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Reduced-order Nonlinear Observer Design for Two-time-scale Systems ⋆

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Abstract: A two-time-scale system involves both fast and slow dynamics. This paper studies the design of nonlinear observers for general nonlinear two-time-scale systems and presents a reduced-order observer design approach. The reduced-order observer is derived based on a lower dimensional model to reconstruct the slow states, along with the algebraic slow-motion invariant manifold function to reconstruct the fast states. Through an error analysis, it is shown that even though the observer is designed based on the reduced model by neglecting the fast dynamics, it is capable of providing accurate estimation of the states of the original detailed system. It will render a vanishing estimation error, with exponential convergence rate governed by the subsystem of fast dynamics and the chosen observer design parameters.

Keywords: Nonlinear observer design; nonlinear model reduction; estimation in biological systems; continuous time system estimation.

INTRODUCTION

Sensors are widely used in chemical processes for safe operation and product quality monitoring purposes. Nevertheless, in many process industry applications, only a part of the variables critical for safety or quality control are available for measurement. Furthermore, the unavailability of information for some variables prevents the implementation of certain feedback control policies (Soroush (1998)). This is frequently encountered in bioreactor systems, where the concentration of some important species is either unavailable due to the lack of or the high price of sensors, or only available with significant measurement delays (Bastin (2013)). In these circumstances, state estimation techniques play an important role in giving accurate estimates for the unknown state variables. The model-based state observer is one of the widely applied state estimation strategies that reconstructs the unmeasurable state variables by using the process model along with the available measurements. The Luenberger observer was first proposed by Luenberger (1964, 1971) for linear systems, which formed the basis for the subsequent emerging observer approaches to address more comprehensive linear state estimation problems. As most of the chemical and physical process dynamics are governed by nonlinear differential equations, it is well recognized that the linear observers designed based on the linearization of process model with only local validity can be inadequate in the presence of nonlinearities. For this reason, many research efforts have been made on developing nonlinear state observer design approaches to handle process nonlinearities (Kazantzis and Kravaris (1998); Kazantzis et al. (2000)).

For many chemical and biochemical processes, the simultaneous occurrence of reactions, heat, mass and momentum transport phenomena, and the interaction with sensor and controller in different time scales can lead to two-time-scale or multi-time-scale dynamic behavior. In particular, many bioreactor systems in bioprocess applications involve both fast and slow dynamics due to the existence of multiple microbial cultures with different metabolic rates (Stamatelatou et al. (2009); Duan et al. (2017)). Modeling these processes can lead to dynamic systems with both fast and slow modes. A system with both fast and slow modes exhibits stiffness in numerical simulation and may cause commonly used numerical methods to be unstable, unless taking extremely small step sizes or using special stiff numerical methods. This inevitably increases the calculation cost and complexity. To address this issue, one option is to apply model reduction to the multi-time-scale models, by applying time analysis tools and only keeping the desired time-scale dynamics.

It is recognized that controllers designed for two-time-scale systems can be at risk of inducing instability. Many researchers have studied control design problems particularly for these systems, mainly from the point of view of singular perturbation methods (Christofides and Daoutidis (1996); Christofides (1998, 2000); Kumar et al. (1998); Khalil (1987); Duan and Kravaris (2018)). Indeed, similarly to the control design, the observer design for systems with both fast and slow modes could also be theoretically challenging. This is due to the fact that the stiff system dynamics may result in ill-conditioned observer gains, and the established convergence properties of the observer can thereby be potentially undermined (Kazantzis et al. (2005)). Similar to the control design, the observer problem has been thoroughly studied for both linear and nonlinear systems within the singular pertur-

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bation framework. Kazantzis et al. (2005) has proposed an observer design approach for the slow states of a singularly perturbed system, by first designing observers for the reduced slow system and then applying it back to the original system. Their work assumed the measurements to be linear and rendered a non-vanishing estimation error of order of ϵ, where ϵ is the perturbation parameter.

In the present work, we will consider general nonlinear two-time-scale systems and propose an improved reduced-order observer design approach. The proposed approach combines the observer design method developed on the basis of exact error linearization with eigenvalue assignment by Kazantzis and Kravaris (1998) and the model reduction method by Kazantzis et al. (2010). Particularly, we will design the nonlinear observer for a reduced-order system comprised of only slow states, which is achieved by using the slow-motion invariant manifold method, and apply it back to the original full order system. The estimation for fast states is established using algebraic equations in terms of the estimated slow states. We will show that this observer derived from the reduced system will also converge to the states from the detailed model.

PRELIMINARIES

Consider a nonlinear autonomous dynamic system

$$\begin{aligned} \frac{dx}{dt} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in R^n$ and $y \in R^m$ are the state and output variable vectors. It is assumed that $f(x)$ and $h(x)$ are real analytic vector functions $f : R^n \to R^n$ and $h : R^n \to R^m$, and that the origin is an equilibrium point: $f(0) = 0$ and $h(0) = 0$. The local linearization approximation of system (1) around origin is of the form:

$$\begin{aligned} \frac{dx}{dt} &= Fx \\ y &= Hx \end{aligned} \tag{2}$$

where $F = \frac{\partial f}{\partial x}(0)$ and $H = \frac{\partial h}{\partial x}(0)$. The following assumptions are made:

Assumption 1. The matrix

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

has rank n.

This assumption guarantees the local observability of the system (1).

Assumption 2. The Jacobian matrix F is Hurwitz, *i.e.* with all its eigenvalues to have negative real parts.

This assumption states that the origin is a locally asymptotically stable equilibrium point for system (1). It is a restriction imposed by the invariant manifold based model reduction method.

Assumption 3. The spectrum of the Jacobian matrix $\sigma(F)$ is comprised of two distinct subsets, a "slow" subset $\sigma_s(F)$ of size p and a "fast" subset $\sigma_f(F)$ of size q: $\sigma(F) = \sigma_s(F) \cup \sigma_f(F)$ and $n = p + q$, where $\sigma_s(F)$ contains "slow" eigenvalues whose real parts are a few orders of magnitude smaller than those of the "fast" eigenvalues in $\sigma_f(F)$ (in absolute value).

Assumption 4. The eigenvalues λ_{si} $(i = 1, ..., p)$ of $\sigma_s(F)$ are not related to the eigenvalues λ_{fj} $(j = 1, ..., q)$ of $\sigma_f(F)$ through any equation of the form with $m_i \in \mathbb{Z}^+$:

$$\sum_{i=1}^{p} m_i \lambda_{si} = \lambda_{fj} \quad \text{where} \quad \sum_{i=1}^{p} m_i > 0.$$

The last two assumptions essentially imply that the system (1) exhibits a two-time-scale feature as the system dynamics consists of both fast modes and slow modes, and also pose a non-resonance condition on the nonlinear system as a prerequisite for the existence of a unique real analytic slow invariant manifold of (1).

As a background before stating the main result, this section briefly outlines the nonlinear observer design approach based on error linearization with eigenvalue assignment, and the invariant manifold based model reduction method.

Nonlinear Observer Design

The nonlinear observer proposed by Kazantzis and Kravaris (1998) is designed using an exact error linearization method. The idea of this design approach is to have the resulting observer error follow a linear dynamics with a pre-specified rate of decay, in curvilinear coordinates.

A full-order nonlinear identity observer for the general nonlinear system (1) is of the form:

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + L(\hat{x})(y - h(\hat{x})) \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the vector of estimated states, and $L(\hat{x})$ is a state-dependent gain. We want to properly choose the state gain so that there would be a locally analytic mapping $z = T(x)$, with $z \in \mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$, that maps (1) to:

$$\dot{z} = Az + By \tag{4}$$

where A and B are two constant matrices of appropriate dimensions (design parameters), subject to the following restrictions: i. A is Hurwitz; ii. its eigenvalues k_i $(i = 1, ..., n)$ are not related to the eigenvalues λ_i $(i = 1, ..., n)$ of F through any equations of the form $\sum_{i=1}^{n} m_i \lambda_i = k_j$ where $\sum m_i > 0$; iii. $\{A, B\}$ is a controllable pair. Then the observer gain $L(x)$ can be calculated from:

$$L(x) = \left[\frac{\partial T}{\partial x}(x)\right]^{-1} B$$

where the mapping $z = T(x)$ is the locally analytic solution of the system of first-order non-homogeneous PDEs:

$$\frac{\partial T}{\partial x} f(x) = AT(x) + Bh(x). \tag{5}$$

With the above choice of the state-dependent observer gain, the observer (3) leads to the linear dynamics in the transformed coordinates $z = T(x)$:

$$\frac{d}{dt}(T(\hat{x}) - T(x)) = A(T(\hat{x}) - T(x)). \tag{6}$$

Because A is a design parameter that can be arbitrarily selected, the error convergence speed could be then adjusted. As long as the matrix A is chosen to be Hurwitz, the estimation error exponentially decays to zero and the estimated states $\hat{x}$ asymptotically converge to the actual states x.

Remark 1. This nonlinear observer design approach involves the calculation of the solution of a system of PDEs. However, a closed form solution of (5) is seldom available. Therefore, in order to implement this methodology, an approximate solution should be calculated. It is possible to approximate $T(x)$ with a truncated multivariable Taylor series around the origin. After expanding functions and unknonwns in Taylor series up to a finite truncation order, the approximate solution can be obtained by equating the coefficients of each side of the PDEs.

Model Reduction with Slow Invariant Manifold

In the model reduction method of Kazantzis et al. (2010), the fast dynamics of the system is considered to be instantaneous and the fast states are approximated through the algebraic slow invariant manifold functions in terms of the slow states. The existence of a unique analytic slow invariant manifold and the effectiveness of this approximation have been validated, and a systematic method to solve for this specific slow invariant manifold has also been introduced in the aforementioned work.

For the general nonlinear system (1), a set $\Omega = \{x \in \mathbb{R}^n | \Phi(x) = 0\}$ where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $\Phi(0) = 0$, is called invariant manifold if for each $x(0)$ such that $\Phi(x(0)) \in \Omega$, $\Phi(x(t)) \in \Omega$ for all $t > 0$.

It can be easily induced that the map Φ needs to satisfy the invariance equation:

$$\frac{\partial \Phi}{\partial x}(x) f(x) = 0, \qquad \forall x \in \Omega. \tag{7}$$

Note that this invariance equation admits multiple solutions as all possible invariant manifolds for the dynamics (1) will satisfy the PDE (7).

In order to explicitly express the fast and slow modes, we will start from the linearized system (2). This system can be transformed into the block-triangular form through a standard linear coordinate transformation (Wiggins (2003)):

$$\begin{aligned} \frac{dx_s}{dt} &= F_s x_s \\ \frac{dx_f}{dt} &= F_{fs} x_s + F_f x_f \\ y &= H' \begin{bmatrix} x_s & x_f \end{bmatrix} \end{aligned} \tag{8}$$

where $x_s \in \mathbb{R}^p$ and $x_f \in \mathbb{R}^q$ of (8) are the slow and fast state vectors with the spectra of F_s and F_f being $\sigma(F_s) = \sigma_s(F)$ and $\sigma(F_f) = \sigma_f(F)$ respectively. The same linear coordinate transformation can be applied to the nonlinear system (1). And under the new coordinate system, the original nonlinear system dynamics (1) takes the form:

$$\begin{aligned} \frac{dx_s}{dt} &= f_s(x_s, x_f) \\ \frac{dx_f}{dt} &= f_f(x_s, x_f) \\ y &= h'(x_s, x_f). \end{aligned} \tag{9}$$

Based on the previous assumptions, it can be easily inferred that $f_s(x_s, x_f)$ and $f_f(x_s, x_f)$ are also real analytic vector functions with origin being an equilibrium point: $f_s(0,0) = 0$, $f_f(0,0) = 0$, and also $\frac{\partial f_s}{\partial x_f}(0,0) = 0$.

Now considering the transformed two-time-scale system (9), it has been proved that as long as the Assumptions $2 - 4$ hold true, there is a unique local analytic invariant manifold that corresponds the slow-motion invariant manifold of the system (9) of following form:

$$\Omega = \{(x_s, x_f) \in R^n | x_f - \pi(x_s) = 0\} \tag{10}$$

where the map π satisfies the invariance equation below:

$$\frac{\partial \pi}{\partial x_s} f_s(x_s, \pi(x_s)) = f_f(x_s, \pi(x_s)). \tag{11}$$

For this slow-motion invariant manifold, the following lemma holds:

Lemma 1. (Kazantzis et al. (2010)). Consider the nonlinear dynamical system (9) and let the Assumptions 2, 3 and 4 hold true. Furthermore, let Ω (10) be an invariant manifold of (9), where $\pi(x_s)$ is the unique locally analytic solution of the invariance PDE (11) and $(x_s(t), x_f(t))$ a solution curve of (9). There exists a neighbourhood U^0 of the origin and real numbers $k > 0$ and $a > 0$ such that, if $(x_s(t_0), x_f(t_0)) \in U^0$, then:

$$||x_f(t) - \pi(x_s(t))||_2 \leq k \exp(-a(t - t_0)) ||x_f(t_0) - \pi(x_s(t_0))||_2. \tag{12}$$

Furthermore, the rate of decay of the dynamics of the off-manifold coordinate $z = x_f - \pi(x_s)$ is governed by the fast eigenvalues of matrix $F_f = \frac{\partial f_f}{\partial x_f}(0,0)$.

This lemma implies that, once the fast dynamics dies out, the slow invariant manifold would attract all system's trajectories. For a two-time-scale system, if the spectral gap between σ_s and σ_f is huge, it is reasonable to assume the fast dynamics to be instantaneous, and the fast states can be then expressed as algebraic functions of the slow states using the slow invariant manifold calculated from (11). Therefore, a reduced-order model can be used to describe the system dynamics (9) by ignoring the fast dynamics as follows:

$$\begin{aligned} \frac{d\bar{x}_s}{dt} &= f_s(\bar{x}_s(t), \pi(\bar{x}_s)) = \bar{f}(\bar{x}_s(t)) \\ \bar{y}(t) &= h'(\bar{x}_s(t), \pi(\bar{x}_s)) = \bar{h}(\bar{x}_s(t)) \\ \bar{x}_f &= \pi(\bar{x}_s) \end{aligned} \tag{13}$$

where the overbar is used to distinguish the variables of the approximate system (13) from the exact system (9).

MAIN RESULTS

For linear two-time-scale systems, the standard observer design is usually troubled with peaking behavior, ill-conditioned error dynamics and stiff observer equations. The high gain observer, which is sometimes used as an alternative approach, could inevitably cause the system to be highly sensitive to measurement noises. These difficulties can get aggravated in the nonlinear cases. First, the nonlinear system usually shows complex stability patterns. Since the stability properties for most observers are only locally established, the observer may not work effectively far away from the local region. This is more likely to happen in the presence of inaccurate estimates of fast states involving peaking. Second, for nonlinear systems, it is widely recognized that the linear constant-gain observer may be inadequate. But the nonlinear observer has state-dependent gains, and when these gains are ill-conditioned vector functions, stiffness problems may get worse. Furthermore, for a high order system with multiple measurements, the A and B matrices in (4) are of high dimensions. With more degrees of freedom, the selection of such design parameters becomes a big challenge.

The difficulty of observer design for two-time-scale systems lies on the presence of both fast and slow dynamics. Intuitively, if we are able to design an observer containing only one time-scale dynamics, the problems related to stiffness can be avoided. Particularly, in the case where the fast dynamics is of less concern or unobservable, an observer can be designed based only on the slow dynamics by applying model reduction techniques and ignoring the fast dynamics. Along this direction, in the current section, we study a slow invariant manifold based reduced-order observer design method for nonlinear two-time-scale systems.

To design such an observer, the original model can be reduced to a lower order model by projecting the dynamics on the slow invariant manifold, i.e. considering the fast dynamics to be instantaneous. A standard nonlinear observer is then designed to only estimate the slow states of this reduced model. The fast states can be thereafter reconstructed using the calculated invariant manifold, as a function of the estimated slow states. But as the fast dynamics is neglected, a rigorous analysis needs to be conducted to investigate whether the estimated states from this observer, designed on the basis of the reduced model, will asymptotically approach the actual states of the original system. This section details the proposed method and studies convergence properties through an error analysis.

Without loss of generality, the system (9) of fast and slow states is the point of departure. Following the slow invariant manifold method, the fast dynamics can be projected on the invariant manifold $x_f = \pi(x_s)$, and the original model is reduced to a lower order model with slow states only, as in (13).

Next, we could design a full-order identity nonlinear observer for this reduced order model with the exact linearization design approach:

$$\frac{d\hat{\bar{x}}_s}{dt} = \bar{f}(\hat{\bar{x}}_s) + L(\hat{\bar{x}}_s)(\bar{y} - \bar{h}(\hat{\bar{x}}_s)). \tag{14}$$

Note that this is a full-order observer for the reduced model, but this reduced-order observer will be applied to the original model. The observer gain $L(\bar{x}_s)$ is computed as:

$$L(\bar{x}_s) = \left[\frac{\partial T}{\partial \bar{x}_s}(\bar{x}_s)\right]^{-1} B_p \tag{15}$$

where $z = T(\bar{x}_s)$ is the solution of the system of PDEs:

$$\frac{\partial T}{\partial \bar{x}_s}\bar{f}(\bar{x}_s) = A_p T(\bar{x}_s) + B_p \bar{h}(\bar{x}_s) \tag{16}$$

where A_p and B_p denote the design parameter matrices in (16). Because there is no spectral gap problem for the reduced system, one can arbitrarily select a matrix A_p with desired eigenvalues (e.g. an order of magnitude larger than those for the reduced slow system), and a matrix B_p of proper dimension with (A_p, B_p) a controllable pair.

The fast states $\hat{\bar{x}}_f$ can be reconstructed using the slow invariant manifold as an algebraic equation of $\hat{\bar{x}}_s$, i.e. $\hat{\bar{x}}_f = \pi(\hat{\bar{x}}_s)$.

Because of the established convergence properties of the observer error linearization method, the estimate $\hat{\bar{x}}_s$ in (14) will asymptotically approach the slow state $\bar{x}_s$ of the reduced model (13). Our ultimate goal is to apply this observer back to the original system (9) with the actual outputs y, which leads to the following reduced-order observer:

$$\begin{aligned} \frac{d\hat{x}_s}{dt} &= \bar{f}(\hat{x}_s) + L(\hat{x}_s)(y - \bar{h}(\hat{x}_s)) \\ \hat{x}_f &= \pi(\hat{x}_s), \end{aligned} \tag{17}$$

where $\hat{x}_s$, $\hat{x}_f$, and y are corresponding state estimates and measurements for the system described by (9). However, it is unclear whether the convergence properties of observer will hold for observer (17) since the observer is originally designed based on the reduced system (13). The answer is given by the following theorem.

Theorem 1. Consider observer (17), which has been derived from the reduced order system (13), where $L(\hat{\bar{x}}_s)$ is given by (15) with $T(\hat{\bar{x}}_s)$ being the unique locally analytic solution of the PDEs (16), and $\bar{f}(\hat{\bar{x}}_s)$ and $\bar{h}(\hat{\bar{x}}_s)$ are defined by (13) with $\pi(\bar{x}_s)$ being the unique locally analytic solution of the invariance equation (11). Let all the aforementioned assumptions hold true. There exists a neighbourhood U of the origin such that, if $(x_s(t_0), x_f(t_0)) \in U$, then for the system (9), the estimation error $e(t) = T(x_s(t)) - T(\hat{x}_s(t))$ induced by the observer (17) satisfies the following equation:

$$e(t) = \exp(A_p(t - t_0))e(t_0) + H(t) \tag{18}$$

where $H(t)$ is an exponentially decaying term that vanishes as $t \to \infty$.

Proof:

$$\begin{aligned}
\frac{de}{dt} =& \frac{d}{dt}\left[T(x_s) - T(\hat{x}_s)\right] \\
=& \frac{\partial T}{\partial x_s}(x_s) f_s(x_s, x_f) - \frac{\partial T}{\partial x_s}(\hat{x}_s)\left[L(\hat{x}_s)(y - \bar{h}(\hat{x}_s))\right. \\
&\left. + \bar{f}(\hat{x}_s)\right] (\Leftarrow (17)) \\
=& \frac{\partial T}{\partial x_s}(x_s) f_s(x_s, x_f) - \frac{\partial T}{\partial x_s}(x_s)\bar{f}(x_s) + \frac{\partial T}{\partial x_s}(x_s)\bar{f}(x_s) \\
&- \frac{\partial T}{\partial x_s}(\hat{x}_s)\bar{f}(\hat{x}_s) - B_p\left[h'(x_s, x_f) - \bar{h}(\hat{x}_s)\right] \\
&(\Leftarrow (15), (9)) \\
=& \frac{\partial T}{\partial x_s}(x_s) f_s(x_s, x_f) - \frac{\partial T}{\partial x_s}(x_s)\bar{f}(x_s) \\
&+ A_p T(x_s) + B_p \bar{h}(x_s) - A_p T(\hat{x}_s) - B_p \bar{h}(\hat{x}_s) \\
&- B_p\left[h'(x_s, x_f) - \bar{h}(\hat{x}_s)\right] (\Leftarrow (16)) \\
=& \frac{\partial T}{\partial x_s}(x_s) f_s(x_s, x_f) - \frac{\partial T}{\partial x_s}(x_s)\bar{f}(x_s) \\
&+ A_p\left[T(x_s) - T(\hat{x}_s)\right] + B_p\left[\bar{h}(x_s) - h'(x_s, x_f)\right] \\
=& A_p e + \frac{\partial T}{\partial x_s}(x_s)\left[f_s(x_s, x_f) - \bar{f}(x_s)\right] + B_p\left[\bar{h}(x_s)\right.
\end{aligned}$$

$-h'(x_s, x_f)]$

Therefore the estimation error is

$$
\begin{aligned}
e(t) =& \exp(A_p(t-t_0))e(t_0)+ \\
& \int_{t_0}^{t} \exp(A_p(t-\bar{t})) \left\{ \frac{\partial T(x(\bar{t}))}{\partial x} \left[f_s(x_s(\bar{t}), x_f(\bar{t})) \right.\right. \\
& \left.\left. -\bar{f}(x_s(\bar{t})) \right] + B_p \left[\bar{h}(x_s(\bar{t})) - h'(x_s(\bar{t}), x_f(\bar{t})) \right] \right\} d\bar{t}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
H(t) =& -\int_{t_0}^{t} \exp(A_p(t-\bar{t})) \left\{ \frac{\partial T(x(\bar{t}))}{\partial x} \left[f_s(x_s(\bar{t}), x_f(\bar{t})) \right.\right. \\
& \left.\left. -\bar{f}(x_s(\bar{t})) \right] + B_p \left[\bar{h}(x_s(\bar{t})) - h'(x_s(\bar{t}), x_f(\bar{t})) \right] \right\} d\bar{t}
\end{aligned}
$$

Then the norm of $H(t)$ is bounded by

$$
\begin{aligned}
\|H(t)\| \leq & \int_{t_0}^{t} \| \exp(A_p(t-\bar{t})) \| \times \\
& \left\{ \left\| \frac{\partial T(x(\bar{t}))}{\partial x} \right\| \times \| f_s(x_s(\bar{t}), x_f(\bar{t})) - \bar{f}(x_s(\bar{t})) \| \right. \\
& \left. + \|B_p\| \times \| \bar{h}(x_s(\bar{t})) - h(x_s(\bar{t}), x_f(\bar{t})) \| \right\} d\bar{t}.
\end{aligned}
$$

Assuming the matrix A_p is chosen to be Hurwitz, there exist positive constants k_0, a_0 such that

$$
\| \exp(A_p(t-\bar{t})) \| \leq k_0 \exp(-a_0(t-\bar{t})). \tag{19}
$$

If we denote by w the off-manifold coordinate: $w(t) = x_f(t) - \pi(x_s(t))$, then from Lemma 1, one obtains

$$
\begin{aligned}
\|w(\bar{t})\| &= \|x_f(\bar{t}) - \pi(x_s(\bar{t}))\| \\
&\leq k_1 \|x_f(t_0) - \pi(x_s(t_0))\| \exp(-a_1(\bar{t}-t_0))
\end{aligned} \tag{20}
$$

for some positive constants k_1 and a_1, in a neighbourhood U^0 of the origin.

Next, denote:

$$
\begin{aligned}
F(x_s, w) &= f_s(x_s, w + \pi(x_s)) \\
G(x_s, w) &= h(x_s, w + \pi(x_s))
\end{aligned}
$$

and we get the bounds

$$
\begin{aligned}
& \|\bar{f}(x_s(\bar{t})) - f_s(x_s(\bar{t}), x_f(\bar{t}))\| \\
& = \|f_s(x_s(\bar{t}), x_f(\bar{t})) - f_s(x_s(\bar{t}), \pi(x_s(\bar{t})))\| \\
& = \|F(x_s, w) - F(x_s, 0)\| \\
& \|\bar{h}(x_s(\bar{t})) - h(x_s(\bar{t}), x_f(\bar{t}))\| \\
& = \|h(x_s(\bar{t}), \pi(x_s(\bar{t}))) - h(x_s(\bar{t}), x_f(\bar{t}))\| \\
& = \|G(x_s, 0) - G(x_s, w)\|.
\end{aligned}
$$

Due to the analyticity of the vector functions $F(x_s, w)$ and $G(x_s, w)$ around the origin, there exist positive constants L_1 and L_2 in a compact neighbourhood U^1 of the origin, such that:

$$
\begin{aligned}
\|F(x_s, w) - F(x_s, 0)\| &< L_1 \|w\| \\
&\leq L_1 k_1 \|x_f(t_0) - \pi(x_s(t_0))\| \exp(-a_1(\bar{t}-t_0)) \\
\|G(x_s, 0) - G(x_s, w)\| &< L_2 \|w\| \\
&\leq L_2 k_1 \|x_f(t_0) - \pi(x_s(t_0))\| \exp(-a_1(\bar{t}-t_0)).
\end{aligned}
$$

Similarly, the analyticity of the map $T(x)$ around the origin implies that there exists a positive constant L_3 in a compact neighbourhood U^2 such that $\|\partial T / \partial x\| \leq L_3$.

Based on the foregoing bounds and defining $U = U^0 \cap U^1 \cap U^2$, the following bound can be established for $H(t)$:

$$
\begin{aligned}
\|H(t)\| \leq & k_0 k_1 \|x_f(t_0) - \pi(x_s(t_0))\| (L_2 \|B_p\| + L_1 L_3) \\
& \int_{t_0}^{t} \exp(-a_0(t-\bar{t})) \exp(-a_1(\bar{t}-t_0)) d\bar{t} \\
\leq & \frac{k_0 k_1}{a_1 - a_0} \|x_f(t_0) - \pi(x_s(t_0))\| (L_2 \|B_p\| + L_1 L_3) \\
& \left[\exp(-a_0(t-t_0)) - \exp(-a_1(t-t_0)) \right].
\end{aligned}
$$

Therefore, $H(t)$ is exponentially decaying as $t \to +\infty$.

The theorem implies that although the reduced-order observer is designed based on the reduced model, when applying it back to the detailed model by using the actual outputs, it is capable of reconstructing the actual state variables for the original system, with an exponentially decaying estimation error. The rate of decay is governed by both a_0 and a_1, which reflect the observer error dynamics determined by the design parameter A_p and the off-manifold dynamics affected by the system fast dynamics, respectively. □

Remark 2. Using similar arguments, based on Lemma 1 and Theorem 1, it can be easily inferred that the fast state estimation error

$$
e_f = x_f - \hat{x}_f = x_f - \pi(\hat{x}_s) = [x_f - \pi(x_s)] + [\pi(x_s) - \pi(\hat{x}_s)]
$$

exponentially converges to zero within a neighbourhood of the origin. Thereby, the fast estimate $\hat{x}_f$ in (17) will also asymptotically converge to the actual fast states x_f of the system (9).

The methodology for designing this observer is depicted in Figure 1. In this approach, the fast dynamics is excluded from the designed observer. There is no fast dynamics and the ill-conditioned observer gains can be avoided. Consequently, there will be no stiffness issues for the implementation of the proposed observer. However, the observer does not account for the fast dynamics.

Remark 3. Compared with the reduced-order observer design method proposed by Kazantzis et al. (2005), which is based on the singular perturbation system with the measurement linear in state variables, the current method is based on a general two-time-scale system and does not pose restrictions to the measurement (other than the observability requirement in Assumption 1). Furthermore, the method based on singularly perturbed system renders a non-vanishing estimation error of $O(\epsilon)$, while Theorem 1 implies that the estimation error exponentially decays in the current approach.

CONCLUSION

In this work, we proposed a reduced-order nonlinear observer design approach for general nonlinear two-time-scale systems, based on the slow system dynamics. To design this observer, the original detailed model is reduced to a lower order model by projecting the dynamics on the slow invariant manifold, i.e. considering the fast dynamics to be instantaneous. An identity nonlinear observer based on error linearization is then designed to only estimate the slow states for this reduced model. The fast states are reconstructed by using the calculated invariant manifold, as a function of the estimated slow states. When applying this observer back to the actual detailed model using the actual output measurement, it has been proved that the estimation given by the proposed reduced-order observer,

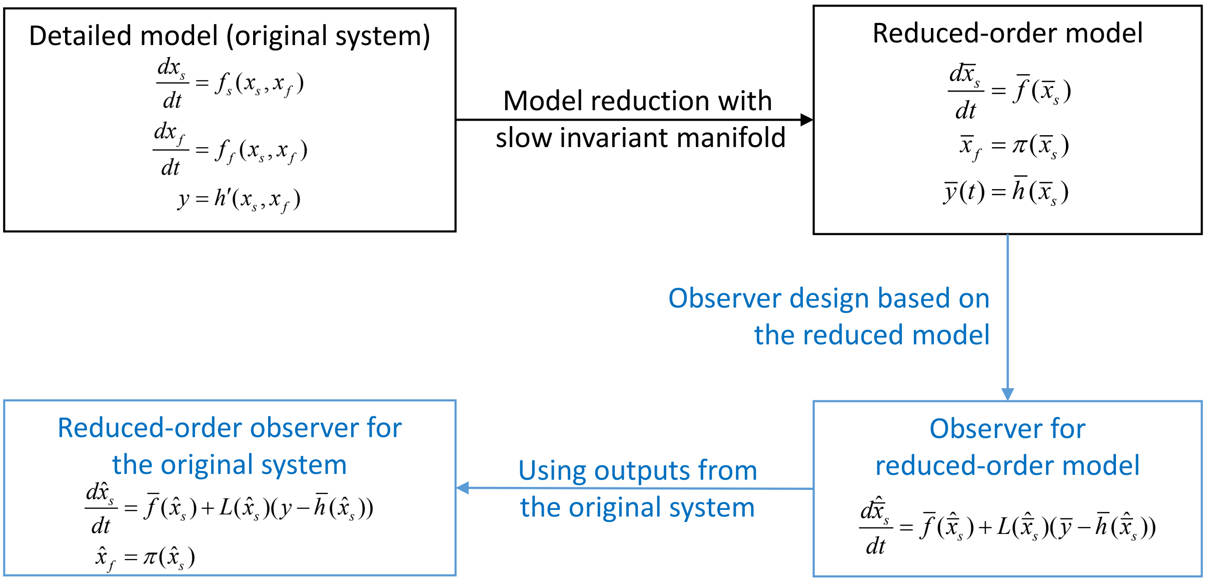

Fig. 1. Reduced-order observer design based on reduced model

will asymptotically approach the exact states of the original system.

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