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Passive observers for distributed port-Hamiltonian systems ⋆

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Abstract: The observer design for 1D boundary controlled infinite-dimensional systems is addressed using the port-Hamiltonian approach. The observer is defined by the same partial differential equations as the original system and the boundary conditions depend on the available information from sensors and actuators. The convergence of the observers is proved to be asymptotically or exponentially under some conditions. The vibrating string and the Timoshenko beam are used to illustrate the observer convergence in different scenarios.

Keywords: Observer design, port-Hamiltonian distributed parameter systems.

1. INTRODUCTION

An observer is a dynamical system which takes all the information available from sensors and actuators and uses the model to reconstruct the state of the system of interest (Luenberger, 1964). For systems governed by Ordinary Differential Equations (ODEs) (or Lumped Parameter Systems (LPSs)), the observer problem has been intensively investigated especially for linear systems. Nevertheless, for systems governed by Partial Differential Equations (PDEs) (or Distributed Parameter Systems (DPSs)), the observer design is more complex and it has drawn the attention of the researchers in recent years (Hidayat et al., 2011).

A natural observer is presented for LPSs and DPSs in (Demetriou, 2004), where for mechanical systems, the estimated velocity is the time derivative of the estimated position for all time. In (Deguenon et al., 2006) is introduced a simple observer for elastic systems, while in (Smyshlyaev and Krstic, 2005; Meurer, 2013) are given observers for a class of parabolic systems, where the backstepping strategy is used for the design. On the other hand, in (Castillo et al., 2013) it is presented an observer for hyperbolic systems with dynamic controllers for flow control applications. In this work, we try to group all these observer in the so-called class distributed boundary port-Hamiltonian (pH) systems.

In the last years, the pH approach has proven to be well-suited for modeling and control of DPSs, and more specifically for Boundary Control Systems (BCSs) (Fattorini, 1968). The well-posedness was developed using semigroup theory in (Le Gorrec et al., 2005), while the stability and control analysis was well developed in (Villegas, 2007; Augner and Jacob, 2014; Ramírez et al., 2014; Villegas et al., 2009; Macchelli, 2013). Nevertheless, pH observers have still not been developed for BCSs. In this note, taking the results already mentioned about asymptotic and exponential stability, different observers are presented, depending on the available measurement. Several controllers for BCSs (Guo and Xu, 2007; Guo and Guo, 2009; Krstic et al., 2008) have been developed using infinite-dimensional observers (Deguenon et al., 2006; Demetriou, 2004). The main idea of this work is to cast all these observers into one general class under the pH framework.

The paper is organized as follow: a brief background on distributed pH systems is presented in Section 2, then in Section 3 is introduced the main problem of this paper. Section 4 shows the convergence conditions for the observer when the sensors are co-energy variables. Section 5 shows a more complex observer, where the sensors are not co-energy variables anymore. Section 6 shows the Timoshenko beam example, while along the paper the string equation is used to exemplify the observer design. Finally, Section 7 shows some conclusions of this work.

In this paper, $M_n(\mathbb{R})$ denotes the space of $n \times n$ square matrices whose entries lie in the space $\mathbb{R}$ and I denotes the identity matrix of appropriate dimension. By $\langle \cdot, \cdot \rangle_{L_2}$ or only $\langle \cdot, \cdot \rangle$ we denote the standard inner product on $L_2([a,b];\mathbb{R}^n)$ and the Sobolev space of order p is denoted by $H^p([a,b],\mathbb{R}^n)$.

2. DISTRIBUTED PORT-HAMILTONIAN SYSTEMS

The class of BCSs that we consider in this paper has the form

$$\mathcal{P} \begin{cases} \dfrac{\partial x}{\partial t}(\zeta,t) = P_1 \dfrac{\partial}{\partial \zeta}(\mathcal{H}x(\zeta,t)) + P_0(\mathcal{H}x(\zeta,t)), \\ W_{\mathcal{B}}\left(\begin{smallmatrix} f_\partial(t) \\ e_\partial(t) \end{smallmatrix}\right) = u(t), \\ y(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} f_\partial(t) \\ e_\partial(t) \end{smallmatrix}\right), \;\; y_m(t) = Cy(t) \end{cases} \tag{1}$$

where $x(\zeta,t) \in \mathbb{R}^n$ is the state variable defined for all $t \geq 0$ and $\zeta \in [a,b]$ with initial condition $x(\zeta,0) = x_0(\zeta)$, $u(t) \in \mathbb{R}^n$ is the input, $y(t) \in \mathbb{R}^n$ is the output and

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$y_m(t) \in \mathbb{R}^p$ is the measured part of the output $y(t)$. $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$ is a non-singular matrix, $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $\mathcal{H}(\cdot) \in M_n(L_2([a,b]))$ is a bounded and continuously differentiable matrix-valued function satisfying for all $\zeta \in [a,b]$, $\mathcal{H}(\zeta) = \mathcal{H}^\top(\zeta)$ and $mI < \mathcal{H}(\zeta) < MI$ with $M > m > 0$ both scalars independent on ζ and $C \in \mathbb{R}^{p\times n}$ is a constant matrix of rank p with $p \leqslant n$. The state space is $X = L_2([a,b];\mathbb{R}^n)$ with inner product $\langle x_1, x_2\rangle_{\mathcal{H}} = \langle x_1, \mathcal{H}x_2\rangle$ and norm $\|x\|^2_{\mathcal{H}} = \langle x, x\rangle_{\mathcal{H}}$, which is related with the Hamiltonian as $H(t) = \frac{1}{2}\|x\|^2_{\mathcal{H}}$. Since X is a Hilbert space and the norm $\|\cdot\|^2_{\mathcal{H}}$ is proportional to the stored energy of the system, hence $x(\zeta,t)$ is called energy variable and $\mathcal{H}(\zeta)x(\zeta,t)$ is called co-energy variable. For simplicity, sometimes we write x and $\mathcal{H}x$ instead of $x(\zeta,t)$ and $\mathcal{H}(\zeta)x(\zeta,t)$. $f_\partial(t) \in \mathbb{R}^n$ and $e_\partial(t) \in \mathbb{R}^n$ are the so called boundary port variables defined by

Definition 1. Let $\mathcal{H}x \in H^1([a,b];\mathbb{R}^n)$. Then, the boundary port variables associated with (1) are the vectors $f_\partial \in \mathbb{R}^n$ and $e_\partial \in \mathbb{R}^n$, defined by

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} \mathcal{H}x(b,t) \\ \mathcal{H}x(a,t) \end{pmatrix}.$$

Matrices $W_{\mathcal{B}}$, $W_{\mathcal{C}} \in \mathbb{R}^{n\times 2n}$ are obtained in the following theorem which ensures the existence and uniqueness of solutions of the PDE in (1).

Theorem 2. (Le Gorrec et al., 2005) Let $W_{\mathcal{B}}$ be a $n \times 2n$ real matrix. With this $W_{\mathcal{B}}$, we define the input maping $\mathcal{B} : H^1([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ and the input

$$u(t) = W_{\mathcal{B}} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} := \mathcal{B}x(t). \tag{2}$$

If $W_{\mathcal{B}}$ has full rank and satisfies $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^\top \geq 0$, with $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then the system (1) with input (2) is a BCS on X. Furthermore, the operator $\mathcal{A}x = P_1 \frac{\partial}{\partial\zeta}(\mathcal{H}x) + P_0\mathcal{H}x$ with domain

$$D(\mathcal{A}) = \left\{ \mathcal{H}x \in H^1([a,b];\mathbb{R}^n) \middle| \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in ker\, W_{\mathcal{B}} \right\}$$
$$= \left\{ \mathcal{H}x \in H^1([a,b];\mathbb{R}^n) \middle| \mathcal{B}x(t) = 0 \right\}$$

generates a contraction semigroup on X. Moreover, let $W_{\mathcal{C}}$ be a full rank $n \times 2n$ matrix such that $[\, W_{\mathcal{B}}^\top \; W_{\mathcal{C}}^\top \,]^\top$ is invertible and let P be given by

$$P = \begin{pmatrix} W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^\top & W_{\mathcal{B}}\Sigma W_{\mathcal{C}}{}^\top \\ W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^\top & W_{\mathcal{C}}\Sigma W_{\mathcal{C}}{}^\top \end{pmatrix}^{-1}.$$

Define the output of the system as the linear mapping $\mathcal{C} : H^1([a,b];\mathbb{R}) \longrightarrow \mathbb{R}^n$

$$y(t) = W_{\mathcal{C}} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} := \mathcal{C}x(\zeta,t)$$

Then, for $u \in C^2([0,\infty);\mathbb{R}^n)$, $\mathcal{H}x(\zeta,0) \in H^1([a,b];\mathbb{R}^n)$ and $u(0) = W_{\mathcal{B}} \begin{pmatrix} f_\partial(0) \\ e_\partial(0) \end{pmatrix}$ the following energy balance equation is satisfied: $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^\top P \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}$.

Definition 3. The class of systems (1) is called impedance energy preserving if $\dot{H}(t) = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_{\mathcal{H}} = u(t)^\top y(t)$.

System (1) is impedance energy preserving when $W_{\mathcal{B}}$ and $W_{\mathcal{C}}$ satisfy $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^\top = W_{\mathcal{C}}\Sigma W_{\mathcal{C}}^\top = 0$ and $W_{\mathcal{B}}\Sigma W_{\mathcal{C}}^\top = I$.

Example 4. The one-dimensional (1D) string, clamped at one side and controlled with a force actuator at the other side can be written in the pH representation (1) with matrices

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P_0 = 0, \mathcal{H}(\zeta) = \begin{bmatrix} E(\zeta) & 0 \\ 0 & \rho(\zeta)^{-1} \end{bmatrix},$$

where $E(\zeta)$ and $\rho(\zeta)$ are the Young's modulus and the mass density, respectively. The state variables are

$$x(\zeta,t) = \begin{bmatrix} q(\zeta,t) \\ p(\zeta,t) \end{bmatrix} := \begin{bmatrix} \frac{\partial w}{\partial\zeta}(\zeta,t) \\ \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) \end{bmatrix},$$

where $w(\zeta,t)$ is the deformation of the string defined for $\zeta \in [a,b]$ and $t \geq 0$. The boundary port variables are

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\rho(b)}p(b,t) - \frac{1}{\rho(a)}p(a,t) \\ E(b)q(b,t) - E(a)q(a,t) \\ E(b)q(b,t) + E(a)q(a,t) \\ \frac{1}{\rho(b)}p(b,t) + \frac{1}{\rho(a)}p(a,t) \end{pmatrix}.$$

The input and output matrices can be chosen as

$$W_{\mathcal{B}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, W_{\mathcal{C}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

which gives input and output variables

$$u(t) = \begin{bmatrix} \frac{1}{\rho(a)}p(a,t) \\ T(b)q(b,t) \end{bmatrix}, y(t) = \begin{bmatrix} -T(a)q(a,t) \\ \frac{1}{\rho(b)}p(b,t) \end{bmatrix}.$$

The Hamiltonian energy of the system is $H(t) = \frac{1}{2}\|x(\zeta,t)\|_{\mathcal{H}} = \frac{1}{2}\int_a^b x(\zeta,t)^\top \mathcal{H}x(\zeta,t)$ and the system is impedance energy preserving since $\dot{H}(t) = u(t)^\top y(t)$.

3. PROBLEM STATEMENT

The following system (3) is the observer to be designed such that its states converge to the states of the plant (1)

$$\hat{\mathcal{P}} \begin{cases} \dfrac{\partial \hat{x}}{\partial t}(\zeta,t) = P_1 \dfrac{\partial}{\partial\zeta}(\mathcal{H}\hat{x}(\zeta,t)) + P_0(\mathcal{H}\hat{x}(\zeta,t)), \\ W_{\mathcal{B}}\left(\begin{smallmatrix} \hat{f}_\partial(t) \\ \hat{e}_\partial(t) \end{smallmatrix}\right) = \hat{u}(t), \\ \hat{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} \hat{f}_\partial(t) \\ \hat{e}_\partial(t) \end{smallmatrix}\right), \;\; \hat{y}_m(t) = C\hat{y}(t) \end{cases} \tag{3}$$

with P_1, P_0, $\mathcal{H}$, $W_{\mathcal{B}}$, $W_{\mathcal{C}}$ and C defined in (1). $\hat{x}(\zeta,t)$ is the estimate of the real state $x(\zeta,t)$ in (1) and $\left(\begin{smallmatrix} \hat{f}_\partial(t) \\ \hat{e}_\partial(t) \end{smallmatrix}\right)$ is the observer boundary port variables defined in the same way as system (1). Since the system $\hat{\mathcal{P}}$ in (3) is virtual, its input $\hat{u}(t)$ can be designed with all the available information, i.e. $\hat{u}(t) = f(u(t), y_m(t), \hat{x}(\zeta,t))$, where $u(t)$ and $y_m(t)$ come from (1) and $f(\cdot)$ is a function to define. Defining the error between the plant (1) and the observer (3) as $\tilde{x}(\zeta,t) := x(\zeta,t) - \hat{x}(\zeta,t)$, then the error system is given by

$$\tilde{\mathcal{P}} \begin{cases} \dfrac{\partial \tilde{x}}{\partial t}(\zeta,t) = P_1 \dfrac{\partial}{\partial\zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_{\mathcal{B}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right) = \tilde{u}(t), \\ \tilde{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix}\right), \;\; \tilde{y}_m(t) = C\tilde{y}(t) \end{cases} \tag{4}$$

where $\tilde{u}(t) = u(t) - \hat{u}(t)$. The aim is to design $\hat{u}(t)$ such that $\tilde{x}(\zeta,t) \to 0$ when $t \to \infty$. The error system is a boundary control pH system with input $\tilde{u}(t)$ defined under Theorem 2 and output $\tilde{y}(t)$ defined such that the system is impedance energy preserving (Definition 3) with energy

$\tilde{H}(t) = \frac{1}{2}\int_a^b \tilde{x}(\zeta,t)^\top \mathcal{H}\tilde{x}(\zeta,t)$ and $\dot{\tilde{H}} = \frac{1}{2}\frac{d}{dt}\|\tilde{x}(t)\|^2_{\mathcal{H}} = \tilde{u}(t)^\top \tilde{y}(t)$.

Remark 5. Note that $\tilde{H}(t) \neq H(t) - \hat{H}(t)$. Where $H(t)$ is the Hamiltonian of the plant and $\hat{H}(t)$ is the estimated Hamiltonian.

4. PASSIVE OBSERVER DESIGN

Choosing the observer input $\hat{u}(t)$ as in the classical Luenberger observer formulation, i.e. the input of the plant $u(t)$ plus an error injection from the measurement, we obtain $\hat{u}(t) = u(t) + L(y_m(t) - \hat{y}_m(t)) = u(t) + LC\tilde{y}(t)$ then,

$$\tilde{u}(t) = -LC\tilde{y}(t) \tag{5}$$

where $L \in \mathbb{R}^{n\times p}$ is a matrix to design. Note that this is exactly the classical damping injection ensuring that the error system (4) converges to zero asymptotically. Now, the error system is described by

$$\tilde{\mathcal{P}} \begin{cases} \dfrac{\partial \tilde{x}}{\partial t}(\zeta,t) = P_1 \dfrac{\partial}{\partial \zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_L\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t)\end{smallmatrix}\right) = 0, \\ \tilde{y}(t) = W_{\mathcal{C}}\left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t)\end{smallmatrix}\right), \end{cases} \tag{6}$$

where

$$W_L = W_{\mathcal{B}} + LCW_{\mathcal{C}}. \tag{7}$$

It is possible to prove that $W_L \Sigma W_L^\top = LC + (LC)^\top$, which implies, by Theorem 2, that the error system (4) with $\tilde{u}(t) = L\tilde{y}_m(t)$ is a boundary control system since LC is positive semi-definite, i.e. $LC \geq 0$. See (Villegas, 2007).

In the following we present two different scenarios: the first one is the ideal case, corresponding to full sensing of (1), i.e. $p = n$ and $C = I$ which implies $y_m(t) = y(t)$; the second one when not all the output $y(t)$ is available, i.e. $p < n$ and $y_m(t) = Cy(t)$ with $C \in \mathbb{R}^{p\times n}$. In both scenarios, we give the conditions such that the asymptotic or exponential convergence of the observer is ensured.

4.1 Full sensing: $p = n$

In this case, L is a square matrix of size n and C is the identity. The following theorem ensures the asymptotic convergence of the error system.

Theorem 6. Consider the BCS (6) from (4) and (5). The energy $\tilde{H}(t)$ is such that for all $t \geqslant 0$ it satisfies $\frac{1}{2}\frac{d}{dt}\|\tilde{x}(t)\|_{\mathcal{H}} = -\langle L\tilde{y}(t), \tilde{y}(t)\rangle$ and $L \in \mathbb{R}^{n\times n}$. If the matrix W_L defined in (7) satisfies $W_L\Sigma W_L^\top > 0$, or equivalently $L > 0$, the system converge to zero asymptotically.

Proof. The proof is a direct application of Theorem 5.1 of (Villegas, 2007, chapter 5) ∎

Example 7. Following Example 4, now we design the observer for the string using Theorem 6. Note that in this scenario we have full sensing. Consider $L = diag([l_1, l_2])$, which gives

$$W_L = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & l_1 & -l_1 & 1 \\ l_2 & 1 & 1 & l_2 \end{bmatrix}, \quad W_L\Sigma W_L^\top = \begin{bmatrix} 2l_1 & 0 \\ 0 & 2l_2 \end{bmatrix}.$$

Then, the error system converges asymptotically to zero for all $l_1 > 0$ and $l_2 > 0$, i.e., $\hat{x}(\zeta,t) \to x(\zeta,t)$ as $t \to \infty$.

4.2 Partial sensing: $p < n$

In this case, the matrix L is not anymore a square matrix and we can not apply anymore Theorem 6. Yet the follow Corollary 8 from (Villegas, 2007) can be rewritten in order to prove the convergence of the error system (6).

Corollary 8. (Villegas, 2007). Consider the BCS in (6) as described in Theorem 2 and assume that the energy of the system is such that for all $t \geq 0$ satisfy

$$\frac{1}{2}\frac{d}{dt}\|\tilde{x}(t)\|^2_{\mathcal{H}} = -\langle LC\tilde{y}(t), \tilde{y}(t)\rangle$$

where LC is a positive semi-definite matrix, i.e. $LC \geq 0$. If either

$$\begin{aligned} \|\mathcal{H}\tilde{x}(b,t)\|^2_{\mathbb{R}} &\leq k_1\langle LC\tilde{y}(t), \tilde{y}(t)\rangle \quad or \\ \|\mathcal{H}\tilde{x}(a,t)\|^2_{\mathbb{R}} &\leq k_1\langle LC\tilde{y}(t), \tilde{y}(t)\rangle \end{aligned} \tag{8}$$

for some positive constant k_1, then the system is exponentially stable.

Proof. The proof is a direct application of Corollary 5.19 of (Villegas, 2007, chapter 5) ∎

Note that the result of the Corollary 8 is stronger than the one of Theorem 6 in terms of convergence even if we use more restrictive conditions/assumptions for the sensing (partial sensing). It is mainly due to the conservatism of the result state in Theorem 6. The conditions (8) are less conservative but may be more difficult to check.

Example 9. Following Examples 4 and 7, but now considering that we can only measure the strain at the left side of the string, i.e. $y_m = -T(a)q(a,t) \Rightarrow C = [\,1\ 0\,]$ and $L = [\,l_1\ l_2\,]^\top$. We obtain the

$$W_L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & l_1 & -l_1 & 1 \\ 0 & l_2+1 & 1-l_2 & 0 \end{bmatrix}, \quad W_L\Sigma W_L^\top = \begin{bmatrix} 2l_1 & l_2 \\ l_2 & 0 \end{bmatrix}.$$

where $W_L\Sigma W_L^\top \geq 0$ is equivalent to have $l_2 = 0$ and $l_1 \geq 0$ which is a sufficient condition to prove existence of solutions of the PDE. Now, we need to prove one of the conditions in (8). Considering $l_2 = 0$, we obtain $\langle LC\tilde{y}, \tilde{y}\rangle = l_1(T(a)\tilde{q}(a,t))^2$, $\|\mathcal{H}(a)\tilde{x}(a,t)\|^2 = (T(a)\tilde{q}(a,t))^2 + \left(\frac{1}{\rho(a)}\tilde{p}(a,t)\right)^2 = (l_1^2+1)(T(a)\tilde{q}(a,t))^2$ in where, choosing $k_1 > l_1^2 + 1$, the conditions (8) $\|\mathcal{H}\tilde{x}(a,t)\|^2_{\mathbb{R}} \leq k_1\langle LC\tilde{y}(t), \tilde{y}(t)\rangle$ is satisfied.

5. PASSIVE OBSERVERS WITH DYNAMIC EXTENSION

We consider now the case where the sensors do not measure co-energy variables as in (1). This is the case for example of mechanical systems like waves or beams, where we can not measure the strain or velocity at the boundaries, but only displacement (laser sensor for example). In order to estimate the state of the system one has to add an integrator at the boundaries. Then, the global system is a mix between PDEs and ODEs which are interconnected through the spatial boundaries of the PDEs, where the ODEs are added to the observer to ensure the convergence of the observer. So, consider now that $\tilde{u}(t)$ from system (4) can be designed from a set of ODEs as Fig. 1 shows, where the block $\mathcal{C}$ is given by

$$\mathcal{C}\begin{cases} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) \end{cases} \tag{9}$$

where $x_c \in \mathbb{R}^{n_c}$ and $u_c,\ y_c \in \mathbb{R}^n$. $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times n}$, $C_c \in \mathbb{R}^{n\times n_c}$ and $D_c \in \mathbb{R}^{n\times n}$ such that $A_c = (J_c - R_c)Q_c$, $B_c = G_c - P_c$, $C_c = (G_c + P_c)^\top Q_c$ and $D_c = M_c + S_c$, where $J_c = -J_c^\top$, $R_c = R_c^\top$, $M_c = -M_c^\top$ and $S_c = S_c^\top$, with these further condition satisfied:

$$\begin{pmatrix} R_c & P_c \\ P_c^\top & S_c \end{pmatrix} \geq 0 \quad and \quad Q_c = Q_c^\top > 0.$$

The error system (4) is interconnected with the finite-

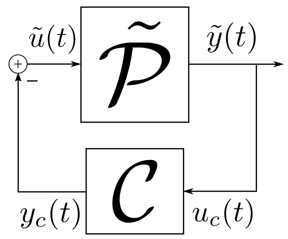

Fig. 1. Passive observer with dynamic extension.

dimensional system (9) as in Fig. 1 that can be written as

$$\begin{pmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_c(t) \\ y_c(t) \end{pmatrix}. \quad (10)$$

The closed-loop system can be characterized by the total Hamiltonian

$$\tilde{H}_{cl}(\tilde{x}(t), x_c(t)) = \frac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}} + \frac{1}{2}x_c^\top(t)Q_c x_c(t) \quad (11)$$

and it can be compactly written as

$$\begin{cases} \dot{\tilde{z}} & = \mathcal{A}_{cl}\tilde{z} \\ 0 & = (\mathcal{B} + D_c\mathcal{C}\ C_c)\,\tilde{z} := \mathcal{B}_{cl}\tilde{z} \end{cases} \quad (12)$$

where

$$\tilde{z} = \begin{pmatrix} \tilde{x} \\ x_c \end{pmatrix} \in Z := X \times \mathbb{R}^{n_c}$$

is the state variable of the new augmented system and $\mathcal{A}_{cl} : D(\mathcal{A}_{cl}) \subset Z \to Z$ is the following linear operator

$$\mathcal{A}_{cl}\begin{pmatrix} \tilde{x} \\ x_c \end{pmatrix} := \begin{pmatrix} \mathcal{A} & 0 \\ B_c\mathcal{C} & A_c \end{pmatrix}\begin{pmatrix} \tilde{x} \\ x_c \end{pmatrix} \quad (13)$$

with domain

$$D(\mathcal{A}_{cl}) = \left\{ \tilde{z} \in Z \;\middle|\; \tilde{x} \in D(\mathcal{A}),\ \mathcal{B}_{cl}\tilde{z} = 0 \right\}. \quad (14)$$

Proposition 10. Consider the infinite dimensional pH system (4) interconnected with the finite dimensional pH system (9) through the passive interconnection (10) as in Fig. 1. The augmented system (12) with $\mathcal{A}_{cl}$ defined in (13) with domain (14) is a BCS and the operator $\mathcal{A}_{cl}$ generates a contraction semigroup.

Proof. The proof is a direct application of Proposition 1 in (Macchelli, 2013). ■

In what follows we give two conditions that have to satisfy the observer in order to asymptotically or exponentially reconstruct the state of the system (1). Before that, we call these two technical Lemma and Corollary

Lemma 11. (Lefschetz-Kalman-Yakubovich) [More details in (Tao and Ioannou, 1988)]. Assume for the system (9) that (A_c, B_c) is controllable and (A_c, C_c) is observable. Then, the transfer matrix $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$ is Strictly Positive Real (SPR) if and only if there exist real matrices $P = P^\top > 0$, L, W and a scalar $\varepsilon > 0$ such that

$$PA_c + A_c^\top P = -L^\top L - \varepsilon P \quad (15a)$$

$$C_c - B_c^\top P = W^\top L \quad (15b)$$

$$D + D^\top = W^\top W \quad (15c)$$

Corollary 12. The system (9) with $A_c = (J_c - R_c)Q_c$, $C_c = B_c^\top Q_c$ and $D_c = 0$ is SPR if $J_c = -J_c^\top$, $R_c = R_c^\top > 0$ and $Q_c = Q_c^\top > 0$.

Proof. From Lemma 11, choose $P = Q_c$ and $W = 0$, then (15c) is trivial, (15b) is $C_c = B_c^\top Q_c$ and (15a) becomes $L^\top L = 2Q_cR_cQ_c - \varepsilon Q_c$, then for $R_c > 0$ there exists a constant $\varepsilon > 0$ such that the matrix $2Q_cR_cQ_c - \varepsilon Q_c$ is positive definite, giving a solution for L, using for instance Cholesky factorization. ■

The following theorem ensures the asymptotic convergence of the observer

Theorem 13. Consider the system (4) with $\tilde{u}(t)$ and $\tilde{y}(t)$ defined according to Theorem 2 such that is an impedance energy preserving system (Definition 3). Consider also a finite dimensional system $\mathcal{C}$ (9) (Fig. 1) such that its transfer matrix between y_c and u_c is SPR. Then, with the passive inteconnecion (10) the error system (4) is well-posed and converges asymptotically to zero.

Proof. The proof is a direct application of Theorems 5.8, 5.9 and 5.10 in (Villegas, 2007, chapter 5) ■

Example 14. Consider the same example as 4, but now the string is free at b, we can act at a and we can only measure the displacement at b

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix}(\zeta, t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\dfrac{\partial}{\partial \zeta}\begin{pmatrix} Tq \\ \frac{1}{\rho}p \end{pmatrix}(\zeta, t) \\ \frac{1}{\rho(a)}p(a,t) = u(t),\ T(b)q(b,t) = 0, \\ y_m(t) = w(b,t). \end{cases}$$

The following observer estimates asymptotically the state variables $q(\zeta, t)$ and $p(\zeta, t)$.

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}(\zeta, t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\dfrac{\partial}{\partial \zeta}\begin{pmatrix} T\hat{q} \\ \frac{1}{\rho}\hat{p} \end{pmatrix}(\zeta, t) \\ \frac{1}{\rho(a)}\hat{p}(a,t) = u(t), \\ T(b)\hat{q}(b,t) = l_1[w(b,t) - \hat{w}(b,t) + \theta(t)], \\ \dot{\theta}(t) = -l_2[w(b,t) - \hat{w}(b,t) + \theta(t)]. \end{cases}$$

In fact, the error system can be written as a dynamical block $\mathcal{C}$ in (9). Take as matrices $A_c = -l_2$, $B_c = 1$, $C_c = l_1$ and $D_c = 0$. Take as variables $x_c(t) = \tilde{w}(b,t) + \theta(t)$, $u_c(t) = \frac{1}{\rho(b)}\tilde{p}(b,t)$ and thus $y_c(t) = l_1 x_c(t) = l_1(\tilde{w}(b,t) + \theta(t))$. Note that, $J_c = 0$, $R_c = \frac{l_2}{l_1}$, $Q_c = l_1$, $P_c = 0$, $G_c = B_c = 1$ and $M_c = S_c = 0$. Also $\begin{pmatrix} R_c & P_c \\ P_c^\top & S_c \end{pmatrix} = \begin{pmatrix} \frac{l_2}{l_1} & 0 \\ 0 & 0 \end{pmatrix}$. Then we have existence of solution for all $l_1 > 0$ and $l_2 \geqslant 0$ (Proposition 10) and we can prove the asymptotic convergence of the observer using Theorem 13. Indeed, using Corollary 12 we know that the block $\mathcal{C}$ is SPR if $l_1 > 0$ and $l_2 > 0$. Note that the observer presented in Example 14 is exactly the same observer as the one proposed in (Guo and Guo, 2009).

The next propositions give the conditions one has to check to ensure the exponential convergence of the error system.

Proposition 15. Consider the error system (4) interconnected with a dynamic extension (9) through the interconnection (10) as Fig. 1 shows. Moreover, denote by $\tilde{H}(t) := \frac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}}$ the energy of the error system (4). If for all $T_1, T_2 > 0$ such that $0 \leq T_2 << T_1 < +\infty$ we have that

$$\int_{T_2}^{T_1} \tilde{H}(t)dt \neq 0 \quad (16)$$

then the total energy (11) of the closed-loop system satisfies for τ large enough

$$\tilde{H}_{cl} \leq c(\tau) \int_0^\tau \|\mathcal{H}(b)\tilde{x}(b,t)\| \ and$$
$$\tilde{H}_{cl} \leq c(\tau) \int_0^\tau \|\mathcal{H}(b)\tilde{x}(a,t)\|$$

where c is a positive constant that depends on τ.

Proof. The proof is a direct application of Proposition 3 in (Macchelli, 2013). ■

Remark 16. Condition (16) is not restrictive and it is natural in an scenario where we need to estimated the state, i.e. $\tilde{x}(\zeta,t) \neq 0$, $x(\zeta,t) \neq \hat{x}(\zeta,t)$ or the error energy is different from zero.

Proposition 17. Under the hypothesis of Proposition 15 if

$$\begin{aligned}\dot{\tilde{H}}_{cl} &\leq -k_1\|\mathcal{H}(b)\tilde{x}(b,t)\|^2 \quad or \\ \dot{\tilde{H}}_{cl} &\leq -k_1\|\mathcal{H}(a)\tilde{x}(a,t)\|^2\end{aligned} \tag{17}$$

for some $k_1 > 0$, then the error state $\tilde{x}(\zeta,t)$ from system (4) converges exponentially to zero.

Proof. The proof is a direct application of Proposition 4 in (Macchelli, 2013). ■

Example 18. Consider now, the same as in Example 9, where we only can measure the strain at a, i.e. $y_m(t) = T(a)q(a,t)$. The following observer estimates exponentially the state variables $q(\zeta,t)$ and $p(\zeta,t)$.

$$\begin{cases}\frac{\partial}{\partial t}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}(\zeta,t) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial\zeta}\begin{pmatrix}T\hat{q}\\ \frac{1}{\rho}\hat{p}\end{pmatrix}(\zeta,t)\\ T(a)\hat{q}(a,t) = \alpha\frac{1}{\rho(a)}\hat{p}(a,t) + \beta\hat{w}(a,t) + T(a)q(a,t),\\ T(b)\hat{q}(b,t) = u(t).\end{cases}$$

Following Proposition 10 the error system is well-posed. Take as matrices $J_c = 0$, $R_c = 0$, $Q_c = \beta$, $P_c = 0$, $G_c = 1$, $M_c = 0$ and $S_c = \alpha$. Take as variables $x_c(t) = -\tilde{w}(a,t)$, $u_c(t) = -\frac{1}{\rho(a)}\tilde{p}(a,t)$ and thus $y_c(t) = -\alpha\frac{1}{\rho(a)}\tilde{p}(a,t) - \beta\tilde{w}(a,t)$. Note that

$$\begin{pmatrix}R_c & P_c\\ P_c^\top & S_c\end{pmatrix} = \begin{pmatrix}0&0\\0&\alpha\end{pmatrix} \geq 0,\ \forall\ \alpha \geq 0.$$

Then the error system can be written as

$$\begin{cases}\frac{\partial}{\partial t}\begin{pmatrix}\tilde{q}\\ \tilde{p}\end{pmatrix}(\zeta,t) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial\zeta}\begin{pmatrix}T\tilde{q}\\ \frac{1}{\rho}\tilde{p}\end{pmatrix}(\zeta,t)\\ T(a)\tilde{q}(a,t) = \alpha\frac{1}{\rho}\tilde{p}(a,t) + \beta\tilde{w}(a,t)\\ T(b)\tilde{q}(b,t) = 0,\end{cases}$$

which is well defined for all $\alpha \geq 0$ and $\beta > 0$. To prove the convergence of the observer we use the Proposition 17. The total energy $\tilde{H}_{cl}$ is such that

$$\frac{dH_{cl}}{dt} = -\alpha\left(\frac{1}{\rho(a)}\tilde{p}(a,t)\right)^2.$$

On the other hand,

$$\begin{aligned}\|\mathcal{H}(a)\tilde{x}(a,t)\|^2 &= (T(a)\tilde{q}(a,t))^2 + (\tfrac{1}{\rho(a)}\tilde{p}(a,t))^2\\ &= (\alpha\tfrac{1}{\rho(a)}\tilde{p}(a,t) + \beta\tilde{w}(a,t))^2 + (\tfrac{1}{\rho(a)}\tilde{p}(a,t))^2\end{aligned}$$

where for β small enough we can always find a scalar $k_1 > 0$ such that $\dot{\tilde{H}}_{cl} \leq -k_1\|\mathcal{H}(a)\tilde{x}(a,t)\|^2$. Note that the observer system used in in Example 18 is the same observer as the one used in (Guo and Xu, 2007). This observer can stabilize exponentially the string through the control law $u(t) = -\alpha\frac{1}{\rho(b)}\hat{p}(b,t)$ (Guo and Xu, 2007).

6. TIMOSHENKO BEAM OBSERVER

Consider the 1D Timoshenko beam, clamped at one side and actuated at the other side with a force and torque actuators. The displacement and rotation angle are measured at the same side of the actuators. The system can be written in the pH representation (1) with

$$P_1 = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\ P_0 = \begin{bmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix},$$

$$\mathcal{H}(\zeta) = \begin{bmatrix}T(\zeta)&0&0&0\\0&\frac{1}{\rho(\zeta)}&0&0\\0&0&EI(\zeta)&0\\0&0&0&\frac{1}{I_\rho(\zeta)}\end{bmatrix},\ x = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

where $x_1(\zeta,t) = \frac{\partial w}{\partial\zeta}(\zeta,t) - \phi(\zeta,t)$, $x_2(\zeta,t) = \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)$, $x_3(\zeta,t) = \frac{\partial\phi}{\partial\zeta}(\zeta,t)$ and $x_4(\zeta,t) = I_\rho(\zeta)\frac{\partial\phi}{\partial t}(\zeta,t)$ are respectively the shear displacement, the transverse momentum distribution, the angular displacement and the angular momentum distribution. $w(\zeta,t)$ and $\phi(\zeta,t)$ are respectively the transverse displacement and the rotation angle of a filament of the beam. All variables are defined for all $t \geq 0$ and $\zeta \in [a,b]$, where the length of the beam is $b-a$. $T(\zeta)$, $\rho(\zeta)$, $EI(\zeta)$ and $I_\rho(\zeta)$ are the parameters of the plant (More details see (Macchelli and Melchiorri, 2004)). The boundary port variables (Definition 1) are

$$\begin{pmatrix}f_\partial(t)\\e_\partial(t)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\rho(b)}x_2(b,t) - \frac{1}{\rho(a)}x_2(a,t)\\ T(b)x_1(b,t) - T(a)x_1(a,t)\\ \frac{1}{I_\rho(b)}x_4(b,t) - \frac{1}{I_\rho(a)}x_4(a,t)\\ EI(b)x_3(b,t) - EI(a)x_3(a,t)\\ T(b)x_1(b,t) + T(a)x_1(a,t)\\ \frac{1}{\rho(b)}x_2(b,t) + \frac{1}{\rho(a)}x_2(a,t)\\ EI(b)x_3(b,t) + EI(a)x_3(a,t)\\ \frac{1}{I_\rho(b)}x_4(b,t) + \frac{1}{I_\rho(a)}x_4(a,t)\end{pmatrix}.$$

Choosing

$$W_{\mathcal{B}} = \begin{pmatrix}-1&0&0&0&0&1&0&0\\0&0&-1&0&0&0&0&1\\0&1&0&0&1&0&0&0\\0&0&0&1&0&0&1&0\end{pmatrix};W_{\mathcal{C}} = \begin{pmatrix}0&1&0&0&-1&0&0&0\\0&0&0&-1&0&0&-1&0\\1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&1\end{pmatrix}$$

we obtain the inputs and outputs

$$u(t) = \begin{pmatrix}\frac{1}{\rho(a)}x_2(a,t)\\ \frac{1}{I_\rho(a)}x_4(a,t)\\ Tx_1(b,t)\\ EIx_3(b,t)\end{pmatrix},\quad y(t) = \begin{pmatrix}-Tx_1(a,t)\\ -EIx_3(a,t)\\ \frac{1}{\rho(a)}x_2(b,t)\\ \frac{1}{I_\rho(a)}x_4(b,t)\end{pmatrix},$$

Taking the measurement of the transverse displacement, the rotation angle at b and the input variables, the following observer is designed

$$\begin{cases}\frac{\partial\hat{x}}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial\zeta}(\mathcal{H}\hat{x}(\zeta,t)) + P_0(\mathcal{H}\hat{x}(\zeta,t))\\ \frac{1}{\rho}\hat{x}_2(a,t) = \frac{1}{\rho(a)}x_2(a,t),\ \frac{1}{I_\rho}\hat{x}_4(a,t) = \frac{1}{I_\rho(a)}x_4(a,t),\\ T\hat{x}_1(b,t) = Tx_1(b,t) + l_1[w(b) - \hat{w}(b) + \theta_1],\\ EI\hat{x}_3(b,t) = EIx_3(b,t) + l_2[\phi(b) - \hat{\phi}(b) + \theta_2],\\ \dot{\theta}_1(t) = -l_3[w(b) - \hat{w}(b) + \theta_1]\\ \dot{\theta}_2(t) = -l_4[\phi(b) - \hat{\phi}(b) + \theta_2]\end{cases}$$

Then, by applying Theorem 13 the observer converges asymptotically to the state of the system as soon as $l_1, l_2, l_3, l_4 > 0$. In order to apply Theorem 13 we can write the error system in the form of (4) interconnected through the boundaries with the system (9) using the interconnection (10) with $J_c = 0$, $R_c = diag(l_3 l_1^{-1}, l_4 l_2{}^{-1})$, $Q_c = diag(l_1, l_2)$, $B_c = (0_{2\times 2}\ I_2)$, $C_c = B_c^\top Q_c$ and $D_c = 0$. Then, using the Corollary 12 the dynamic block is an SPR system for $l_1, l_2, l_3, l_4 > 0$. Now, using Theorem 13 the asymptotic convergence is ensured. Simulations are done for $a = 0$, $b = 1$, $T(\zeta) = \rho(\zeta) = EI(\zeta) = I_\rho(\zeta) = 1$ and $u(t) = 0$ with initial conditions $x_1(\zeta, 0) = 1$, $x_2(\zeta, 0) = 0$, $x_3(\zeta, 0) = b - \zeta$ and $x_4(\zeta, 0) = 0$, while for the observer the initial conditions are all zero. The spatial discretization method used is the one given by (Trenchant et al., 2017), where an staggered grids finite difference allows to preserve the pH structure on the finite-dimensional system. On the other hand, the midpoint method is used for the time discretization using an step time $\delta t = 0.1ms$. Fig. 2 shows the results of the simulation for an space discretization of 40 elements per state variable. Note that even for a smaller discretization (10 elements per state variable) the deformation curve of the observer (Fig. 2 (d)) converge to the real one (Fig. 2 (a)).

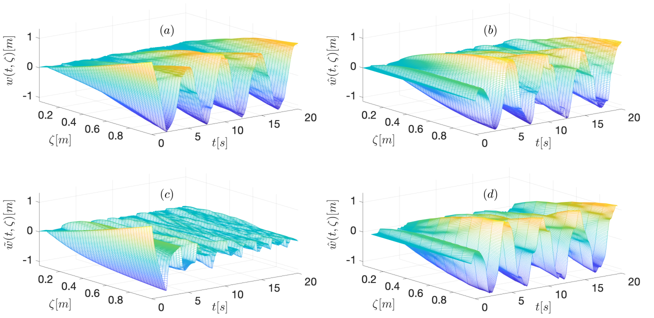

Fig. 2. (a): beam deformation, (b): observed beam deformation, (c): deformation error, (d): observed beam deformation for a low order observer.

7. CONCLUSIONS

Different observers have been addressed for infinite dimensional systems using the pH approach. Under some conditions the asymptotic or exponential convergence of the observer is ensured. The simplest case is when the sensor are co-energy variables like forces and velocities for mechanical systems. But, even in the case when this is not possible, for example when the sensors are displacement variables, the convergence of the observer is guaranteed. A perspective of this work is the implementation of these observers in together with control action like damping injection, energy shaping, among others.

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