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Multistable Energy Shaping of Passive Linear Systems with Hybrid Mode Selector

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Abstract: This paper presents a novel control strategy for stable linear time–invariant systems operating with a finite number of set points. Inspired by the theory of passivity-based control, the proposed method aims at simultaneously and asymptotically stabilize all the desired *working modes* by means of a static nonlinear state feedback law. An asynchronous external signal is then employed to trigger a hybrid controller in order to switch between the different working modes. The proposed approach is validated by means of simulations performed on the ubiquitous mass-spring-damper system.

Keywords: linear systems, nonlinear control, port–Hamiltonian systems, hybrid systems, multistability, passivity–based control.

1. INTRODUCTION

In the last three decades, the fundamental concept of *energy* experienced an impressive growth process in engineering practice and in particular in system theory. The framework of *passivity–based control* (PBC) is now a well–established branch in nonlinear control theory and aims at treating dynamical systems as devices able to exchange energy, rather than to process signals (Ortega et al., 2001). This is possible by equipping dynamical systems with additional structure (e.g. storage functions, supply rates, etc.) by means of which the concepts of energy and input/output characterisation of the system are connected in a unique framework (Sontag, 2008). The other fundamental aspect of this paradigm is *interconnection* of systems by means of *power ports* (Duindam et al., 2009), which led to the definition of *port-Hamiltonian systems* (Maschke and Schaft, 1992; Ortega et al., 2001; Van Der Schaft and Jeltsema, 2014), the mathematical framework in which PBC developed naturally, merging geometry and network theory. Hence, the control problem reduces to the design of a dynamical system (the controller) and an interconnection structure that "shapes", in a desired way, the energy of the original system (Ortega et al., 2001, 2008). This approach allows control engineers to pay particular attention to the performance of the control system and not only to stabilizability (as common in nonlinear control).

In this perspective, the aim of this work is to design a controller for a stable linear time–invariant (LTI) system, able to simultaneously stabilize multiple fixed points of the controlled system and switch among them.

Exponentially stable LTI systems admit a unique equilibrium point, while in many practical situations, they have to operate in a finite number of *working modes* (fixed values of voltages, positions, etc.). Thus, with standard linear control techniques, a continuous exogenous reference signal must be constantly provided in order to achieve the desired behaviour, e.g. asymptotic stabilisation of a desired set points.

In order to embed in the controlled system the information on the desired working modes, a nonlinear controller is employed to simultaneously stabilise multiple points, i.e. achieve *multistability* and avoid the need of external reference signals. The introduction of the nonlinear terms gives rise to interesting properties of the controlled system, e.g. the possibility of shaping the basins of attraction of the different fixed points. Appealing studies on the inverse problem, i.e., turning *monostable* a multistable nonlinear system, have already been presented by Pisarchik and Feudel (2014).

Here, considering that stable LTI system can be made passive through an opportune choice of input and output (Byrnes et al., 1991), a nonlinear controller able to stabilize multiple points is designed following a port-Hamiltonian paradigm, (Ortega and Mareels, 2000; Secchi et al., 2007; Ortega et al., 2008; Van Der Schaft and Jeltsema, 2014). Then, in order to switch among the working modes, a *mode selector* is developed exploiting the theory of hybrid dynamical systems (Van Der Schaft and Schumacher, 2000; Goebel et al., 2009).

Notation: The set $\mathbb{R}$ ($\mathbb{R}^+$) is the the set of real (non negative real) numbers. The set of squared–integrable functions $z : \mathbb{R} \to \mathbb{R}^m$ is $\mathcal{L}_2^m$ while the set of d–times continuously differentiable functions is $\mathcal{C}^d$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ denote the inner product on $\mathbb{R}^m$ and

$\|v\|_2 \triangleq \sqrt{\langle v, v\rangle}$ its induced norm. The origin of $\mathbb{R}^n$ is $\mathbb{0}_n$. Let $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and let $\partial\mathcal{H} \in \mathbb{R}^n$ be its transposed gradient, represented as a column vector. The Hessian of $\mathcal{H}$ is $\partial^2\mathcal{H} \in \mathbb{R}^{n\times n}$.

2. MULTISTABLE ENERGY SHAPING OF LTI SYSTEMS

In this section a nonlinear feedback law for a LTI system is designed to stabilise multiple fixed points. Passivity and the properties of passive LTI systems are briefly discussed.

2.1 Passivity of LTI Systems

Let us consider a controlled affine system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, $y \in \mathcal{Y} \subset \mathbb{R}^m$. $f : \mathcal{X} \to \mathbb{R}^n$, $g : \mathcal{X} \to \mathbb{R}^{m\times n}$ ($\text{rank}(g) = m \leq n$) and $h : \mathcal{X} \to \mathbb{R}^m$ are assumed smooth enough such that the solutions are forward–complete for all initial conditions $x_0 \in \mathcal{X}$ and all inputs $u(t) \in \mathcal{L}_2^m$. Let $\Phi(t, x_0, u)$ denote the state trajectory at time $t \geq 0$.

A *supply rate* is a real valued function ω defined on $\mathcal{Y}\times\mathcal{U}$. The system (1) is said to be *dissipative* with respect to the supply rate ω if there exists a continuous function $\mathcal{H} : \mathcal{X} \to \mathbb{R}^+$, called *storage function* such that, for all $u \in \mathcal{U}$, $x \in \mathcal{X}$ and $t \geq 0$, it holds

$$\mathcal{H}(x(t)) - \mathcal{H}(x(0)) \leq \int_0^t \omega(s)ds.$$

Furthermore, the system is said to be *passive* if it is dissipative with respect to the supply rate $\omega = \langle y, u\rangle$. The supply rate ω and the storage function $\mathcal{H}(x)$ can be thought as the generalized power and the generalized energy [1], respectively. In fact, the pair (u, y) represents the medium by which the system can exchange generalized energy through ω.

Definition 1. (Kalman-Yakubovich-Popov (KYP) Property). System (1) is said to enjoy the KYP property if there exists a storage function $\mathcal{H} : \mathcal{X} \to \mathbb{R}^+$, $\mathcal{H}(x) \in \mathcal{C}^1$, $\mathcal{H}(\mathbb{0}_n) = 0$ such that:

$$\partial^\top\mathcal{H}(x)f(x) \leq 0 \quad \partial^\top\mathcal{H}(x)g(x) = h^\top(x)$$

for all $x \in \mathcal{X}$.

Proposition 2. (Byrnes et al., 1991). System (1) is passive with storage function $\mathcal{H}(x) \in \mathcal{C}^1$ if and only if it enjoys the KYP property.

Let us consider the stable LTI specialization of (1), i.e. the standard system with state space realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2}$$

with system matrices (A, B, C) of appropriate dimensions and A Hurwitz. It is easy to verify that, once system (2) is equipped with a quadratic storage function $\mathcal{H}(x) = \frac{1}{2}x^\top Px$, $P = P^\top > 0$, it enjoys the KPY property, i.e. it is passive, if and only if

$$A^\top P + PA \leq 0 \quad B^\top P = C. \tag{3}$$

Furthermore, for system (2) with A Hurwitz and $u = 0$, the storage function is non increasing along trajectories, i.e.

$$\dot{\mathcal{H}}(x) = x^\top PAx \leq 0 \quad \forall x \in \mathcal{X}.$$

This means that the *natural* dissipation inferred by the choice of P of the autonomous system corresponds to $x^\top PAx$.

In order to present clearly the upcoming concepts, the following prototype linear system has been chosen as example to be invoked throughout the paper.

Example 3. Consider a forced mass–spring–damper system with unitary mass

$$\ddot{q} + b\dot{q} + kq = \varphi$$

where q is the displacement of the mass and $k, b \in \mathbb{R}^+ \setminus\{0\}$ are the stiffness and damping coefficients. By choosing the forcing term φ as input, $u = \varphi$, and letting the state of the system be $x = [q, \dot{q}]^\top$, the state–space form of the system becomes:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u.$$

The autonomous system is exponentially stable if $b > 0$. Let $P = \text{diag}(k, 1)$. Indeed $P = P^\top > 0$ and $A^\top P + PA < 0$. Thus, the system is passified (with storage function $\mathcal{H} = \frac{1}{2}x^\top Px$) choosing the linear output as $y \triangleq B^\top Px = \dot{q}$. Note that $\mathcal{H}$ is the total energy of the system. In general, any mechanical systems is passive with its total energy as storage function by choosing as input(s) the (generalised) forces and as output(s) the (generalised) velocities.

Now let us briefly review port–Hamiltonian systems and their properties, since they represent the framework where PBC has been consistently developed (Ortega et al., 2001).

2.2 Port–Hamiltonian Systems and PBC

A port–Hamiltonian (PH) system (Van Der Schaft and Jeltsema, 2014) has an input–state–output representation [2]:

$$\begin{cases} \dot{x} = [J(x) - R(x)]\,\partial\mathcal{H} + g(x)u \\ y = g^\top(x)\partial\mathcal{H} \end{cases} \tag{4}$$

which is in the form (1) where $J(x)$ is a skew symmetric matrix representing the power preserving interconnections, $R(x)$ is a symmetric positive semi–definite matrix representing the dissipation of the system and $g(x)$ is a matrix representing the way in which external power is distributed into the system. Furthermore, $\mathcal{X}$ is an n–dimensional manifold, $\mathcal{U}$ is a m–dimensional vector space and $\mathcal{Y} = \mathcal{U}^*$ is its dual space; $y^\top u$ has the unit measure of power.

The power conservation property of (4) is defined by the power–balance equation

$$\frac{d}{dt}\mathcal{H} = \partial^\top\mathcal{H}\,\dot{x} = -\partial^\top\mathcal{H}\,R\partial\mathcal{H} + y^\top u \leq y^\top u$$

[1] Without any loss of generality, $\mathcal{H}(x)$ can be taken bounded from below rather than nonnegative, since the properties of storage functions hold regardless of an additive constant.

[2] From now on, the dependency on x of the scalar function $\mathcal{H}$ is hidden for compactness.

i.e., PH systems are passive. Indeed, system (4) has the KYP property automatically satisfied.

Problem 4. (Passivity–based control). Consider a PH system (4). A control action $u = \beta(x) + v$ solves the PBC problem if the closed-loop system satisfies a desired power-balance equation

$$\dot{\mathcal{H}}^* = z^\top v - d^*$$

where $\mathcal{H}^*$ is the desired energy function, d^* the desired dissipation function and $z \in \mathbb{R}^m$ the new power conjugated (passive) output.

The most common solution to the PBC problem is the *energy–balancing PBC* (EB–PBC) proposed by Ortega and Mareels (2000). The controller is obtained directly from the power balance equation by setting the desired dissipation d^* equal to the natural dissipation of the system, i.e, $d^* \triangleq \partial^\top \mathcal{H} R \partial \mathcal{H}$ and keeping the same output $z \triangleq y$. Next proposition gives an operative insight of how to accomplish the EB–PBC control task

Proposition 5. (Secchi et al., 2007). If it is possible to find a function $\beta(x)$ such that

$$\dot{\mathcal{H}}_a = y^\top \beta(x)$$

then the control law $u = \beta(x) + v$ is such that

$$\dot{\mathcal{H}}^* = y^\top v - d^*$$

is satisfied for $\mathcal{H}^* \triangleq \mathcal{H} + \mathcal{H}_a$.

This implies that the state feedback $\beta(x)$ is such that the *added energy* $\mathcal{H}_a$ equals the energy supplied to the system and, consequently, $\mathcal{H}^*$ is the difference between the stored and supplied energy. In Ortega et al. (2008) the closed-form solution of the EB–PBC controller is given by

$$\beta(x) = -g^+ [J - R]^\top \partial \mathcal{H}_a$$

where g^+ is the left pseudo–inverse of matrix g and $\mathcal{H}_a$ satisfies the following matching equations

$$\begin{bmatrix} g^\perp [J - R]^\top \\ g^\top \end{bmatrix} \partial \mathcal{H}_a = \mathbb{0}_{n+m}$$

being $g^\perp$ a left full–rank annihilator of matrix g.

The idea behind this state–feedback control is to "shape" the energy function so that its only minimum translates towards a new minimum, representing the desired working condition of the controlled system (e.g. *PD + gravity compensation* in robot regulation, Secchi et al. (2007)). In the following it is shown how this concept can be extended to produce multiple stable working conditions by means of a nonlinear feedback law applied to an LTI system (2).

2.3 Application to LTI Systems and Multistable PBC

Any passive system (2) with storage function $\mathcal{H}(x) = \frac{1}{2}x^\top Px$ such that null$(P) \subseteq$ null(A) admits a port-Hamiltonian representation[3]:

$$\begin{cases} \dot{x} = [J - R]\, Px + Bu \\ y = B^\top Px \end{cases}$$

where

$$J = \frac{1}{2}(AP^{-1} - P^{-1}A^\top),\ R = -\frac{1}{2}(AP^{-1} + P^{-1}A^\top).$$

[3] Since we choose $P > 0$, this condition does not represent a loss of generality, i.e. any system (2) can be written in PH form.

Thus, for a LTI system, the energy balancing control law becomes

$$\beta(x) = -B^+[J - R]^\top (\partial \mathcal{H}^* - Px) \tag{5}$$

with $\mathcal{H}_a \triangleq \mathcal{H}^* - \frac{1}{2}x^\top Px$ and the matching conditions:

$$\begin{bmatrix} B^\perp [J - R]^\top \\ B^\top \end{bmatrix} (\partial \mathcal{H}^* - Px) = \mathbb{0}_{n+m}. \tag{6}$$

Note that, the closed loop linear PH system becomes $\dot{x} = [J - R]\partial \mathcal{H}^* + gv$. Our purpose, is to simultaneously asymptotically stabilize N desired set points of the system through EB–PBC. Hereafter, we show that if we design a $\mathcal{H}^*$ with N isolated minima in the desired set points, those are locally asymptotically stable.

Lemma 6. Consider the EB-PBC controlled system. Every closed set $\mathcal{M} \subset \mathbb{R}^n$ such that

$$\forall x \in \mathcal{M},\ \partial \mathcal{H}^* = \mathbb{0}_n,\ \partial^2 \mathcal{H}^* > 0,\ \partial^\top \mathcal{H}^* R \partial \mathcal{H}^* = 0$$

which is contained in an open neighborhood $\mathbb{U} \supset \mathcal{M}$ satisfying

$$\forall x \in \mathbb{U} \setminus \mathcal{M},\ \partial \mathcal{H}^* \neq \mathbb{0}_n$$

is locally stable. Furthermore, if $\forall x \in \mathbb{U} \setminus \mathcal{M}$ $\partial^\top \mathcal{H}^* R \partial \mathcal{H}^* < 0$, then $\mathcal{M}$ is asymptotically stable.

Proof. For all $x \in \mathcal{M}$, $\dot{x} = \mathbb{0}_n$ by construction and, thus, $\mathcal{M}$ is forward invariant. Let $\mathbb{U} \supset \mathcal{M}$ be an open neighborhood of $\mathcal{M}$ such that $\forall x \in \mathbb{U} \setminus \mathcal{M}$, $\partial \mathcal{H}^* \neq \mathbb{0}_n$ and let $V(x) = \mathcal{H}(x)$ be a candidate Lyapunov function. It holds: $\forall x \in \mathbb{U}$ $\dot{V} = \partial^\top \mathcal{H}^* R \partial \mathcal{H}^* \leq 0$ and $\forall x \in \mathcal{M}$ $\dot{V} = 0$. Thus $\mathcal{M}$ is stable. In addition, if $\forall x \in \mathbb{U} \setminus \mathcal{M}$ $\partial^\top \mathcal{H}^* R \partial \mathcal{H}^* = \dot{V} < 0$, $\mathcal{M}$ is asymptotically stable.

Thus, it follows:

Proposition 7. If, moreover, $\mathcal{H}^*$ possesses solely isolated minima $\{x_i^*\}_{i \in \mathbb{N},\ i \leq N}$, those are locally stable in the sense of Lyapunov. If this is the case, by further assuming asymptotic stability, the minima x_i^* are also attractive, i.e. $\exists x_i^*$: $\lim_{t \to \infty} \Phi(t, x_0, \beta(x)) = x_i^*$ for almost all initial condition $x_0 \in \mathcal{X}$ (but local maxima or stable eigendirections of saddle points).

Proof. The proof follows directly from Lemma 6 noticing that the attractiveness of the minima is guarandeed for, since $\forall x \in \mathcal{X}, \dot{\mathcal{H}}^*(x) \leq 0$.

Remark 8. Deeper evaluations and considerations on Lyapunov functions for multistable nonlinear systems are reported in Efimov (2012). It has to be underlined that, in order to have an energy function with multiple minima, it is necessary to have the presence of local maxima, which however do not affect global behaviour of the system, since those are just unstable invariants of the closed loop system.

Furthermore, if the system is detectable, the control law $u = \beta(x) + v$ with $v = -K_d y = -K_d B^\top Px$ ($K_d \in \mathbb{R}^{m \times m}$, $K_d = K_d^\top > 0$), will asymptotically stabilise any minimum of $\mathcal{H}^*$ (see Secchi et al., 2007). This control law is known as *damping injection* and the constant matrix K_d is often defined as the *dissipation rate*. Note that, the damping injection term in closed loop is equivalent to renaming the dissipation $R = R + gg^\top K_d > 0$. This implies that damping injection can asymptotically stabilise in closed–loop points that are not affected by natural dissipation. A block diagram picturing the overall control scheme is represented in Fig. 1.

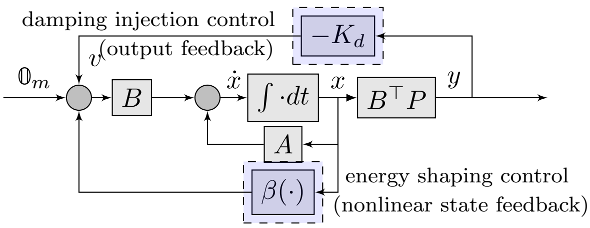


Fig. 1. A block representation of the controlled system.

2.4 Application Example 3

Consider the system in Example 3 and let the desired energy function have two symmetrically distributed minima on the displacement axes, e.g.,

$$\mathcal{H}^* = \lambda q^4 - \mu q^2 + \frac{1}{2}\dot{q}^2 - \frac{\mu^2}{2\lambda} \qquad \lambda, \mu > 0$$

which has two minima in $[\pm\sqrt{\mu/2\lambda}, 0]^\top$ and a local maximum in $[0, 0]^\top$. Thus,

$$\mathcal{H}_a = \mathcal{H}^* - \mathcal{H} = \lambda q^4 - (\mu + \frac{1}{2}k)q^2 - \frac{\mu^2}{2\lambda}$$

and, therefore

$$\partial \mathcal{H}_a = [4\lambda q^3 - (2\mu + k)q, 0]^\top .$$

It is easy to proof that the matching conditions (6) of the EB–PBC are satisfied for $\mathcal{H}_a$. The energy shaping control law becomes

$$\begin{aligned} \beta(x) &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} \begin{bmatrix} 4\lambda q^3 - (2\mu + k)q \\ 0 \end{bmatrix} \\ &= -4\lambda q^3 + (2\mu + k)q. \end{aligned} \tag{7}$$

A numerical simulation of the proposed control scheme has been performed with $k = 1$, $b = 0.5$. The parameters λ and μ have been set to 2 and 1 respectively, placing the minima of $\mathcal{H}^*$ in $[\pm 0.5, 0]^\top$. The dissipation rate $k_d \in \mathbb{R}$ has been set to zero as the asymptotic stability is already guaranteed by the natural dissipation of the autonomous linear system. Starting in the initial position $x_0 \triangleq [-0.9, 0]^\top$ the system has been simulated for both the autonomous and the multistable EB–PBC controlled system for $40s$. The resulting phase–space portraits are reported in Fig. 2.

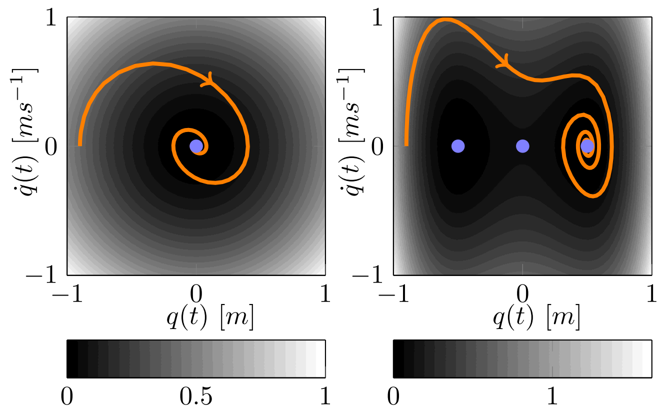


Fig. 2. Comparison of the phase–space portraits of the autonomous and the multistable EB–PBC controlled system with $k = 1$, $b = 0.5$, $\lambda = 2$, $\mu = 1$ and $k_d = 0$. The phase–space portraits are represents over contour plots of the corresponding energy functions, i.e., $\mathcal{H} = \frac{1}{2}(q + \dot{q})$ and $\mathcal{H}^* = 2q^4 - q^2 + \frac{1}{2}\dot{q} + \frac{1}{8}$. The blue dots indicates the critical points of $\mathcal{H}$ and $\mathcal{H}^*$.

2.5 Choice of the Dissipation Rate: Shaping the basins of attraction

In this section it is shown how, by tuning the dissipation rate K_d, it is possible to shape the basins of attraction of the designed stable fixed points of the closed–loop system.

Definition 9. The *basin of attraction* $\mathcal{B}$ of a fixed point x^* of a system (1) is the set of all initial conditions x_0 leading to long-time behaviour that approaches that fixed point, i.e.

$$\mathcal{B} \triangleq \left\{ x_0 \in \mathcal{X} \,|\, \lim_{t\to\infty} \Phi(t, x_0, u) = x^* \right\}.$$

The designed feedback control law (7) allows to fix multiple stable points for the closed–loop system. The damping injection component of the overall control action can be used to "shape" their basins of attraction. This property allows to have interesting control actions which will be qualitatively shown.

Consider the system of Example 3 controlled by the energy shaping control law (7). The closed–loop system in the form (1) can be expressed as

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{q} \\ -4\lambda q^3 + 2\mu q - b\dot{q} \end{bmatrix} + Bv \\ y = B^\top P x \end{cases} .$$

If $v = 0$ the system has two asymptotically stable fixed points in $[\pm\sqrt{\mu/2\lambda}, 0]^\top$. The output feedback controller $v = -k_d y = -k_d \dot{q}$ does not change the location of the fixed points. In facts, it only changes the overall dissipation of the system from $b\dot{q}$ to $(b + k_d)\dot{q}$. Besides, the respective basins of attraction strongly depend on the choice of the controller, i.e., on the value of k_d. In Fig. 3 the basins of attraction of the two fixed points are shown for different values of k_d in the region $[-1, 1] \times [-1, 1]$ of the state space with $b = 0$, $\lambda = 2$, $\mu = 1$. It is evident that the higher the dissipation rate is, the lower is the number of transitions between basins of attraction.

Therefore, given an initial condition x_0, and a desired set point x_i^*, corresponding to one of the minima of $\mathcal{H}^*$, we would be interested in choosing a dissipation rate $K_d = K_d^\top \geq 0$ such that x_0 belongs to the basin of attraction $\mathcal{B}_i$ of x_i^*. In order to choose among the possibly infinite values of K_d such that $x_0 \in \mathcal{B}_i$, one could minimise both the approaching time and the damping injection control effort needed to bring $x(t)$ from x_0 to x_i^*, i.e.,

$$\begin{aligned} \underset{K_d}{\text{minimize}} \quad & \int_0^\infty \Big((x(s) - x_i^*)^\top Q\, (x(s) - x_i^*) + \\ & \quad + x^\top(s) PBK_d SK_d B^\top P x(s) \Big)\, ds \\ \text{subject to} \quad & \lim_{t\to\infty} \Phi(t, x_0, \beta(x) - K_d B^\top P x) = x_i^* \end{aligned}$$

with $Q = Q^\top \geq 0$ and $S = S^\top \geq 0$. Further investigations on the implementation of this optimal controller are left as future work.

3. HYBRID MODE SELECTOR

Once the feedback law and the damping injection are designed to produce the desired working modes and to shape their basins of attraction, the proposed strategy aims to switch from a working mode to another. In particular, considering the system to be in one of the

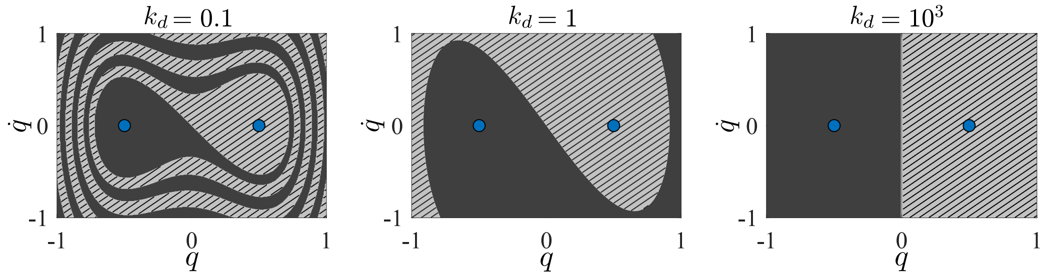


Fig. 3. Basins of attraction of the fixed points of the system for different values of k_d (κ in the figure) in the region $[-1, 1] \times [-1, 1]$. The basin of attraction of the minima [blue points] are represented in dark gray ($[-0.5, 0]^\top$) and light hatched gray ($[0.5, 0]^\top$).

working modes x_i^*, a control action which moves the system to another desired mode, x_j^*, is designed. The strategy reckons on the following actions:

1. Switch-off the energy shaping controller (the system turns back linear);
2. Give an impulse to the system to bring the state inside the basin of attraction of x_j^*;
3. Switch on again the energy shaping controller.

3.1 Impulse generation

When the nonlinear controller $u = \beta(x) + v$ is switched off, i.e. $u = 0$, the system turns back in the LTI form (2). Without loss of generality, let $t = 0$ and let the LTI system be controllable. The response of the system to a weighted impulse input

$$u(t) = \nu\delta(t)$$

where $\nu \in \mathbb{R}^m$ distributes the Dirac delta function $\delta(t)$ among the m inputs, is

$$\begin{aligned} x(t) &= e^{tA}x_0 + \int_0^t e^{(t-s)A}B\nu\delta(s)ds \\ &= e^{tA}x_0 + e^{tA}B\nu = e^{tA}\left(x_0 + B\nu\right). \end{aligned} \tag{8}$$

Since the control objective is to move the system from x_i^* to x_j^* in a time t^*, it is tempting to impose the desired behaviour in (8) by requiring:

$$x_j^* \triangleq x(t^*) = e^{t^*A}\left(x_i^* + B\nu\right). \tag{9}$$

Therefore, $\nu = B^+(e^{-t^*A}x_j^* - x_i^*)$. However, unless $m = n$, (9) is overdetermined, i.e. n (scalar) equations with only m unknowns (the components of ν). To overcome this issue, the design of the impulse controller is achieved by solving the following optimisation problem: find t^*, ν such that

$$\begin{aligned} &[t^*, \nu] = \arg\min_{t^*,\nu} \gamma\|\nu\|_2^2 + \rho\|x_j^* - e^{t^*A}\left(x_i^* + B\nu\right)\|_2^2 \\ &\text{subject to } \ \Phi(t^*, x_i^*, \nu\delta(t)) \in \mathcal{B}_j \end{aligned} \tag{10}$$

where $\gamma,\ \rho \in \mathbb{R}^+$ are two arbitrary weights and $\mathcal{B}_j$ is the basin of attraction of x_j^*. The solution of (10), provides an impulsive input $u = \nu\delta$ which guarantee that the system will arrive in $\mathcal{B}_j$ in a time t^*. For the sake of a latter numerical implementation, the constraint was defined as

$$\|\hat{\Phi}\left(T, e^{t^*A}\left(x_i^* + B\nu\right), \beta(x) - K_dB^\top Px\right) - x_j^*\|_2 \leq \varepsilon$$

where $\hat{\Phi}$ is the numerically integrated trajectory of the system, $T \gg 1$ is the integration time and $0 \leq \varepsilon \ll 1$ is a chosen threshold.

Remark 10. Assuming that the switch is performed only at steady–state, the optimisation of ν and t^* for each pair of fixed points x_i, x_j can be performed off–line.

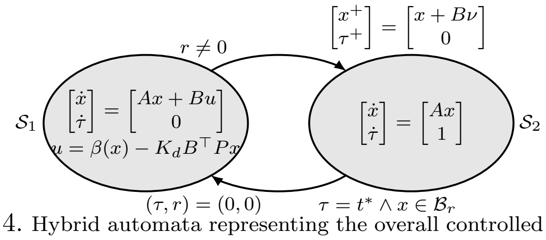


Fig. 4. Hybrid automata representing the overall controlled system with the hybrid mode selector.

3.2 Overall Hybrid System

If an impulse is applied to the system at time t in order to reach the desired destination x_j^*, in the instant of time in which the impulse is applied, the state undergoes to a discontinuous jump

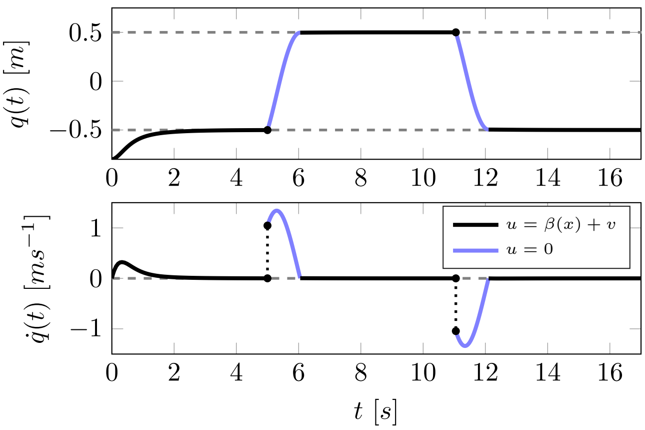
$$x^+ = x(t) + B\nu. \tag{11}$$

Therefore, the controlled system can be described as an *hybrid automata*, (Van Der Schaft and Schumacher, 2000), with two logic states $\mathcal{S}_1$ and $\mathcal{S}_2$. In $\mathcal{S}_1$ the system is controlled with the multistable EB–PBC and in $\mathcal{S}_2$ the system is completely uncontrolled. Thus, by introducing a timer τ and an external asynchronous signal r (initialised to 0), the transition from $\mathcal{S}_1$ to $\mathcal{S}_2$ will happen when r changes from 0 to the index of the desired fixed point, i.e., $r \in \{0, 1, \ldots, N\}$, with a state jump described by (11) and resetting the timer τ to 0. Then the system will remain uncontrolled (and thus linear) for a time t^*, i.e. $\tau = t^*$ after which the logic state will switch back from $\mathcal{S}_2$ to $\mathcal{S}_1$ and the timer and the external signal r will be reset. A graphical representation of the designed hybrid automata is given in Fig. 4.

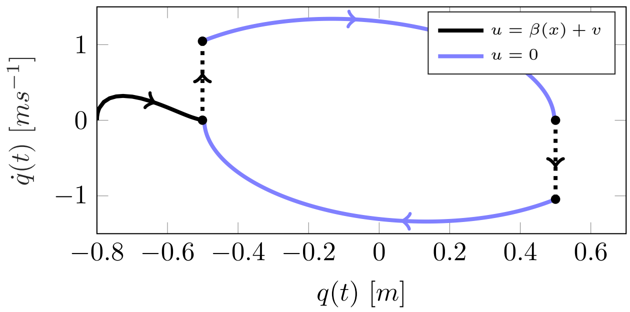
3.3 Numerical Simulation

A numerical simulation of the overall controlled system has been performed to validate the proposed control scheme. The whole procedure has been implemented in the MATLAB® [4] environment. The system introduced in Example 3 has been controlled with the multistable energy shaping (7) and the damping injection $v = -k_d\dot{q}$. The system parameters has been chosen as $k = 5$, $b = 0.5$ and, as in the example in Section 2, the minima of $\mathcal{H}^*$ have been positioned in $[\pm 0.5, 0]^\top$ by setting $\lambda = 2$ and $\mu = 1$. To implement the asynchronous external control signal r, the two fixed points have been denoted with $x_1^* = [-0.5, 0]^\top$ and $x_2^* = [0.5, 0]^\top$. The dissipation rate k_d has been set to 4.5. The *fmincon* solver of the *global optimization*

[4] The source code is available at `https://github.com/massastrello/MultistableControl`.

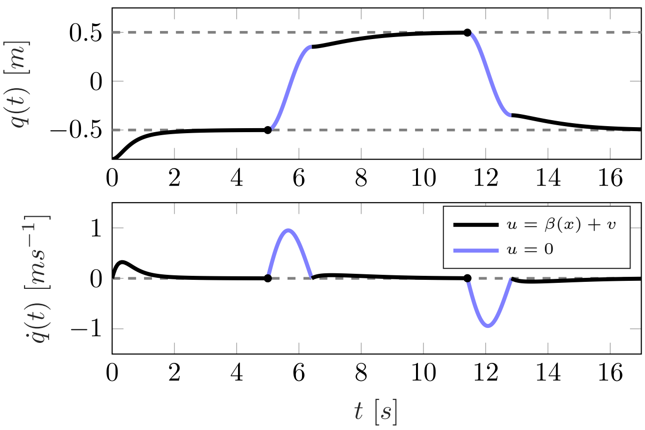


(a) Time evolution of the states $x = [q, \dot{q}]^\top$.

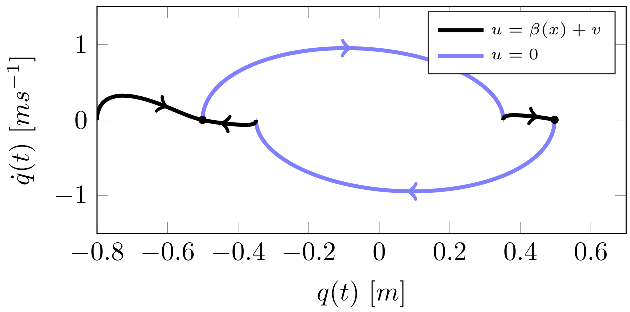


(b) Phase–space portrait.

Fig. 5. Simulation experiment of the overall hybrid system carried out by settling $\gamma = 10^{-3}$.



(a) Time evolution of the states $x = [q, \dot{q}]^\top$.



(b) Phase–space portrait.

Fig. 6. Simulation experiment of the overall hybrid system carried out by settling $\gamma = 2$.

toolbox of MATLAB® has been employed to solve the optimisation problem (10). The optimisation parameters have been chosen as $T = 10^3$, $\varepsilon = 10^{-5}$, considering an absolute tolerance of 10^{-6} for the numerical integration (ODE45). Starting from the initial state $x_0 \triangleq [-0.8, 0]^\top$, the system, controlled with the nonlinear state feedback and the damping injection, has been integrated until secured convergence to x_1^* ($5s$). Then, r has been set to 2 in order to trigger the change of working mode, i.e. to bring the state to x_2^*. After the jump, the system has been let flowing uncontrolled for a time t^* and then the nonlinear controller has been turned on again. After other $5s$ the procedure has been repeated, by setting r to 1 and bring the state back in x_1^*. This simulation has been performed twice, with different values of γ for the impulse design process (performed off–line); ρ has been set to 1 in both cases. First, γ has been set to 10^{-3} to emphasise the minimisation of the norm of the error $\|e\|_2^2 = \|x_j^* - e^{t^* A}(x_i^* + B\nu)\|_2^2$. Then, γ has been set to 2, accentuating the minimisation of the squared norm $\|\nu\|_2^2$ of the impulse weights vector (in this case $\nu \in \mathbb{R}$).

The numerical results of the optimisation are reported in Table 1 while the system trajectories are shown in Figs. 5 and 6. In the first case ($\gamma = 10^{-3}$), the transient from x_1^* to x_2^* (and vice versa) is very fast and without any oscillation in the displacement, due to the high dissipation rate and the minimised error norm: when the EB–PBC controller is switched–on again the state is very close to the desired energy minimum. However, the price of this performance is the impulse, i.e. state jump, that has to be generated. On the other hand, when $\gamma = 2$, no state discontinuity is needed to change working mode, but at the cost of a slower transient.

4. CONCLUSIONS AND FUTURE WORK

In this paper a novel technique for controlling stable linear time invariant systems which operate in a finite number of working modes has been presented. The theory of passivity–based control and port-Hamiltonian systems has been used to stabilize multiple fixed points of the closed-loop system. The proposed method allows then to switch among the desired working modes by engaging a hybrid mode selector triggered asynchronously by an external logic signal. Simulations have been performed to prove the validity of the control scheme. Future work will include investigations of the theoretical aspects linked to the multistable controller, the shaping of the basins of attraction and the stability of the overall hybrid system. A real implementation of the control system will also be explored.

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Table 1. Hybrid controller optimisation results.

(a) 1st impulse ($x_1^* \to \mathcal{B}_2$)

γ	$\|e\|_2^2$	ν	t^*
10^{-3}	0.00	1.05	1.04
2	0.15	0.00	1.41

(b) 2nd impulse ($x_2^* \to \mathcal{B}_1$)

γ	$\|e\|_2^2$	ν	t^*
10^{-3}	0.00	-1.05	1.05
2	0.15	0.00	1.42

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