Annaswamy, 2005)

$$\dot{\theta}(t) = -b_p e^T(t) P x_p(t), \quad \theta(0) = \theta_0 \qquad (9)$$

where $e(t) = x_p(t) - x_m(t)$ is the control error and $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix such that $A_m^T P + P A_m = -Q$, with $Q \in \mathbb{R}^{n \times n}$ a positive definite matrix.

Also, fractional adaptive laws have been used for this problem (see for instance Aguila-Camacho & Duarte-Mermoud (2016b)), resulting

$${}_0D^\alpha \theta(t) = -b_p e^T(t) P x_p(t), \quad \theta(0) = \theta_0 \qquad (10)$$

where $\alpha \in (0, 1]$.

One of the advantages that has been reported when using fractional adaptive laws (10) with respect to classic adaptive laws (9) is that a lower control energy (e.g. integral of the squared control signal) can result in the fractional case (Aguila-Camacho & Duarte-Mermoud, 2016). This is a great outcome, specially given the importance of reducing energy consumption in control systems. However, the introduction of the fractional adaptive laws (10) also can lead to a slower convergence speed of the control error, which in some applications could be not desirable.

This paper proposes alternative fractional adaptive laws with switched fractional orders, given by

$$\begin{gathered} D^{\alpha_s} \theta(t) = -b_p e^T(t) P x_p(t), \quad \theta(0) = \theta_0 \\ \alpha_s \in (0, 1) \; if \; t \in (t_i, t_i + T] \\ \alpha_s = 1 \; if \; t \in (t_i + T, t_{i+1}], \end{gathered} \qquad (11)$$

where $T > 0$ is a constant finite value and t_i is an event-triggered sequence of time with $t_1 = 0$. The rest of the elements of the sequence t_i are arbitrary, with the only restriction that $t_{i+1} - t_i > T$ and that the sequence is finite, that is, $t_i < T_f < \infty$. For instance, elements of the sequence can be given by an external condition, such as every time a step change in the reference signal occurs.

Time T is a finite number and an additional design parameter, thus its exact value will depend on the specific system under control and on the designer's choice. Since this is a preliminary study, still there is not any results or recommendations regarding how to choose T, but it is considered in the foregoing research.

The main goal of using switched adaptive law (11) is getting benefits from both orders; fractional order in transients to spent less control energy and integer order in steady state for good convergence of the control error.

In order to avoid discontinuity issues, integer order adaptive law ($\alpha_s = 1$) will be activated only when $t \in (t_i + T, t_{i+1}]$, using as initial conditions $\theta(t_i + T)$, that is, the last value obtained with the fractional adaptive laws. In the same way, fractional adaptive laws will be activated only when $t \in (t_i, t_i + T]$, using the last value of θ obtained with the integer order adaptive law as initial value, and also neglecting the past values of θ (resetting fractional order adaptive law).

Note that adaptive law (11) has been proposed such that, since $t_1 = 0$, switching will start with the fractional order, thus necessarily fractional order will be active only during bounded constant time intervals of length T, and without taking into account the past history of the signal. These characteristics will be used in the proof of boundedness in section 3.3.

Summarizing, the control strategy proposed in this paper to control plant (5) is given by (6),(8),(11).

3.2 Analytical description of the controlled system

Since the control error $e(t) = x_p(t) - x_m(t)$, substracting (6) from (5) and using (8), it can be obtained that

$$\dot{e}(t) = A_m e(t) + b_p(t)\left[\theta^T(t) - \theta^{*^T}\right] x_p(t). \tag{12}$$

Let us define the parameter error are as the difference between the controller estimated parameters and the real unknown controller parameters, that is

$$\phi(t) = \theta(t) - \theta^*. \tag{13}$$

Using (13) in (12) it is obtained that

$$\dot{e}(t) = A_m e(t) + b_p \phi^T(t) x_p(t), \quad e(0) = e_0. \tag{14}$$

Equation (14) describes the evolution of the control error in time. Also, according to (13), it holds that $D^{\alpha_s}\phi = D^{\alpha_s}\theta$, thus, the parameter errors evolution in time is described by the differential equation

$$\begin{gathered} D^{\alpha_s}\phi(t) = -b_p e^T P x_p(t), \quad \phi(0) = \phi_0 \\ \alpha_s \in (0,1) \ if \ t \in (t_i, t_i + T] \\ \alpha_s = 1 \ if \ t \in (t_i + T, t_{i+1}] \end{gathered} \tag{15}$$

Equations (14),(15), completely describe the controlled system.

3.3 Boundedness of the signals and convergence of the control error in the control scheme

As it was mentioned in previous section, the adaptive law will be reset every time the order switchs from integer order to fractional order. Thus, proof of boundedness needs to be made only for the first two intervals $(0, T]$ and $(T, t_2]$, and it will be valid also for the rest of the intervals $(t_i, t_{i+1}]$, $i = 2, \ldots$. Note that the intervals where fractional order is active are always bounded and of length T, while intervals where integer order is active can be of infinite length (if no more switching commands are received).

Thus, proof of boundedness will be divided in two parts. First, boundedness of the signals will be proved for the interval $t \in [0, T]$ and after that, the proof for the interval $t \in (T, t_2)$ will be provided.

Proof. If we use the following notation

$$\begin{aligned} f_1(e, \phi) &= A_m e + b_p \phi^T x_p \\ f_2(e, \phi) &= -b_p e^T P x_p \end{aligned} \tag{16}$$

then when $t \in [0, T]$, the controlled system is described by the equations

$$\dot{e}(t) = f_1(e, \phi), \ e(0) = e_0 \tag{17}$$

$$\begin{aligned} {}_0D^{\alpha_s}\phi(t) &= f_2(e, \phi), \ \alpha_s \in (0,1) \\ &\phi(0) = \phi_0 \end{aligned} \tag{18}$$

Let us apply the integer order integral to Eq. (17) and the I^{α_s} integral to Eq. (18). Using properties of fundamental calculus and properties of the Caputo derivative (Kilbas et al., 2006) we obtain

$$\begin{aligned} e(t) &= e_0 + \int_0^t f_1(e, \phi) \\ \phi(t) &= \phi_0 + I^{\alpha_s} f_2(e, \phi) \end{aligned} \tag{19}$$

Note that f_1, f_2 are $\mathcal{C}$ functions $\forall t \in [0, T]$ and Lipschitz continuous functions on e, ϕ. Also, initial values e_0, ϕ_0 are constant bounded values. Thus, Eqs. in (19) are in the form of Theorem 1 with p_1 and p_2 constant values, and we can conclude from Theorem 1 that e, ϕ are continuous in $[0, T]$. Since e, ϕ are continuous in a closed interval $[0, T]$, then it follows that e, ϕ remain bounded in $[0, T]$.

Les us now see what happens in $t \in (T, t_2]$. In this interval (not necessarily of finite length if it is the last switch), controlled system (14),(15) corresponds to classic error model 2 (Narendra & Annaswamy, 2005), since $\alpha_s = 1 \ \forall t \in (T, t_2]$. Initial conditions for this classic adaptive system have been proved to be bounded above, thus, boundedness of $e(t), \phi(t)$ in $(T, t_2]$ follows from Narendra & Annaswamy (2005).

Since the sequence t_i is finite, thus using a recursive argument, the above analysis will be also valid for the rest of the switching intervals. Thus, we can conclude that e, ϕ remain bounded $\forall t$. Since the reference signal r is assumed to be bounded and the reference model (6) is asymptotically stable, thus x_m is also bounded. Thus, according to the definition $e = x_p - x_m$ and the fact that e is bounded, it can be concluded that x_p also remains bounded. Regarding the controller estimated parameters, using (13) and the fact that ϕ is bounded, then it is concluded that θ remains bounded, and consequently u remains bounded as well. Thus, all the signals in the scheme remain bounded for all t.

Regarding the convergence to zero of the control error, once the last switch is made (and consequently last order is $\alpha_s = 1$), then it is proved in Narendra & Annaswamy (2005) that

$$\lim_{t\to\infty} e(t) = 0.$$

Remark 2. Although the convergence of the error can be assured only when no more switches take place, if the time between switches is large, then the time using $\alpha_s = 1$ will also be large and in practice we can expect a good convergence of the control error in every interval as well.

Remark 3. Foregoing research includes more complex aspects such as: infinite time sequence t_i, different values of α_s for every element of the adaptive law, recommendations for choosing switching interval T, non-constant switching time T, among others.

4. SIMULATIONS RESULTS

This section presents simulation results for a second order system using switched adaptive laws (11) proposed in this paper, in order to show the advantages given by the additional degree of freedom in the design (switched fractional order in time $t_i + T$). Also, it aims to present potential advantages of their use compared to classic integer order adaptive laws (9) and non-switched fractional order adaptive laws (10).

System to be controlled is given by (5), with known $b_p = [1 \quad 1]^T$ and

$$A_p = \begin{bmatrix} 4 & -1 \\ 5 & -3 \end{bmatrix}$$

the unknown plant matrix. Note that this is an unstable plant. Initial values for the plant states are used as $x_{p_0} = [0 \quad 1]^T$.

Model reference is given by (6), with known

$$A_m = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad b_m = [1 \quad 1]^T.$$

Initial values for the model reference states are used as $x_{m_0} = [1 \quad 5]^T$.

Note that selected values of A_m, b_m, A_p, b_p guarantee that $k^* = 1$ and $\theta^* = [-5 \quad 1]^T$. Thus, control signal used corresponds to (8), and adaptive law for estimating controller parameter θ is used as (11), with initial value for the estimated parameter in the adaptive law (11) as $\theta_0 = [0 \quad 0]^T$. Also, matrix P in the adaptive law is chosen as $P = [1\,0; 0\,1]$, which implies that $Q = [2\,0; 0\,4]$.

Since this is a preliminary research study, still there is no clarity about the best way to select parameter T and order α_s for those time intervals where fractional order is active. Thus, we decided exploring different values of $T \in (0, 5]$. Since the main goal of the control scheme is the plant states following the reference model states, this selection somehow tries to use fractional order during a relevant part of the transient, and then switching to integer order adaptive law to guarantee a fast convergence. Regarding the fractional order, we will explore some different values of $\alpha_s \in (0, 1)$.

For comparison purposes, classic integer order adaptive laws (9) and non-switched fractional adaptive laws (10) with same α are used as well, starting from the same initial value θ_0 specified above.

In order to evaluate the system behavior, the integral of the time weighted squared control error norm will be used, that is

$$ITSE = \int_0^{T_f} t\|e(t)\|^2 dt, \tag{20}$$

where T_f is the final simulation time. $ITSE$ is a very useful functional when bringing attention to small errors that remain in time, thus here it will help us to evaluate the convergence speed of the control error in the schemes. $ITSE$ also gives small weight to initial errors, thus the analysis of the transient stage in the schemes will be carried out using not only $ITSE$ but also observation from figures.

In order to quantify the control energy used, the integral of the squared control signal will be used, that is

$$ISI = \int_0^{T_f} u^2(t)\, dt. \tag{21}$$

The reference signal used corresponds to a step reference of magnitude 3 in $t = 0$ and later in $t = 15$ seconds another step from magnitude 3 to 2 occurs, all along a simulation time window of $T_f = 40$ seconds. Thus, the time sequence t_i in (10) will be used as $t_1 = 0$, $t_2 = 15$. This implies that every time a step change occurs in the reference, the fractional order adaptive law will be used during a time window of 1 second (according to the selected T), while the integer order adaptive law will be active the rest of the time.

First, we will explore the influence of switching time T in the scheme. To that extent, different values of $T \in (0, 5]$ will be used, for the same fractional order $\alpha = 0.7$. Fig. 1 shows the evolution of the control error for this case, while Fig. 2 shows the corresponding control signal $u(t)$. The first 6 seconds of the simulation have been zoomed in the lower plots, in order to check more clearly what happens in the transient.

As can be seen from Fig. 1, switched adaptive law guarantees that the control error converges to zero in a similar time to that of the classic adaptive law, while non-switching fractional adaptive law has a lower convergence speed. On the other hand, from Fig. 2 it can be seen that initial values of the control signal are pretty similar for all the adaptive laws, but fractional order non-switched adaptive law (cyan) and switched fractional order with $T \geq 2$ give smoother control signals. Thus, somehow the switched fractional adaptive laws proposed in this paper achieve both, a fast convergence to zero of the control error and a smooth control signal. We most note that there are some discontinuity in the control signal when switching occurs. This is an implementation issue, since adaptive laws are being handled in such a way that no discontinuity

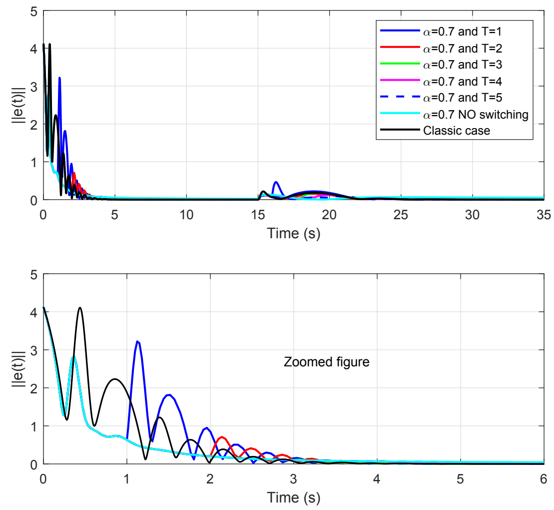

Fig. 1. Evolution of the norm of the control error $\|e(t)\|$ for step references and different values of switching time T.

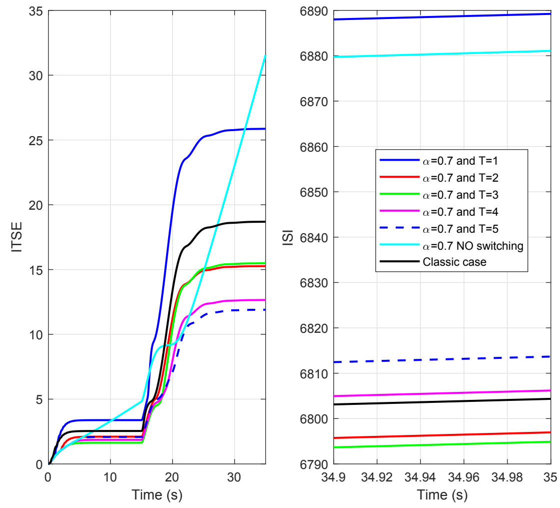

Fig. 3. Evolution of the functionals ITSE (20) and ISI (21) for step references and different values of switching time T.

occurs in θ. This implementation issue is being analyzed in order to improve the scheme behavior in future works.

Let us see the plots for $ITSE$ and ISE, which give a more precise idea about the behavior of the controlled system. Fig. 3 shows both indexes for this simulation scenario.

Let us focus our attention first in the evolution of $ITSE$ and ISI for the classic adaptive laws (black line) and the non-switched fractional adaptive laws (cyan line). These are the adaptive laws that can be founded with analytical basis in technical literature up to date. As can be seen, $ITSE$ is lower for the non-switched fractional adaptive laws at the beginning of the simulations, compared to the classic case, but after some time both functionals became worst for the non-switching fractional adaptive laws than for the classic case.

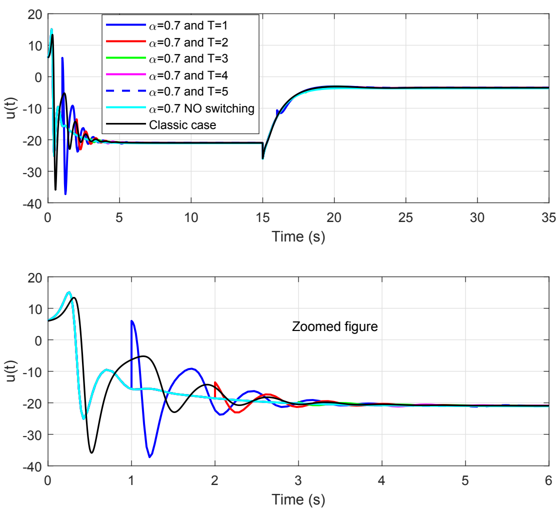

Fig. 2. Evolution of the control signal $u(t)$ for step references and different values of switching time T.

However, switched fractional order adaptive laws proposed in this paper (blue line) using $T = 2$ (red line) sec and $T = 3$ sec (green line) have lower values for both, $ITSE$ and ISI, compared to the classic case. This means that using the switching rule, choosing the correct value for the switching time T, we can keep system behavior as desired (or even improve it) and at the same time spent a lower amount of control energy to achieve it.

The question now is, what happens with the fractional order α? Up to here it has been used as $\alpha = 0.7$ but, how is it correlated to the switching time T and the functionals $ITSE$ and ISE? In order to have some idea about the influence of these two parameters (T, α_s) in the obtained results, let us do some additional experiments.

As a simple approach to address this question, several simulation studies were carried out using different combinations of switching times T and fractional orders α_s. Results are summarized in Fig. 4 and Fig. 5, where last values of $ITSE$ and ISI have been respectively plotted, for different combinations of T, α_s. Numerical results used to construct plots in Fig. 4 and Fig. 5 are also included in Tables 1 and 2, together with the corresponding results for the classic case and the non-switched case, which are not included in Fig. 4 and Fig. 5.

It can be seen from Fig. 4 that improvement in the behavior of the control error ($ITSE$) is not linearly related to switching time T nor to the fractional order. The pair of parameters which offer the minimum $ITSE$ (black dot point) corresponds to $T = 5$, $\alpha_s = 0.5$, $\forall t \in [0, T]$. Still, it can be seen from Table 1 that case with lowest $ITSE$ is a switched case, nor the classic or non-switched fractional cases.

In the case of the control energy, dependence is not linear neither. It can be seen from Fig. 5 that values of $1 < T \leq 4$ are those with lower ISI, being the case $T = 2$, $\alpha_s = 0.5$, $\forall t \in [0, T]$ the one with lowest ISE (black dot point). Again, it can be seen from Table 2 that case with lowest

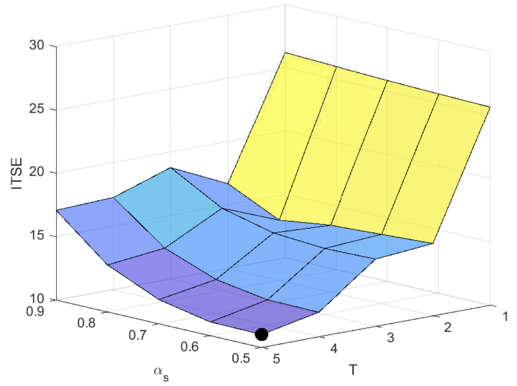

Fig. 4. Evolution of the functional ITSE (20) for step references, using different values for α_s when $t \in [0, T]$ and switching time T.

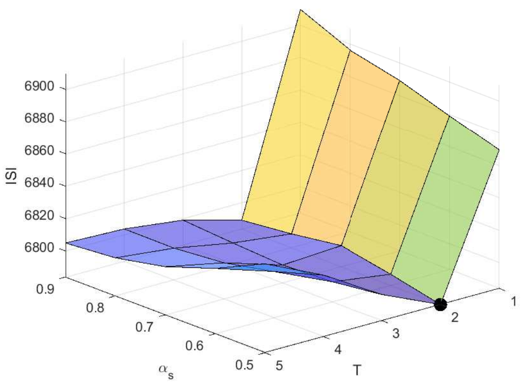

Fig. 5. Evolution of the functional ISI (21) for step references, using different values for α_s when $t \in [0, T]$ and switching time T.

ISI is a switched case, nor the classic or non-switched fractional cases.

Summarizing, even when it is not straightforward to find the best pair of T, α_s for the design of the switched fractional order adaptive laws proposed in this paper, it has been clearly stated that switched fractional order adaptive laws (when carefully designed) allow obtaining better system performance and management of the control energy, compared to the classic adaptive laws and non-switched fractional order adaptive laws.

Remark 4. More exhaustive simulation studies using also $\alpha < 0.5$ are being conducted in order to state more general conclusions. Also, research is being conducted about possible correlation between switching time and other design parameters such as α, plant and reference model parameters.

Table 1. Numerical results for functional ITSE, used to construct plot in Fig. 4

	0.5	0.6	0.7	0.8	0.9	1
T=1	25.62	25.73	25.86	26.03	26.21	X
T=2	15.67	15.47	15.26	14.73	16.68	X
T=3	15.28	15.19	15.48	16.50	18.78	X
T=4	11.93	11.97	12.65	14.19	17.26	X
T=5	**11.01**	11.06	11.89	13.70	17.08	X
Non-switched	97.15	58.80	31.55	19.17	15.39	X
Classic	X	X	X	X	X	18.69

Table 2. Numerical results for functional ISI, used to construct plot in Fig. 5

	0.5	0.6	0.7	0.8	0.9	1
T=1	6869.6	6879.0	6889.2	6896.3	6909.6	X
T=2	**6784.0**	6791.0	6797.0	6792.5	6789.2	X
T=3	6800.1	6796.2	6794.9	6796.8	6799.2	X
T=4	6822.0	6812.5	6806.2	6803.5	6803.7	X
T=5	6838.9	6824.3	6813.7	6807.5	6804.9	X
Non-switched	7028.3	6941.8	6881.1	6841.8	6819.1	X
Classic	X	X	X	X	X	6804.3

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