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Hierarchical dynamic control for robust attitude tracking

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Abstract: In this paper robust attitude tracking for fully actuated rigid bodies is addressed. By exploiting the cascade structure of the underlying mathematical model, a hierarchical framework including a large number of dynamic feedback controllers is proposed. The closed loop results in error dynamics comprising an inner loop associated with the angular velocity error, and an outer loop associated with the attitude error. We then establish sufficient conditions for solving the attitude tracking problem from an (almost) global perspective by leveraging recent results on stabilization of nonlinear cascades and invariance principles for differential inclusions. The modular nature of the proposed approach allows one to conclude stronger stability properties than those available for existing dynamic control laws, such as PID loops, often employed in applications.

1. INTRODUCTION

The attitude control problem for systems that can be modeled as fully actuated rigid bodies by a control torque has been addressed by a huge body of literature. Applications enjoying this structure range from aerospace, robotics and underwater vehicles. Most of the works dealing with attitude control from a global perspective are concerned with the design of stabilizing controllers in an ideal setting, which is an already demanding challenge: due to the topology of SO(3), global stabilization by means of continuous state feedback is structurally impossible [Koditschek 1989, Casau et al. 2019]. When accounting for parametric uncertainties and exogenous disturbances, the attitude control problem becomes even more involved.

Robust attitude control designs have begun to emerge in recent years by extending techniques developed for Euclidean spaces [Camblor et al. 2015, Goodarzi et al. 2013, Maithripala and Berg 2015]. Exploiting the full actuation assumption, the control torque is designed to compensate for the gyroscopic torque and to enforce a decrease condition of suitable Lyapunov functions along the flow of the closed-loop system. However, the resulting control laws are typically complex and not easy to tune due to nontrivial conditions on the gains: one may not be able to enforce a desirable level of performance around the desired attitude without violating sufficient conditins guaranteeing global stability.

In this work we appeal to ideas coming from reduction theory [Maggiore et al. 2019] to propose a hierarchical control design based on an inner-outer loop paradigm that exploits the cascade structure of the dynamical attitude control model. In this way, we split the challenges of the control design in two simpler sub-problems: A) stabilization of the attitude error kinematics, which evolves in a boundaryless compact manifold and B) stabilization of the angular velocity error dynamics, which evolves in a Euclidean space. Although inner-outer loop paradigms already exist for attitude control [Rudin et al. 2011, Forbes 2013], the proposed approach is different in several ways including the fact that it encompasses a more general class of dynamic stabilizing laws. We start by developing a modular cascade control law based on static and dynamic feedback for attitude and angular velocity tracking, respectively. By requiring boundedness of the outer loop output (a desirable property), we are able to embed the closed-loop solutions in the funnel generated by a differential inclusion enjoying a cascaded property, without *a priori* restricting the class of allowable inner loop controllers. Recent results about Input-to-State Stability (ISS) for multistable systems [Angeli and Praly 2011] allow us to point out basic properties that each component of the control law should satisfy to guarantee closed-loop stability. In particular, we show that the widely adopted attitude stabilizer corresponding to the gradient of the modified trace function on SO(3) [Koditschek 1989] fits within our framework. Finally, we suggest a rather general class of dynamic controllers for angular velocity tracking, combining passivity concepts and invariance principles for differential inclusions [Seuret et al. 2019]. For instance, our control design makes it possible to prove almost global tracking, in the presence of constant disturbances and regardless of the gain values, for a control law behaving like a P/PI cascaded linear controller near the desired attitude motion.

2. NOTATIONS AND MATHEMATICAL PRELIMINARIES

$\mathbb{R}\,(\mathbb{R}_{>0},\mathbb{R}_{\geq 0})$ denotes the set of (positive, nonnegative) real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices. The i-th vector of the canonical basis in $\mathbb{R}^n$ is denoted as e_i (vector with 1 in the *i-th* position and zeros elsewhere) and the identity matrix in $\mathbb{R}^{n\times n}$ is $I_n := [e_1 \cdots e_i \cdots e_n]$. Given $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, we denote $(x,y) := [x^\top y^\top]^\top \in \mathbb{R}^{n+m}$. For $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, $\|x\| := \sqrt{x_1^2+\ldots+x_n^2}$ is its Euclidean norm, $\|x\|_\infty := \max_{i\in\{1,\ldots,n\}}(|x_i|)$ is its Euclidean norm while for a matrix

$A \in \mathbb{R}^{n\times n}$, tr(A) is its trace, $\|A\|_F := \sqrt{\mathrm{tr}(A^\top A)}$ is the Frobenius norm and $\mathrm{skew}(A) := \frac{A-A^\top}{2}$ is the skew-symmetric part of A. The set $\mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$ denotes the third-order Special Orthogonal group while $\mathbb{S}^n := \{q \in \mathbb{R}^{n+1} : \|q\| = 1\}$ denotes the n-dimensional unit sphere. The tangent space to SO(3) at any $R \in \mathrm{SO}(3)$ is denoted as $T_R\mathrm{SO}(3)$. The normalized distance with respect to I_3, induced by the Frobenius norm, is denoted as $\|R\|_{\mathrm{SO}(3)} := \frac{1}{8}\|R - I_3\|_F = \sqrt{\frac{1}{4}\mathrm{tr}(I_3 - R)} \in [0,1]$. Given $\omega \in \mathbb{R}^3$, the S map $S(\cdot) : \mathbb{R}^3 \to \mathfrak{so}(3) := \{\Omega \in \mathbb{R}^{3\times 3} : \Omega = -\Omega^\top\}$ is such that $S(\omega)y = -S(y)\omega = S(y)^\top \omega = \omega \times y$, $\forall y \in \mathbb{R}^3$ where $\times$ represents the cross product in $\mathbb{R}^3$. The inverse of the S map is the denoted as $S(\cdot)^{-1} : \mathfrak{so}(3) \to \mathbb{R}^3$. We use standard comparison functions from Khalil [2002]: function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is also unbounded. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $r \mapsto \beta(r,s)$ is of class $\mathcal{K}$ for each $s \geq 0$ and $s \mapsto \beta(r,s)$ is decreasing for each $r \geq 0$ and converges to zero as $s \to +\infty$.

3. RIGID BODY ATTITUDE CONTROL PROBLEM

Consider an inertial reference frame and a body fixed frame the origin of which is located at the center of mass of a rigid body. The configuration of the rigid body is described by the orientation of the body fixed frame with respect to the inertial frame, and it is globally and uniquely represented by a rotation matrix $R := [b_1\ b_2\ b_3] \in \mathrm{SO}(3)$ where $b_i \in \mathbb{S}^2$, $i \in \{1,2,3\}$ are three orthogonal vectors with components resolved in the inertial frame forming a right-handed triad. The attitude motion of the rigid body is described by the following well-known dynamics:

$$\dot{R} = RS(\omega) \tag{1}$$
$$J\dot{\omega} = -S(\omega)J\omega + \tau_c + \tau_e \tag{2}$$

where $J = J^\top \in \mathbb{R}^{3\times 3}_{>0}$ is the inertia matrix expressed in the body frame, $\omega \in \mathbb{R}^3$ is the body angular velocity, $\tau_c \in \mathbb{R}^3$ is the control torque exerted by the actuators and $\tau_e \in \mathbb{R}^3$ is the disturbance torque accounting for unknown exogenous effects. Our goal is to introduce dynamic controllers guaranteeing attitude tracking for a fully actuated rigid body in the presence of disturbances. For such a problem, we assume the following standard assumption for the desired attitude trajectory that should be tracked.

Assumption 1. (Smoothness and boundedness of the desired trajectory). The desired trajectory $t \mapsto (R_d(t), \omega_d(t)) \in \mathrm{SO}(3) \times \mathbb{R}^3$ satisfies $\dot{R}_d(t) = R_d(t)S(\omega_d(t))$ $\forall t \geq 0$ and $t \mapsto \omega_d(t)$ is continuously differentiable and uniformly bounded.

The state feedback dynamic attitude tracking problem for a fully actuated rigid body can be then formalized as follows.

Problem 1. Consider the attitude dynamics in equations (1)-(2). Given a desired trajectory $t \mapsto (R_d(t), \omega_d(t)) \in \mathrm{SO}(3) \times \mathbb{R}^3$ satisfying Assumption 1, design a state-feedback dynamic controller delivering a control torque $\tau_c \in \mathbb{R}^3$ such that the trajectory $t \mapsto (R_d(t), \omega_d(t))$ is asymptotically tracked.

It is worthwhile to mention that the attitude kinematics model (1) evolves on SO(3), a boundaryless compact manifold (not diffeomorphic to a Euclidean Space), while on the other hand, the angular velocity dynamics (2) evolves on a linear manifold $\mathbb{R}^3$ and, in practical applications, is affected by exogenous disturbances and parameter uncertainties. The motivation for the solution proposed here is to parametrize hierarchical dynamic control architectures allowing for the use of PI-like robust control solutions independently addressing these uncertainty problems for each one of the two subsystems in (1)-(2).

4. CONTROL LAW AND STABILITY ANALYSIS

We propose here a hierarchical control architecture solving Problem 1 based on an inner-outer loop paradigm. Our framework leaves the door open for different selections of stabilizers of the inner and outer loops. Some examples are given in Section 5.

4.1 Control law and closed-loop dynamics

Let the attitude and angular velocity error coordinates be defined, respectively, as

$$R_e := R_d^T R \in \mathrm{SO}(3) \tag{3}$$
$$\omega_e := \omega_v - \omega \in \mathbb{R}^3, \tag{4}$$

and consider the following control law

$$\dot{x}_c = \gamma_c(x_c, \omega_e, \omega_v(R_e, t)) \tag{5}$$
$$\tau_c := S(\omega)J\omega + J\dot{\omega}_v(R_e, \omega_e, t) + \gamma_\omega(x_c, \omega_e, \omega_v) \tag{6}$$

where

$$\omega_v(R_e, t) := \gamma_R(R_e) + R_e^\top \omega_d(t), \tag{7}$$

whose derivative $\dot{\omega}_v$, exploiting $\dot{R}_e$ in (9), proven in Proposition 1, can be expressed as

$$\dot{\omega}_v(R_e, \omega_e, t) := \dot{\gamma}_R(R_e, \omega_e) - S(\gamma(R_e) - \omega_e)R_e^\top \omega_d(t) + R_e^\top \dot{\omega}_d(t) \tag{8}$$

and $x_c \in \mathbb{R}^{n_c}$ is the controller state and $\gamma_R(\cdot) : \mathrm{SO}(3) \to \mathbb{R}^3$, $\gamma_\omega(\cdot,\cdot,\cdot) : \mathbb{R}^{n_c} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $\gamma_c(\cdot,\cdot,\cdot) : \mathbb{R}^{n_c} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ are continuous stabilizers to be defined.

The inner-outer loop architecture of controller (5)–(6) is represented in Figure 1: the inner loop (6) is in charge of assigning a suitable torque τ_c to track the angular velocity reference ω_v provided by the outer loop (7). The controller dynamics in (5) is included in our architecture to allow for suitable dynamic approaches, such as PID loops, often necessary in practical applications, where unknown external disturbances or unmodeled dynamics affect the dynamics of ω and may negatively affect closed-loop stability and performance.

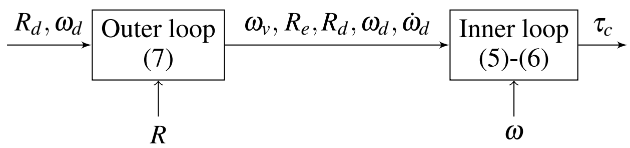

Fig. 1. Proposed inner-outer loop dynamic control architecture.

The following proposition well characterizes the dynamics of the closed-loop system.

Proposition 1. Consider the plant dynamics (1)-(2) and the dynamic controller (5)-(6). Using the tracking errors in (3)-(4), the closed-loop dynamics reads

$$\dot{R}_e = R_e S(\gamma_R(R_e)) + R_e S(\omega_e)^\top \tag{9}$$
$$J\dot{\omega}_e = -\gamma_\omega(x_c, \omega_e, \omega_v(R_e, t)) - \tau_e \tag{10}$$
$$\dot{x}_c = \gamma_c(x_c, \omega_e, \omega_v(R_e, t)). \tag{11}$$

Proof. Taking the time derivative of (3) along (1), and using the properties of ω_d and R_d in Assumption 1, we get

$$\begin{aligned}\dot R_e &= \dot R_d^\top R + R_d \dot R = -S(\omega_d) R_d^\top R + R_d^\top R S(\omega)\\ &= -S(\omega_d) R_e + R_e S(\omega) = R_e (S(\omega) - R_e^\top S(\omega_d) R_e)\end{aligned} \tag{12}$$

Recall now that $S^{-1}(RS(x)R^\top) = Rx\ \forall x \in \mathbb{R}^3$, which implies $S(Rx) = RS(x)R^\top$, and add and subtract ω_v from the previous equation, to get, based on (7),

$$\begin{aligned}\dot R_e &= R_e(S(\omega_v) + S(\omega) - S(\omega_v) - R_e^\top S(\omega_d) R_e)\\ &= R_e(S(\gamma_R(R_e)) + R_e^\top S(\omega_d) R_e - S(\omega_e) - R_e^\top S(\omega_d) R_e)\\ &= R_e S(\gamma_R(R_e)) - R_e S(\omega_e) = R_e S(\gamma_R(R_e)) + R_e S(\omega_e)^\top.\end{aligned} \tag{13}$$

Finally, computing $J\dot\omega_e = J(\dot\omega_v - \dot\omega)$ and substituting equations (2), (7) and (6) into (4), we get (10). ◦

Remark 2. As shown in Figure 2, the error dynamics (9)–(11) presents a nontrivial feedback structure perturbed by the (time-varying) desired velocity $t \mapsto \omega_d(t)$. It is emphasized that one could transform this feedback structure into an autonomous cascaded form by selecting $\gamma_\omega(x_c, \omega_e, \omega_v) = \bar\gamma_\omega(x_c, \omega_e)$ and $\gamma_c(x_c, \omega_e, \omega_v) = \bar\gamma_c(x_c, \omega_e)$. While mathematically elegant, such a selection can result in limited degrees of freedom in the design of stabilizers that do not cancel non-harmful nonlinearities (for example only partially canceling the term $-S(\omega)J\omega$ as done in Invernizzi et al. [2020]). Such stabilizers are also used in our design of Section 5. Our general formulation results in the feedback interconnection of Figure 2: while being more challenging from the point of view of stability analysis, this non cascaded solution gives more flexibility to the control designer.

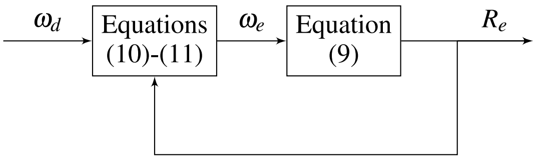

Fig. 2. Closed-loop error dynamics: feedback interconnection.

4.2 Stability analysis

In this section we address the stability analysis of the feedback interconnection structure in Figure 2 of the closed-loop dynamics (9)–(11) characterized in the previous section. The main goal is to guarantee asymptotic stability of the equilibrium [1] $(R_e, \omega_e, x_c) = (I_3, 0, 0)$, which captures the tracking requirements in Problem 1. Due to the presence of the external perturbation $t \mapsto \omega_d(t)$ in the error system a possible strategy for the study of asymptotic stability of the time-varying error dynamics (9)–(11) is to ensure that γ_R in (7) is uniformly bounded over SO(3) and exploit the uniform boundedness of ω_d in Assumption 1, to conclude that there exists a uniform bound $\omega_M > 0$ on $t \mapsto \omega_v(t)$ along all solutions to (9)–(11), and then embed these solutions in the larger funnel of solutions of the following differential inclusion:

$$\dot R_e = R_e S(\gamma_R(R_e)) + R_e S(\omega_e)^\top \tag{14}$$

$$\begin{bmatrix}\dot\omega_e\\ \dot x_c\end{bmatrix} \in F_\omega(\omega_e, x_c), \tag{15}$$

[1] It is assumed hereafter that one can write the perturbed closed-loop error dynamics as (9)-(11) for a given perturbation τ_e with an equilibrium point having $x_c = 0$, possibly after introducing a suitable change of coordinates (see for instance Example 1).

where

$$F_\omega(\omega_e, x_c) := \mathrm{co} \bigcup_{\|\omega_v\| \le \omega_M} \begin{bmatrix} -J^{-1}\gamma_\omega(x_c, \omega_e, \omega_v)\\ \gamma_c(x_c, \omega_e, \omega_v)\end{bmatrix}, \tag{16}$$

where co denotes the convex hull and where, to the end of studying stability, we select the input $\tau_e = 0$. It is instructive to observe that embedding the solution set of (9)–(11) in the larger funnel of solutions to (14), (15) allows us to restore the cascaded structure obtained in Remark 2 when restricting the degrees of freedom of the control architecture.

Remark 3. A basic requirement for the well posedness (in terms of existence and sequential compactness of the set of solutions) of the differential inclusion (14), (15) is that the map γ_R be a continuous function and that F_ω be outer semicontinuous and locally bounded relative to $\mathbb{R}^3 \times \mathbb{R}^{n_c}$, $F_\omega(\omega_e, x_c)$, in addition, to being nonempty and convex for each $(\omega_e, x_c) \in \mathbb{R}^3 \times \mathbb{R}^{n_c}$. These properties are automatically satisfied if γ_c and γ_ω in the original control law (5), (6) are continuous functions.

To analyze asymptotic stability of the equilibrium $(R_e, \omega_e, x_c) = (I_3, 0, 0)$, we use the following result, which is a slightly modified version of [Angeli 2004, Theorem 2], adapted to the closed-loop system (9)-(11) evolving on $\mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^{n_c}$. This lemma is instrumental to prove our main result.

Lemma 4. Let $x \in \mathbb{R}^n$ and $R \in \mathrm{SO}(3)$ and consider the cascade interconnection $\dot x = f(x)$, $\dot R = Q(R) + G(R, x)$ where $f : \mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$, $Q : \mathrm{SO}(3) \to T\mathrm{SO}(3)$ and $G : \mathrm{SO}(3) \times \mathbb{R}^n \to T\mathrm{SO}(3)$ are smooth vector fields with $f(0) = 0$, $Q(I_3) = 0$ and $G(I_3, 0) = 0$. Suppose that the equilibrium $R = I_3$ is almost global Input-to-State Stable (aISS) for $\dot R = Q(R) + G(R, d)$ with respect to d, namely, that $R = I_3$ is locally asympotically stable for $d = 0$ and that there exists $\gamma \in \mathcal{K}$ such that for each essentially bounded and measurable $d : \mathbb{R} \to \mathbb{R}^n$, there exists a zero volume set $\mathcal{B}_d \subset \mathrm{SO}(3)$ such that, for all $R(t_0) \in \mathrm{SO}(3) \setminus \mathcal{B}_d$,

$$\limsup_{t \to +\infty} \|R(t)\|_{\mathrm{SO}(3)} \le \gamma(\|d\|_\infty). \tag{17}$$

If the equilibrium $x = 0$ is GAS for $\dot x = f(x)$, then $(x, R) = (I_3, 0)$ is almost globally asymptotically stable.

Remark 5. The notion of almost global ISS, originally introduced in Angeli [2004], is exploited here to deal with the fact that, due to topological obstructions, there exists no continuous function $\gamma_R(\cdot)$ globally asymptotically stabilizing the desired attitude. In the context of nonlinear observer design [Vasconcelos et al. 2011], this notion has been exploited to show that the widely adopted stabilizer $\gamma_R(R_e) := -\frac{k_R}{2} S^{-1}(\mathrm{skew}(R_e))$, $k_R \in \mathbb{R}_{>0}$, makes the equilibrium $R_e = I_3$ aISS for (9) with respect to ω_e for a sufficiently large gain.

The proof of the Lemma 4 follows exactly the same steps as the proof given in Angeli [2004] and is therefore omitted. Note that herein we assume the GAS property (instead of almost GAS) for the perturbing subsystem (a more desirable property for our purposes), which is not hard to obtain as both the angular velocity and the controller dynamics evolve in Euclidean spaces.

In the spirit of a modular control design [Invernizzi et al. 2018], we now present some basic properties that the functions $\gamma_R(\cdot)$, $\gamma_\omega(\cdot,\cdot,\cdot)$ and $\gamma_c(\cdot,\cdot,\cdot)$ must satisfy in order to guarantee desirable stabilizing properties for the closed-loop system.

Property 1. (Outer loop). The attitude stabilizer $R_e \mapsto \gamma_R(R_e)$ is at least C^2, uniformly bounded, and such that

- $R_e = I_3$ is a locally asymptotically stable equilibrium point for $\dot{R}_e = R_e S(\gamma_R(R_e))$;
- subsystem (9) is aISS with respect to ω_e as in (17).

Remark 6. We emphasize that the uniform boundedness assumption of γ_R comes for free when γ_R is the gradient of a smooth function on SO(3).

Property 2. (Inner loop). Given a uniform bound $\bar{\gamma}_R > 0$ on function γ_R, and a uniform bound $\bar{\omega}_d > 0$ from Assumption 1, define $\omega_M := \bar{\gamma}_R + \bar{\omega}_d$. The angular velocity error stabilizer $(x_c, \omega_e, \omega_v) \mapsto \gamma_\omega(x_c, \omega_e, \omega_v)$ and the controller $(x_c, \omega_e, \omega_v) \mapsto \gamma_c(x_c, \omega_e, \omega_v)$ are continuous functions such that $(\omega_e, x_c) = (0,0)$ is a GAS equilibrium point for (15), (16).

Based on the above properties and Lemma 4, the main result of the paper is given by the following theorem.

Theorem 1. Consider the closed-loop system described by (1)-(2) controlled by (5)-(6). If $\gamma_R(\cdot), \gamma_\omega(\cdot,\cdot,\cdot)$ and $\gamma_c(\cdot,\cdot,\cdot)$ satisfy Properties 1 and 2, then for any desired trajectory $t \mapsto (R_d(t), \omega_d(t)) \in \mathrm{SO}(3) \times \mathbb{R}^3$ satisfying Assumption 1, the control law (7)-(6) solves Problem 1. In particular, the point $(R_e, \omega_e, x_c) = (I_3, 0, 0)$ is almost globally asymptotically stable.

Proof. From the definition of ω_M in Property 2 and expression (7), we have that along all the closed-loop solutions it holds that $|\omega_v(R_e,t)| \le \bar{\gamma}_R + \bar{\omega}_d = \omega_M$. Hence, the solutions of the error dynamics (9)–(11) are contained in the solution funnel of the differential inclusion (14)-(15). The cascaded structure of (14)-(15) comprises an upper subsystem (15), for which the equilibrium point $(\omega_e, x_c) = (0,0)$ is GAS due to Property 2, and a lower subsystem (14), for which the stabilizer $\gamma_R(\cdot)$ guarantees local asymptotic stability of $R_e = I_3$ and aISS due to Property 1. The proof is then completed by applying Lemma 4 to these two subsystems. ◦

Before concluding this section, we emphasize that an extension of our main result (Theorem 1) to global asymptotic stability could be achieved by employing suitable hybrid attitude stabilizers Casau et al. [2019] guaranteeing ISS-like properties.

5. SAMPLE STABILIZERS DESIGN

In this section we present a sample selection of the basic components of our control design, namely functions $\gamma_R(\cdot)$, $\gamma_\omega(\cdot,\cdot,\cdot)$ and $\gamma_c(\cdot,\cdot,\cdot)$. Then, we prove that they satisfy Properties 1 and 2, thereby guaranteeing the applicability of Theorem 1. For the outer loop controller (7), we employ a widely adopted solution for attitude stabilization [Koditschek 1989]. As for the inner loop controller, we propose a structure involving an important linear dynamic component, and then use passivity arguments to satisfy Property 2. Our general selection includes robust control solutions such as PI control loops. As an important outcome of these results (and of Theorem 1) our hierarchical construction allows proving almost global tracking results in the presence of constant disturbances τ_e in (2), without imposing conditions on the PI controller gains [Maithripala and Berg 2015, Goodarzi et al. 2013].

5.1 Outer loop stabilizer

The stabilization of the attitude kinematics in a global sense is a challenging problem due to the non-Euclidean nature of SO(3) [Koditschek 1989]. Among the large number of (nonlinear) stabilizers that have been developed in the literature, we select one of the most common solutions, for which proving the aISS of Property 1 is straightforward, thanks to the tools developed in Angeli and Praly [2011].

Proposition 1. Given $\gamma_R(R_e) := -\frac{1}{2}S^{-1}(\mathrm{skew}(K_R R_e))$, where $K_R \in \mathbb{R}^{3\times 3}$ is a symmetric matrix with distinct eigenvalues satisfying $\mathrm{tr}(K_R)I_3 - K_R \in \mathbb{R}^{3\times 3}_{>0}$, Property 1 is satisfied. [2]

Proof. The proof stems from the results of [Angeli and Praly 2011, Proposition 2] combined with those of Koditschek [1989]. As shown in the latter, $\gamma_R(R_e)$ makes the equilibrium $R_e = I_3$ locally asymptotically stable for $\dot{R}_e = R_e \gamma_R(R_e)$ while the other equilibria $R_e \in \Omega_R := \{R \in \mathrm{SO}(3) \setminus I_3 : \gamma_R(R) = 0\}$ are isolated and exponentially unstable, *i.e.*, at least one eigenvalue of the tangent map of $\gamma_R(R_e)$ restricted to $R_e \in \Omega_R$ has positive real part. Thus, the sufficient conditions A1-A2 of [Angeli and Praly 2011, Proposition 2] are matched. Finally, aISS holds upon verification that the solutions to (9) are ultimately bounded according to [Angeli and Praly 2011, Definition 2], which is trivial for our case where the state evolves on a compact manifold. By [Angeli and Praly 2011, Proposition 2], subsystem (9) with $\gamma_R(\cdot)$ defined as in Proposition 1 is aISS with respect to ω_e and thereby satisfies Property 1. ◦

5.2 Inner loop stabilizer

The inner loop stabilizer comprises the stabilizer $\gamma_\omega(\cdot,\cdot,\cdot)$ for the angular velocity dynamics and the function $\gamma_c(\cdot,\cdot,\cdot)$, which shapes the controller dynamics. Consider the selection

$$\gamma_c(x_c, \omega_e, \omega_v) := A_c x_c + B_c \omega_e \tag{18}$$

$$\gamma_\omega(x_c, \omega_e, \omega_v) := C_c x_c + D_c \omega_e + K_\omega \omega_e + S(\omega_e) J(\omega_v - \omega_e) \tag{19}$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times 3}$, $C_c \in \mathbb{R}^{3 \times n_c}$, $D_c \in \mathbb{R}^{3\times 3}$ and $K_\omega \in \mathbb{R}^{3\times 3}$. The control torque obtained by substituting (19) in (5), *i.e.*, $\tau_c = S(\omega_v)J\omega + J\dot{\omega}_v + C_c x_c + (D_c + K_\omega)\omega_e$, does not cancel the gyroscopic term $-S(\omega)J\omega$: this may be helpful in practical implementations to reduce the risk of actuators saturation, *e.g.*, when the desired angular velocity ω_d is much smaller than the initial angular velocity. It is emphasized that the dynamics in (18)-(19) comprises some linear dynamics. The following proposition proves Property 2 whenever that linear dynamics is passive (a special case of this being PI controllers).

Proposition 2. Consider the inner loop controller (5),(6) with the selections (18)-(19). Suppose that $G_\omega(s) := C_c(sI_{n_c} - A_c)^{-1}B_c + D_c$ is a $n_c \times n_c$ positive real transfer function, where (A_c, B_c) and (C_c, A_c) are controllable and observable pairs, respectively. [3] Then, for any $K_\omega \in \mathbb{R}^{3\times 3}_{>0}$ and any $\omega_M > 0$, Property 2 is satisfied.

Proof. When using selection (18)-(19), the differential inclusion in (15) can be written as

$$\begin{bmatrix} \dot{\omega}_e \\ \dot{x}_c \end{bmatrix} \in \bigcup_{\|\omega_v\| \le \omega_M} \begin{bmatrix} J^{-1}(S(\omega_e)^\top J(\omega_v - \omega_e) - C_c x_c - (D_c + K_\omega)\omega_e) \\ A_c x_c + B_c \omega_e \end{bmatrix} =: F_\omega, \tag{20}$$

[2] K_R having distinct eigenvalues guarantees isolated solutions to $\gamma_R(R_e) = 0$, a condition required to use the arguments in Angeli and Praly [2011] to prove aISS. The case $K_R = k_R I_3$, $k_R \in \mathbb{R}_{>0}$, has been analyzed in Vasconcelos et al. [2011], whose results can be used to extend the proof of Proposition 1 also for this gain selection.

[3] Note that assuming controllability and observability of (A_c, B_c) and (C_c, A_c) is without loss of generality, as a matter of fact any unobservable or uncontrollable subcomponents would not cause any effect on the closed loop, besides possibly perturbing the initial transient.

indeed F_ω is already convex because ω_v enters linearly in the dynamics. To prove our proposition we rely on an invariance principle for differential inclusions similar to [Invernizzi et al. 2020, Proposition 3], based on the results in Seuret et al. [2019]. Consider the following Lyapunov candidate $V_\omega := \frac{1}{2}\omega_e^\top J\omega_e + \frac{1}{2}x_c^\top P x_c$, where $P \in \mathbb{R}^{n_c\times n_c}_{>0}$. Next, define $\dot{V}_\omega(\omega_e,x_c) := \max_{f\in F_\omega(\omega_e,x_c)} \langle \nabla V(\omega_e,x_c), f\rangle$. From (20), we obtain:

$$\begin{aligned}\dot{V}_\omega(\omega_e,x_c) &= \omega_e^\top(-C_c x_c - (D_c+K_\omega)\omega_e)\\ &\quad + \frac{1}{2}x_c^\top\left(PA_c^\top + A_c^\top P\right)x_c + x_c^\top PB_c\omega_e\end{aligned} \tag{21}$$

where we exploited the property $\omega_e^\top S(\omega_e)^\top J(\omega_v-\omega_e) = -\omega_e^\top(\omega_e \times J(\omega_v-\omega_e)) = 0\ \forall\omega_v$. Since $G_\omega(s)$ is positive real, from the positive real lemma [Khalil 2002, Lemma 6.2], there exist $P\in\mathbb{R}^{n_c\times n_c}_{>0}$, L, W such that $PA_c^\top + A_c^\top P = -L^\top L$, $PB_c = C_c^\top - L^\top W$ and $W^\top W = D_c + D_c^\top$. Hence, equation (21) can be written as:

$$\begin{aligned}\dot{V}_\omega(\omega_e,x_c) &= -\omega_e^\top K_\omega\omega_e - \omega_e^\top C_c x_c - \frac{1}{2}\omega_e^\top W^\top W\omega_e\\ &\quad -\frac{1}{2}x_c^\top L^\top L x_c + x_c^\top C_c^\top\omega_e - x_c^\top L^\top W\omega_e\\ &= -\omega_e^\top K_\omega\omega_e - \frac{1}{2}(Lx_c + W\omega_e)^\top(Lx_c+W\omega_e) \le 0,\end{aligned} \tag{22}$$

$\forall(\omega_e,x_c)\in\mathbb{R}^3\times\mathbb{R}^{n_c}$. Combining this result and the fact that V_ω is radially unbounded, compact sets of the form $\Omega_c := \{(\omega_e,x_c)\in\mathbb{R}^3\times\mathbb{R}^{n_c} : V_\omega(\omega_e,x_c)\le c\}$, $c\in\mathbb{R}_{>0}$, are forward invariant. Finally, since the viability condition of [Goebel et al. 2012, Pag. 124] is satisfied, solutions starting in Ω_c are complete. Consider now any solution $t\mapsto\phi(t) := (\omega_e(t),x_c(t))$ starting in Ω_c at $t=t_0\ge 0$ such that $V_\omega(\phi(t_0)) = a \ne 0$. Then, if $\phi(t_0)\in\{(\omega_e,x_c)\in\mathbb{R}^3\times\mathbb{R}^{n_c} : \omega_e\ne 0\}$, the function $V_\omega(\phi(t))$ has to decrease in time by continuity and by the negative upper bound in (22). Instead, if $\phi(t_0)\in\mathcal{G} := \{(\omega_e,x_c)\in\mathbb{R}^3\times\mathbb{R}^{n_c} : \omega_e = 0\}\setminus\{(0,0)\}$, then, according to the closed-loop dynamics restricted to $\mathcal{G}$,

$$\begin{bmatrix}\dot{\omega}_e\\ \dot{x}_c\end{bmatrix} \in F_\omega(\omega_e,x_c)|_{\mathcal{G}} = \bigcup_{\|\omega_v\|\le\omega_M}\begin{bmatrix}-J^{-1}C_c x_c\\ A_c x_c\end{bmatrix} = \begin{bmatrix}-J^{-1}C_c x_c\\ A_c x_c\end{bmatrix} \tag{23}$$

$\phi(t)$ will exit the set $\mathcal{G}$ for some small $t\ge t_0$. Indeed, if the solution were to evolve in $\mathcal{G}$, due to linearity of the dynamics (23), $x_c(t) = \exp(A_c(t-t_0))x_c(t_0)$, and remaining in $\mathcal{G}$ would also impose $J\dot{\omega}_e(t) = C_c\exp(A_c(t-t_0))x_c(t_0) = 0$ for all $t\ge 0$. Since (C_c,A_c) is observable, this establishes a contradiction, thus proving all the items of the invariance principle and establishing Property 2.

Example 1. (PI-like controller.) Consider the dynamic part of the controller corresponding to an integral action, *i.e.*, $A_c = 0$, $B_c = I_3$, $C_c = \mathrm{diag}(k_{I_1},k_{I_2},k_{I_3})\in\mathbb{R}^{3\times 3}_{>0}$, $D_c = 0$. Due to the decoupled structure for each axis, checking positive realness (as required by Proposition 2) is trivial because $G_\omega(s)$ is diagonal and each element $G_{\omega_{ii}}(s) = k_{I_i}/s$ is positive real. The overall control law corresponds to a PI-like loop, comprising the proportional term K_ω. This solution is particularly interesting as it allows rejecting a constant unknown torque $\tau_e\in\mathbb{R}^3$ in (2). This can be shown by referring to the perturbed closed-loop dynamics $J\dot{\omega}_e = S(\omega_e)^\top J(\omega_v-\omega_e) - C_c x_c - K_\omega\omega_e + \tau_e$ and by applying the change of coordinates $\tilde{x}_c = x_c - C_c^{-1}\tau_e$: the closed-loop dynamics in the new coordinates has the same structure as equation (20). Hence, we can conclude that $(\omega_e,x_c) = (0,C_c^{-1}\tau_e)$ is GAS from the proof of Proposition 2.

Remark 7. More complex inner loop controllers achieving desirable performance and disturbance rejection capabilities could be considered. For instance, PID control, typically employed in aircraft applications and multirotor UAV control [PX4-Community 2018], fits within the class of controllers outlined in Proposition 2 . Disturbance rejection of harmonic disturbances with known frequency but unknown phase and amplitude could be also considered, by including an internal model along the lines of Camblor et al. [2015].

6. NUMERICAL RESULTS

We present a simulation example to show that the control law (5)-(6) with the inner and outer loop stabilizers selected as in Propositions 1 and 2 solves the robust tracking problem in the presence of constant disturbances τ_e in (2). We employ a PI-like controller for the inner loop corresponding to the selection $A_c = 0$, $B_c = I_3$, $C_c = \mathrm{diag}(k_{I_1},k_{I_2},k_{I_3})\in\mathbb{R}^{3\times3}_{>0}$, $D_c = 0$, $K_\omega = \mathrm{diag}(k_{\omega_1},k_{\omega_2},k_{\omega_3})\in\mathbb{R}^{3\times3}_{>0}$, from which we expect asymptotic tracking in the considered scenario, following Example 1.

The desired attitude motion is characterized by a polynomial spin up maneuver with a steady-state angular velocity $\omega_{d_{ss}} = (0.5,0.5,0.5)\mathrm{rad/s}$ (see Figure 3). The initial state that we select corresponds to a largely misplaced configuration $R_e(0) = I_3 + \sin(\pi)S(e_1) + (1-\cos(\pi))S(e_1)^2$ and to a significant angular velocity error $e_\omega(0) := \omega(0) - R_e^\top(0)\omega_d(0) = (3,3,3)\mathrm{rad/s}$ with respect to the target velocity. The rigid body that we use in the numerical simulation has inertia matrix $J = \mathrm{diag}(1,2,3)\mathrm{kgm}^2$ and is affected by a piece-wise constant disturbance torque $\tau_e = (1,1,1)\mathrm{Nm}$ for $t < 15\mathrm{s}$ and $\tau_e = (3,3,3)\mathrm{Nm}$ for $t\ge 15\mathrm{s}$. The gains of the controller are tuned to have a predefined behavior of the closed-loop system in the proximity of the desired attitude motion. Specifically, the gains of the inner loop stabilizers $G_{c_i}(s) = k_{\omega_i} + \frac{k_{I_i}}{s}$ are chosen to have the linearized inner loop error dynamics [4], namely, $J\dot{\omega}_{e_i} = -k_{I_i}x_i - k_{\omega_i}\omega_{e_i}$, $\dot{x}_{I_i} = \omega_{e_i}$, behaving like first order systems with bandwidth 10rad/s ($k_{\omega_3} = 2k_{\omega_2} = k_{\omega_1} = k_{I_3} = 2k_{I_2} = 3k_{I_1} = 3.33$). The outer loop gains are tuned to have a 1rad/s bandwidth for the linearized outer loop error [5] dynamics ($K_R = \mathrm{diag}(1,1.001,0.999)$). The attitude tracking performance of the proposed controller $\tau_c^{\mathrm{prop}} := S(\omega_v)J\omega + J\dot{\omega}_v + C_c x_c + K_\omega\omega_e$ is illustrated in Figure 4 (black dotted line) where it is compared to the results obtained with a feedback linearizing version of the control law (red solid line), namely, $\tau_c^{\mathrm{fl}} := S(\omega)J\omega + J\dot{\omega}_v + C_c x_c + K_\omega\omega_e)$ and with a saturated version (blue dash-dotted line) $\tau_c^{\mathrm{sat}} := \sigma_M(\tau_c^{\mathrm{prop}})$ with bound $M = 3\mathrm{Nm}$ on each component. [6] Finally, we also considered for comparison a PD-like controller $\tau_c^{\mathrm{pd}} := S(R_e^\top\omega_d)JR_e^\top\omega_d + JR_e^\top\dot{\omega}_d + \gamma_R(R_e) + K_\omega(R_e^\top\omega_d - \omega)$ (magenta dotted line) borrowed from the literature, tuned to have a similar transient response ($K_R = \mathrm{diag}(25,12.5,0)$, $K_\omega = \mathrm{diag}(10,20,30)$). From Figures 4-5, one sees that the feedback linearizing controller achieves a faster tracking at the expense of a larger actuation

[4] Note that we assumed $\omega_d = \mathrm{const}\approx 0$.

[5] Considering small error angles $\theta_e\in\mathbb{R}^3$ such that $R_e\approx I_3 + S(\theta_e)$, the linearized outer loop error dynamics is given by $\dot{\theta}_e = -\tilde{K}_R\theta_e$, where $\tilde{K}_R := \frac{1}{2}\mathrm{diag}\left(k_{R_2}+k_{R_3}, k_{R_1}+k_{R_3}, k_{R_1}+k_{R_2}\right)$. Slightly different gains are used to respect the conditions of Proposition 1. See Footnote 2 about the use of equal gains in the attitude stabilizer $\gamma_R(\cdot)$.

[6] $\sigma_M(x) := (\mathrm{sat}_M(x_1),\mathrm{sat}_M(x_2),\mathrm{sat}_M(x_3))$ for $x := (x_1,x_2,x_3)$, where $\mathrm{sat}_M(\cdot) := \min(\max(\cdot,-M),M)$ for $M\in\mathbb{R}_{>0}$

effort than the other PI-like controllers. The PD-like controller cannot compensate for the disturbance effects as shown by the non-null steady state error in Figure 4 and furthermore, it requires much more actuation effort than the other controllers while showing a similar transient (Figure 5). All the PI-like controllers promptly react to the abrupt change of the disturbance torque occurring at $t = 15$s. It is worth mentioning that the saturated controller is capable of asymptotically tracking the desired motion but has the worst transient performance.

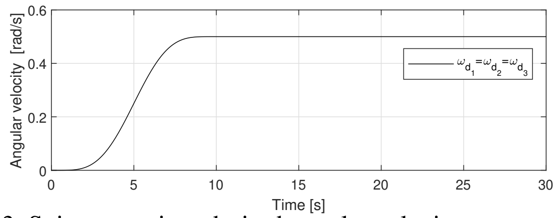

Fig. 3. Spin up motion: desired angular velocity ω_d.

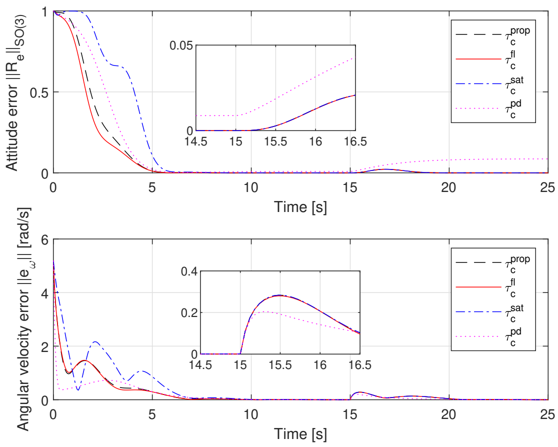

Fig. 4. Tracking errors: attitude $\|R_e\|_{SO(3)}$ (top) and angular velocity error $\|e_\omega\|$ (bottom).

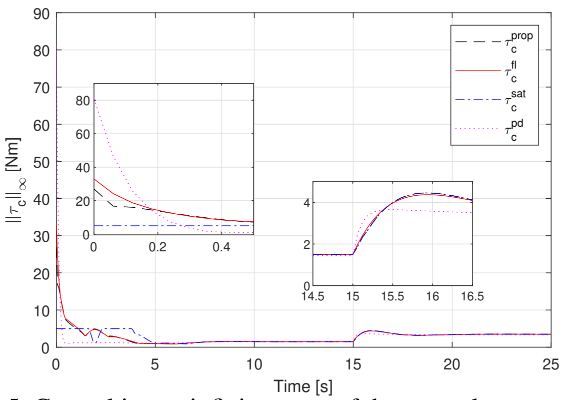

Fig. 5. Control input: infinity norm of the control torque τ_c.

7. CONCLUSIONS

In this paper we revisited the attitude tracking control problem and proposed an inner-outer loop design achieving almost global tracking results. By means of the proposed approach one can exploit dynamic controllers for the inner loop, such as PID-like loops, to achieve desirable performance and disturbance rejection capabilities. Future work is oriented to obtaining global tracking results and to showing the potential of the proposed approach in handling model uncertainties and saturation effects.

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