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Local state-feedback stabilization of continuous-time Takagi-Sugeno fuzzy systems

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Abstract: This paper deals with the problem of local state-feedback stabilization for continuous-time nonlinear systems represented by Takagi–Sugeno (T–S) fuzzy models. The approach is based on a polytopic representation for the gradient of the membership functions but, differently from most of the available methods, bounds for the time-derivatives of the membership functions are not required. A two step strategy is proposed for the control design. First, a sufficient condition provides a stabilizing state-feedback gain for the dual system. Although there is no guarantee of stability for the original system, the controller is used as an initial condition for the second step of the method. If a feasible solution is found, a stabilizing state-feedback controller and an estimate of the domain of attraction are certified by means of a fuzzy Lyapunov function with polynomial dependence on the membership functions. The proposed conditions, given in terms of parameter-dependent linear matrix inequalities (LMIs) with a scalar search, can be solved by LMI relaxations with optimization variables considered as homogeneous polynomials of fixed degree. Examples based on T–S models borrowed from the literature illustrate that the method performs better than other existing approaches in terms of providing stabilizing gains associated with larger estimates for the domain of attraction.

Keywords: Continuous-time Takagi-Sugeno fuzzy systems; local state-feedback stabilization; fuzzy Lyapunov functions; region of attraction; LMI relaxations

1. INTRODUCTION

The control of nonlinear systems based on Takagi-Sugeno (T–S) fuzzy models (Takagi and Sugeno, 1985) has been an important topic of research in the last two decades. Thanks to the exact representation of nonlinear systems in a compact set of the state-space, by the so-called *sector nonlinearity* approach (Tanaka and Wang, 2001), stability analysis and control design can be performed using quadratic Lyapunov functions, that also provide level sets yielding compact and invariant regions in the state space as estimates of the domain of attraction of the origin. Most of the works in the T–S literature performs the stability analysis and the control synthesis by means of quadratic Lyapunov functions with a fixed (membership-independent) Lyapunov matrix, yielding the so-called Parallel Distributed Compensation (PDC) control law (see Feng (2006) and references therein). The main difficulty of using fuzzy (i.e., membership-dependent) Lyapunov functions (Tanaka et al., 2003) comes from the presence of the time-derivatives of the membership functions (that depend on the states) in the design conditions. The simplest approach is to guess upper bounds for the time-derivatives of the membership functions that hold in the domain of validity of the T–S model (Lam, 2009; Mozelli et al., 2009; Tognetti et al., 2011; Xie et al., 2015). However, this strategy is not efficient in the design of controllers, since the actual bounds cannot be known in advance and, therefore, an *a posteriori* test must be performed to verify if the domain of validity of the model is entirely contained in the region where such bounds are valid.

In a sense, this drawback is overcome in the works of Lee et al. (2012); Pan et al. (2012b). In Lee et al. (2012); Lee and Kim (2014), the intersection of the domain of validity of the T–S model with the space defined by the bounds on the time-derivatives of the membership functions is taken into account in the design conditions, providing sharper estimates for the domain of attraction. However, there is no recipe for guessing the values of the upper bounds used in the design. In Pan et al. (2012b,a); Márquez et al. (2017), guesses for the upper bounds are proposed, while indirect bounds are established in Bernal and Guerra (2010); Guerra et al. (2012) by considering constrained input signals. Although some works concerning stability analysis avoid the use of bounds on the time-derivative of the membership functions (Lee et al., 2014; Gomes et al., 2019), extensions for control design are still an open issue.

This paper follows a different strategy to cope with the control design problem for continuous-time T–S fuzzy systems, in the sense that the explicit knowledge of bounds on the time-derivative of membership functions is not required. Local stability analysis conditions from (Lee et al., 2014; Campos, 2015; Gomes et al., 2019), that explicitly consider a polytope representing the domain of validity of the T–S model in the stability

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conditions, are extended to provide synthesis conditions for the T–S fuzzy system. Due to the difficulty of manipulating the conditions in order to obtain a linear matrix inequality (LMI) optimization problem for control design, a two-step procedure is proposed. At the first step, a stabilizing state-feedback control law is computed for the dual system. Although there exists no guarantee of stability for the original system, the obtained controller is used as input data in a second condition that, if satisfied, assures the closed-loop stability of the T–S fuzzy original system, providing a stabilizing state-feedback controller and an estimate of the domain of attraction inside the region of validity of the model. Both steps are solved by means of LMI relaxations based on homogeneous polynomial matrices of arbitrary degree. In the first step a scalar parameter is introduced as an extra degree of freedom, and in the second step slack variables help to reduce the conservatism for the estimation of the domain of attraction. Numerical examples from the literature illustrate that the proposed technique can provide better results in terms of larger estimates for the domain of attraction when compared with other approaches, with the additional advantage of not requiring the knowledge of bounds for the time-derivatives of the membership functions.

Notation: The set of real matrices with dimension n rows and m columns is denoted by $\mathbb{R}^{n\times m}$, M^T means the transpose of matrix M and $\mathrm{He}(M) = M + M^T$. $M > 0$ ($M < 0$) indicates that matrix M is positive (negative) definite, the symbol $\star$ represents a symmetric block in a matrix.

2. PRELIMINARIES

Consider the nonlinear system

$$\dot{x} = f(x)x + g(x)u \tag{1}$$

where the origin is as equilibrium point for $u = 0$, that is, $f(0) = 0$, $x \in \mathbb{R}^n$, and $f(\cdot)$, $g(\cdot)$ are assumed to be bounded and smooth in a compact set of the state-space. Using the sector nonlinearity approach (Tanaka and Wang, 2001), the nonlinear system (1) can be represented in an exact way by means of the T–S fuzzy system

$$\dot{x} = A(\alpha(z))x + B(\alpha(z))u, \quad \forall x \in \mathscr{X}, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $A(\alpha(z)) \in \mathbb{R}^{n\times n}$, $B(\alpha(z)) \in \mathbb{R}^{n\times m}$ and $\mathscr{X}$ is a compact set in the state-space. Variable $z \in \mathbb{R}^p$, the *premise variable* of the T–S fuzzy model, depends on the states (i.e., $z = z(x)$) and represents the p nonlinearities of the original system. Moreover,

$$A(\alpha(z)) = \sum_{i=1}^{N} \alpha_i(z)A_i, \quad B(\alpha(z)) = \sum_{i=1}^{N} \alpha_i(z)B_i,$$

where A_i, B_i, $i = 1,\ldots,N$ are supposed to be known and $\alpha(z) = [\alpha_1(z)\cdots\alpha_N(z)]^T \in \Lambda_N$,

$$\Lambda_N = \left\{\xi \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0,\ i = 1,\ldots,N\right\}.$$

The weighting functions $w_{\ell j}$, $\ell = 1,\ldots,p$, $j = 1,2$, are given by

$$w_{\ell 1}(z_\ell) = (z_\ell(x) - \overline{z}_\ell)/(\overline{z}_\ell - \underline{z}_\ell), \quad w_{\ell 2}(z_\ell) = 1 - w_{\ell 1}(z_\ell),$$

where $\overline{z}_\ell = \max(z_\ell)$ and $\underline{z}_\ell = \min(z_\ell)$. Then, the entries of the membership function $\alpha(z)$ are constructed as

$$\alpha_i(z) = \prod_{\ell=1}^{p} w_{\ell j}(z_\ell),\ i \in \{1,\ldots,N\},\ j \in \{1,2\},\ N = 2^p.$$

The T–S fuzzy model (2) is valid inside the following polytope (Boyd et al., 1994)

$$\mathscr{X} = \left\{x \in \mathbb{R}^n : a_k^T x \leq c_k,\ \ k = 1,\ldots,q\right\} \tag{3}$$

where $a_k \in \mathbb{R}^n$ and $c_k \in \mathbb{R}$, $k = 1,\ldots,q$, are given vectors, with the linear constraints that define $\mathscr{X}$ guaranteeing that $0 \in \mathscr{X}$. The set $\mathscr{X}$ can alternatively be written as

$$\mathscr{X} = \mathrm{co}\{h^1, h^2, \ldots, h^\kappa\}, \tag{4}$$

where the vectors h^i, $i = 1,\ldots,\kappa$ can be systematically obtained through the linear constraints in (3) using, for instance, a vertex enumeration algorithm (Avis and Fukuda, 1992). Thus, any $x \in \mathscr{X}$ can be represented by

$$x_\gamma = \sum_{k=1}^{\kappa} \gamma_k h^k, \qquad \gamma \in \Lambda_\kappa. \tag{5}$$

Since the premise variables depend on the states, one has

$$\dot{\alpha}(z) = J(\theta)\dot{x}, \quad J(\theta) = \nabla_x \alpha(z) = \sum_{i=1}^{\vartheta} \theta_i(x)J_i \tag{6}$$

where $\theta(x) \in \Lambda_\vartheta$,

$$[\nabla_x \alpha(z)]_{ij} = \frac{\partial \alpha_i}{\partial x_j}$$

and J_i are matrices obtained from the knowledge of $\alpha(z)$ and set $\mathscr{X}$ (Tanaka and Wang, 2001).

The asymptotic stability of the origin of system (2) (with $u = 0$) can be investigated by means of a fuzzy Lyapunov function [1] given by

$$V(x,\alpha) = x^T P(\alpha)x \tag{7}$$

where $P(\alpha)$ is a positive definite symmetric fuzzy Lyapunov matrix to be determined. The result is presented in next lemma.

Lemma 1. If there exists a parameter-dependent matrix $P(\alpha) = P(\alpha)^T > 0$ such that

$$\mathrm{He}\Big(A(\alpha)^T P(\alpha) + (1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)\Big) < 0, \tag{8}$$

where

$$\nabla_{\alpha\gamma}P(\alpha) = \left[\frac{\partial P(\alpha)x_\gamma}{\partial \alpha_1} \cdots \frac{\partial P(\alpha)x_\gamma}{\partial \alpha_N}\right], \tag{9}$$

holds for all $\alpha \in \Lambda_r$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\nu$, then the origin of system (2) is asymptotically stable.

Proof. For system (2) with $u = 0$, the time-derivative of the Lyapunov function in (7) yields

$$\begin{aligned}\dot{V}(x,\alpha) &= \dot{x}^T P(\alpha)x + x^T P(\alpha)\dot{x} + x^T \dot{P}(\alpha)x \\ &= x^T\left(A(\alpha)^T P(\alpha) + P(\alpha)A(\alpha) + \dot{P}(\alpha)\right)x\end{aligned}$$

Considering only the quadratic term involving $\dot{P}(\alpha)$, using (5), (6) and (9), one has

$$\begin{aligned}x^T \dot{P}(\alpha)x &= x^T\left(\frac{\partial P(\alpha)}{\partial \alpha_1}\dot{\alpha}_1 + \cdots + \frac{\partial P(\alpha)}{\partial \alpha_N}\dot{\alpha}_N\right)x \\ &= x^T\left(\sum_{i=1}^{N} \dot{\alpha}_i \frac{\partial P(\alpha)}{\partial \alpha_i}\right)x = \sum_{i=1}^{N} x^T\left(\frac{\partial P(\alpha)}{\partial \alpha_i}\right)x\dot{\alpha}_i \\ &= x^T \nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)x \\ &= x^T\left(\mathrm{He}\big((1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)\big)\right)x\end{aligned} \tag{10}$$

Therefore, if condition (8) holds with $P(\alpha) = P(\alpha)^T > 0$ for all $\alpha \in \Lambda_r$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\nu$, one has

$$\begin{aligned}\dot{V}(x,\alpha) = x^T\big(\mathrm{He}\big(&A(\alpha)^T P(\alpha) \\ &+ (1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)\big)\big)x < 0\end{aligned}$$

[1] The dependence of $\alpha(z)$ on z is omitted for brevity.

and the proof is completed.

Note that in the third step presented in (10), the state x (that multiplies $\dot{\alpha}_i$) has been replaced by x_γ, valid inside $\mathscr{X}$, which is the main feature of the approach. Remark also that, in order to search for a feasible solution for Lemma 1, the structure of $P(\alpha)$ must be defined (see Section 4 for a discussion concerning numerical implementation of the proposed conditions).

The problem to be addressed in this paper is: find a state-feedback control law $u = K(\alpha)x$ such that the origin of the closed-loop T–S fuzzy system

$$\dot{x} = \big(A(\alpha)+B(\alpha)K(\alpha)\big)x, \quad \forall x \in \mathscr{X}, \tag{11}$$

is asymptotically stable, also providing an estimate for the region of attraction inside $\mathscr{X}$.

3. RESULTS

The first contribution of the paper is a local state-feedback synthesis condition assuring the asymptotic stability of the origin of the dual system of (11).

Theorem 1. If there exist parameter-dependent matrices $P(\alpha) = P(\alpha)^T > 0$, $L(\alpha)$, a matrix S and a scalar ξ such that

$$\begin{bmatrix} \mathrm{He}\left(A(\alpha)P(\alpha)+(1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)^T+B(\alpha)L(\alpha)\right) & \star \\ L(\alpha)^T B(\alpha)^T & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} 0 \\ \xi I \end{bmatrix}\begin{bmatrix} P(\alpha)+(1/2)J(\theta)^T\nabla_{\alpha\gamma}P(\alpha)^T - S & -S \end{bmatrix}\right) < 0 \tag{12}$$

holds for all $\alpha \in \Lambda_N$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\kappa$, then $K_d(\alpha) = L(\alpha)S^{-1}$ is a state-feedback gain assuring that the asymptotic stability of the origin of

$$\dot{v} = \big(\underbrace{A(\alpha)+B(\alpha)K_d(\alpha)}_{A_{cl_d}(\alpha)}\big)^T v, \quad \forall v \in \mathscr{X}. \tag{13}$$

Proof. Multiplying (12) on the right by $\mathscr{B}_\perp$ and on the left by $\mathscr{B}_\perp^T$, with

$$\mathscr{B}_\perp = \begin{bmatrix} I \\ S^{-1}\left(P(\alpha)+(1/2)J(\theta)^T\nabla_{\alpha\gamma}P(\alpha)^T\right) - I \end{bmatrix}$$

one has

$$\mathrm{He}\Big(A(\alpha)P(\alpha)+(1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A(\alpha)^T + B(\alpha)L(\alpha)S^{-1}P(\alpha) + (1/2)B(\alpha)L(\alpha)S^{-1}J(\theta)^T\nabla_{\alpha\gamma}P(\alpha)^T\Big) < 0$$

Considering $K_d(\alpha) = L(\alpha)S^{-1}$ yields

$$\mathrm{He}\left(A_{cl_d}(\alpha)P(\alpha)+(1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A_{cl_d}(\alpha)^T\right) < 0$$

and, therefore,

$$\mathrm{He}\left(A_{cl_d}(\alpha)P(\alpha)\right)+\dot{P}(\alpha) < 0$$

which proves the asymptotic stability of the origin of system (13).

The first remark about the conditions of Theorem 1 is that the proposed algebraic manipulations would not work (at least, not in an easy way) if the primal system had been considered. In that case, congruence transformations involving the inverse of the Lyapunov matrix would be necessary, and the manipulation of $\dot{P}(\alpha)$, using the model given in (10), would be much more difficult to cope with. The different strategy used in this paper, that is, to address the stabilization of the dual system (13), is motivated by the fact that the stability of the original system (11) is equivalent in the case of membership functions with arbitrary rates of variation (Hu and Blanchini, 2010). Although nothing can be guaranteed about the equivalence in the case of bounded rates, the numerical experiments show that this heuristic provides a good starting point for the main result of the paper, presented next.

Theorem 2. Let $K_d(\alpha)$ be a parameter-dependent state-feedback gain. If there exist parameter-dependent matrices $P(\alpha) = P(\alpha)^T > 0$, $L(\alpha)$, $F(\alpha)$, $G(\alpha)$, and a matrix H such that

$$Q + \mathrm{He}\big(X\mathscr{B}\big) < 0, \tag{14}$$

holds for all $\alpha \in \Lambda_N$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\kappa$, with Q, X and $\mathscr{B}$ given in (15)-(16) (top of next page), then $K(\alpha) = H^{-1}L(\alpha)$ is a state-feedback gain that assures that the origin of the closed-loop T–S fuzzy system (11) is asymptotically stable.

Proof. Multiplying (14) on the right by $\mathscr{B}_\perp$ and on the left by $\mathscr{B}_\perp^T$ with

$$\mathscr{B}_\perp = \begin{bmatrix} I & 0 \\ 0 & I \\ H^{-1}L(\alpha)-K_d(\alpha) & 0 \end{bmatrix}.$$

Adopting $K(\alpha) = H^{-1}L(\alpha)$ and $A_{cl}(\alpha) = A(\alpha)+B(\alpha)K(\alpha)$, one has

$$\begin{bmatrix} \mathrm{He}\big(A_{cl}(\alpha)^T F(\alpha)^T + (1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A_{cl}(\alpha)\big) & \star \\ P(\alpha)-F(\alpha)^T+G(\alpha)A_{cl}(\alpha) & -G(\alpha)-G(\alpha)^T \end{bmatrix} < 0 \tag{17}$$

Multiplying (17) by $\mathscr{T}$ on the left and by $\mathscr{T}^T$ on the right with

$$\mathscr{T} = \begin{bmatrix} I & A_{cl}(\alpha)^T \end{bmatrix}$$

yields

$$\mathrm{He}\left(A_{cl}(\alpha)^T P(\alpha)+(1/2)\nabla_{\alpha\gamma}P(\alpha)J(\theta)A_{cl}(\alpha)\right) < 0$$

and, therefore,

$$\mathrm{He}\left(A_{cl}(\alpha)^T P(\alpha)\right)+\dot{P}(\alpha) < 0$$

which proves that $K(\alpha)$ is a stabilizing gain. On the other hand, performing the same steps but using

$$\mathscr{B}_\perp^T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

leads to $\mathrm{He}\big(A_{cl_d}(\alpha)^T P(\alpha)\big)+\dot{P}(\alpha) < 0$, implying that the Lyapunov matrix $P(\alpha)$ also proves that $K_d(\alpha)$ is a stabilizing gain.

If the conditions of Theorem 2 have a feasible solution, the synthesized gain $K(\alpha)$ is actually stabilizing, certified by the Lyapunov matrix $P(\alpha)$. Moreover, interestingly, the same Lyapunov matrix also proves that the gain given *a priori* $K_d(\alpha)$ (computed for the dual system) is also stabilizing. As a consequence, the estimate for the region of attraction, proposed in the sequence, is valid for both gains.

As an interesting property of Theorem 2, note that the resulting gain $K(\alpha)$ could be used as a new input data, giving rise to an iterative procedure, possibly improving the estimates for the domain of attraction.

Regarding the computation of an estimate for the domain of attraction, consider the set

$$\Omega = \left\{x \in \mathbb{R}^n : \ x^T P(\alpha)x \le 1\right\} \tag{18}$$

as the the largest invariant set contained in the polytope $\mathscr{X}$. The constraint $\Omega \in \mathscr{X}$ holds if $a_k^T P(\alpha)^{-1} a_k \le c_k^2$, $k =$

$$Q = \begin{bmatrix} \mathrm{He}\left(\left(A(\alpha)^T + K_d(\alpha)^T B(\alpha)^T\right) F(\alpha)^T + (1/2)\nabla_{\alpha\gamma} P(\alpha) J(\theta)\left(A(\alpha)+B(\alpha)K_d(\alpha)\right)\right) & \star & \star \\ P(\alpha) - F(\alpha)^T + G(\alpha)\left(A(\alpha)+B(\alpha)K_d(\alpha)\right) & -G(\alpha)-G(\alpha)^T & \star \\ B(\alpha)^T\left(F(\alpha)^T + (1/2)J(\theta)^T \nabla_{\alpha\gamma} P(\alpha)^T\right) & B(\alpha)^T G(\alpha)^T & 0 \end{bmatrix}, \quad (15)$$

$$X^T = [0 \ 0 \ I], \quad \mathscr{B} = [L(\alpha) - HK_d(\alpha) \ 0 \ -H], \quad (16)$$

$1,\ldots,q$ (Boyd et al., 1994). Applying a Schur complement, one gets

$$\begin{bmatrix} P(\alpha) & a_k \\ a_k^T & c_k^2 \end{bmatrix} \geq 0, \qquad k = 1,\ldots,q, \ \forall \alpha \in \Lambda_N. \quad (19)$$

Among several other possibilities, the enlargement of Ω can be obtained by employing the following criterion (Kapila and Grigoriadis, 2002)

$$\min \mathrm{Tr}(W), \quad \text{s.t. (19)}, \ P(\alpha) \leq W, \quad (20)$$

Next corollary provides synthesis conditions associated with the maximization of Ω inside the domain of validity of the T–S fuzzy model.

Corollary 1. Let $K_d(\alpha)$ be a given parameter-dependent state-feedback gain. If there exist parameter-dependent matrices $P(\alpha) = P(\alpha)^T > 0$, $L(\alpha)$, $F(\alpha)$, $G(\alpha)$, and a matrix H that solve the optimization problem (20) subject to (14), for all $\alpha \in \Lambda_N$, $\theta \in \Lambda_\vartheta$ and $\gamma \in \Lambda_\kappa$, then $K(\alpha) = H^{-1}L(\alpha)$ is a state-feedback gain that assures that the origin of the closed-loop T–S fuzzy system (11) is asymptotically stable and $\Omega \subseteq \mathscr{X}$ given by (18) is an invariant set that estimates the domain of attraction.

4. FINITE DIMENSIONAL CONDITIONS

The conditions proposed in this paper are presented in terms of parameter-dependent LMIs (in Theorem 1, ξ must be given), which constitute infinite dimensional optimization problems. Solutions computed through a finite set of LMIs can be obtained by employing polynomial approximations for the optimization variables (Bliman, 2004; Oliveira and Peres, 2007). Once a fixed degree (on α) is chosen for all variables, the resulting polynomial inequalities (depending on α, θ and γ) can be tested using the well known “coefficients check approach”, which comprises the application of the Pólya’s relaxations for multiple simplexes (Oliveira et al., 2008). This procedure can be automatically performed by the Robust LMI Parser (Agulhari et al., 2019), which is also capable to compute the partial time-derivatives of $P(\alpha)$ (when a polynomial degree is imposed) with respect to each α_i. After the extraction of the LMIs, any semidefinite programming solver can be used. The notation $X_g(\alpha)$ means that the parameter-dependent matrix $X(\alpha) = X_g(\alpha)$ is a homogeneous polynomial matrix of degree g on α. All scripts implemented to obtain the numerical results presented in the next section were written in Matlab version 9.4.0.813654 (R2018a) employing YALMIP (Löfberg, 2004) and Mosek (Andersen and Andersen, 2000) through the interface ROLMIP (version 3.0) (Agulhari et al., 2019) in an Intel Core (TM) i7-7700 CPU @ 3.60GHz x 8 computer with 16GB RAM.

5. NUMERICAL EXPERIMENTS

Example 1 Consider the nonlinear system

$$\dot{x} = \begin{bmatrix} -\frac{2+a}{2} + \frac{2-a}{2}\sin(x_1) & -4 \\ \frac{19}{2} - \frac{21}{2}\sin(x_1) & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \frac{10+b}{2} + \frac{10-b}{2}\sin(x_1) \end{bmatrix} u$$

that can be exactly represented as the T–S fuzzy system (2) in the compact set $\mathscr{X} = \{x \in \mathbb{R}^2 : |x_i| \leq \pi/2, i = 1,2\}$, with

$$A_1 = \begin{bmatrix} -a & -4 \\ -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$\alpha_1(z) = \frac{1+\sin(x_1)}{2}, \quad \alpha_2(z) = 1 - \alpha_1(z)$$

The choices $a = -4$ and $b = 1$ reproduce (Lee et al., 2012, Example 7) and (Lee and Kim, 2014, Example 3). Figure 1 presents the estimate of the invariant region for the closed-loop system using Corollary 1, when $K_d(\alpha)$ is computed with Theorem 1. In this scenario, the degree g of $P_g(\alpha)$ and $L_g(\alpha)$ is set to $g = 3$, using $\xi = 10$ in the first stage and with Corollary 1 as the second stage. To illustrate the dynamic behavior of the T–S system, a few trajectories for different initial conditions are also plotted.

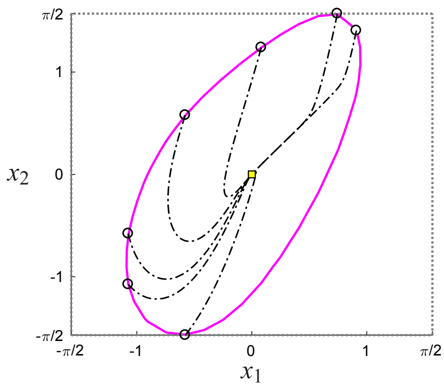

Fig. 1. Estimate of the domain of attraction obtained with the proposed method (solid magenta), domain of validity of the model (dotted gray), and trajectories (dot-dashed black) starting in ∘ and ending at the origin (yellow square) for *Example 1*.

Figure 2 and Table 1 show a comparison in terms of the estimates for the domain of attraction obtained with the proposed method, the ones from (Lee et al., 2012, Theorem 6) (using $\phi_1 = \phi_2 = 5$) and (Lee and Kim, 2014, Theorem 2) (using degree $q = 3$ and $\phi = \phi^* = 172.0994$) and the corresponding areas. Table 1 also shows the number of scalar variables V and of LMI rows R for each method (including Theorem 1). No feasible solution has been obtained with the conditions from (Pan et al., 2012b, Theorem 2) for this example.

As can be seen in Table 1, the proposed approach provides stabilizing gains associated with better estimates for the domain

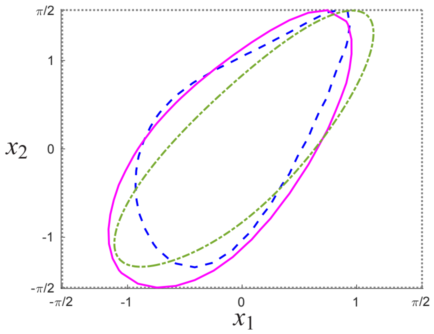

Fig. 2. Estimates of the domain of attraction obtained with the proposed method (solid magenta), (Lee et al., 2012, Theorem 6) (dashed blue), and (Lee and Kim, 2014, Theorem 2) (dot-dashed green), and domain of validity of the model (dotted gray), *Example 1*.

Table 1. Areas of the estimates of the domain of attraction for *Example 1*, number V of scalar variables and R of LMI rows for each condition.

	Area	V	R
Theorem 1	–	24	168
Corollary 1	3.9215	40	392
(Lee et al., 2012, Theorem 6)	2.9982	44	30
(Lee and Kim, 2014, Theorem 2)	2.9432	129	180

of attraction, at the price of increasing (in terms of V and R) the computational complexity.

Example 2 The nonlinear system

$$\dot{x} = \begin{bmatrix} -\frac{7}{2} - \frac{3}{2}\sin(x_1) & -4 \\ \frac{19}{2} - \frac{21}{2}\sin(x_1) & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{13}{2} + \frac{7}{2}\sin(x_1) \end{bmatrix} u$$

can be exactly represented as the T–S fuzzy system (2) in the compact set $\mathscr{X} = \{x \in \mathbb{R}^2 : |x_i| \leq \pi/2, i = 1,2\}$, with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\alpha_1(z) = \frac{1 + \sin(x_1)}{2}, \quad \alpha_2(z) = 1 - \alpha_1(z),$$

as presented in Pan et al. (2012b). Figure 3 shows the estimate for the domain of attraction provided by Corollary 1 (using degree $g = 2$) and Theorem 1 at the first stage with $\xi = 10$ and $g = 2$, while Figure 4 illustrates the different estimates for the domain of attraction obtained with Corollary 1, (Lee et al., 2012, Theorem 6) with $\phi_1 = \phi_2 = 5$, (Pan et al., 2012b, Theorem 2) (reproducing the curve presented in the paper) and (Lee and Kim, 2014, Theorem 2) (using degree $q = 3$ and $\phi = 10000$). The corresponding areas are given in Table 2, with the number V of scalar variables and R of LMI rows for each condition. As in Example 1, the proposed method provides the best estimates for the region of attraction.

6. CONCLUSION

A new strategy to compute stabilizing state-feedback control laws and estimates of the domain of attraction for continuous-time nonlinear systems represented by T–S fuzzy models is proposed. The conditions are given in terms of parameter-dependent LMIs that must be solved in two steps. In the first step, a gain that stabilizes the dual system is obtained. At the second step, using this gain as an input, sufficient conditions provide another state-feedback control gain associated with an estimate for the region of attraction of the closed-loop system. The closed-loop stability of the system and the invariance of the domain are certified by means of a fuzzy Lyapunov function with polynomial dependence of arbitrary degree. The approach provides control laws that guarantee larger estimates of the domain, when compared with other techniques and examples borrowed from the literature. Extensions to cope with decentralized and output feedback control are under investigation.

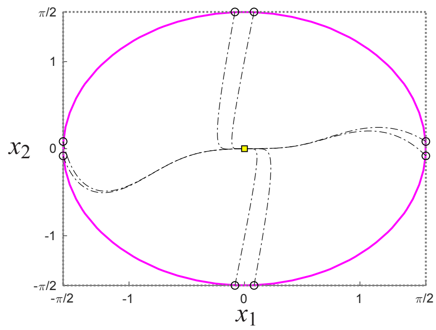

Fig. 3. Estimate of the domain of attraction obtained with the proposed method (solid magenta), domain of validity of the model (dotted gray), and trajectories (dot-dashed black) starting in ∘ and ending at the origin (yellow square) for *Example 2*.

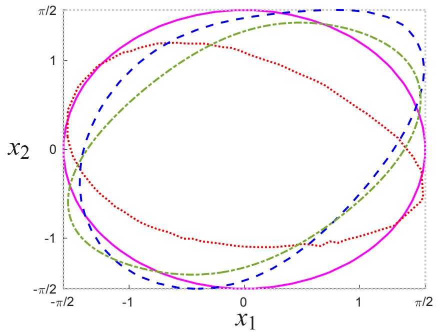

Fig. 4. Estimates of the domain of attraction obtained with the proposed method (solid magenta), (Lee et al., 2012, Theorem 6) (dashed blue), (Lee and Kim, 2014, Theorem 2) (dot-dashed green), and (Pan et al., 2012b, Theorem 2) (dotted red), and domain of validity of the model (dotted gray), *Example 2*.

Table 2. Areas of the estimates of the domain of attraction for *Example 2*, number V of scalar variables and R of LMI rows for each condition.

	Area	V	R
Theorem 1	–	19	134
Corollary 1	7.7244	35	284
(Lee et al., 2012, Theorem 6)	6.8524	44	30
(Lee and Kim, 2014, Theorem 2)	6.6008	129	180
(Pan et al., 2012b, Theorem 2)	5.2738	22	32

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