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Stabilizing unstable biomechanical model to understand sitting stability for persons with spinal cord injury

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Abstract: This paper deals with the stabilization of a new open-loop unstable non-linear biomechanical model to represent a person living with a spinal cord injury. The computational complexity of the model when using the Takagi-Sugeno formalism is increased compared to previous models which is a source of difficulty to find a control law. Several solutions are presented combining robustness, and model simplification from previous works in the form of linear matrix inequalities (LMI) which are then solved using convex optimization technics. Finally, simulations results are presented to show the validity and the effectiveness of different approaches.

Keywords: Spinal cord injury, biomechanical systems, Takagi-Sugeno fuzzy models, nonlinear control

1. INTRODUCTION

Sitting balance is amongst the most important ability of people living with a spinal cord injury (SCI) who usually end up using a wheelchair. Because of their lack of muscular activity in the lumbar region, they are at high risk of falling during activities of daily living which makes sitting stability one of their greatest challenge (Grangeon et al., 2012; Milosevic et al., 2015).

Most of biomechanical models are not valid to study SCI sitting control because they either stabilize the model through an active lumbar joint or rely on linearization technics around equilibrium points (Reeves et al., 2009) which makes them unfit for the large motions measured with SCI experimental data. In order to study sitting stabilization of people living with an SCI, a first non-linear biomechanical model considering the motion of the upper limbs was proposed, the H2AT (Head 2 Arms and Trunk). As it represents subjects with no abdominal or back muscle activation, this model is both under-actuated and unstable in open-loop. In previous papers we addressed mainly the problem of observation for H2AT (Blandeau et al., 2017). The control problem of H2AT was addressed in (Blandeau, 2018) in a continuous framework. Therein we will consider a much more complex model called Seated 3 Segments (S3S) and its stabilization in a discrete-time framework context. Here, the discrete-time control framework especially because there is a large choice of parameter dependent Lyapunov functions that allows to highly reduce the pessimism of LMI constraints conditions obtained using a quadratic Lyapunov functions (Guerra and Vermeiren, 2004; Nguyen et al., 2019).

Even though the final goal is to derive an observer for the unknown inputs, the system being open-loop unstable, it is first necessary to derive a stabilizing control law. The problem depicted here is not a usual one, as this control has to be similar or to mimic the human behavior for stabilization. Therefore, in this case it is a state feedback control, as the human “knows” his on states (position and speed) and what inputs to involve (torques). Nevertheless, for a Person with Reduced Mobility with sensorimotor impairment, it corresponds to an under actuated problem as the principal actuator, i.e. the trunk (in the sense that it should have produced the maximum torque) is absent. Thus, the stability domain will be highly reduced and only small initial conditions are possible. In view of this restricted stabilization area the conditions to derive have to be the less conservative possible, otherwise there will never get any solution.

As in our previous works, we will use the formalism of the so-called Takagi-Sugeno (T-S) models (Takagi and Sugeno, 1985) belonging to the class of quasi-LPV models. Their advantage is twofold, keep a structure close to the nonlinear model, in our case a descriptor form (Taniguchi et al., 1999), and represent exactly the nonlinear model in a compact set defined on the state variables, in our case corresponding to possible movements of the PRM. Nevertheless, for the problem we are faced to, there is another issue corresponding to the complexity burden. A brute force way of doing will give more than 4,7 million of variables and 262 144 LMI constraints, figures that are definitively incompatible with, not only actual solver but also actual computers RAM (Thieffry et al., 2018). Therefore, several problems have to be addressed in order to simplify this burden without increasing (too much) the conservatism. The paper is organized as follows. The first part presents the nonlinear S3S model and its representation as a descriptor T-S discrete model. The second part presents the position of the problem, especially the under actuated issue. The third part proposes the main results including the way to reduce the complexity of the LMI constraints problem, in order to get a feasible solution. Thus, simulations are proposed and discussion and perspectives close the paper.

2. MATERIALS AND METHODS

2.1 The S3S Models

The Seated 3 Segments (S3S, see Fig. 1) model is a variation of a 2D triple inverted pendulum represented in the sagittal plane by the trunk, upper arm and forearm segments (i.e. segments 0, 1 and 2, respectively) and interconnected by revolute joints at points T (trunk), S (shoulder), and E (elbow) whereas point H stands for Hands. Each segment has its own constants ($i \in \{0,1,2\}$), mass (m_i), length (l_i), length from origin to the center of mass (COM) (l_{G_i}), and its moment of inertia about the COM (I_{G_i}). Regression rules are previously used to obtain these constants for a typical 80 kg male (Dumas et al., 2007; Fang et al., 2017).

To derive the motion equations, the Lagrangian $L = K - U$ is computed. The kinetic energy of the system K equals the sum of each segment K_i according to the following formula:

$$K_i = \frac{1}{2}\left(\vec{\omega}_i^{\,T} \mathrm{I}_{G_i} \vec{\omega}_i + m_i \vec{V}_{G_i}^{\,T} \vec{V}_{G_i}\right) \tag{1}$$

where $\overrightarrow{V_{G_i}}$ is the velocity of the COMi and $\vec{\omega}_i$ is the angular velocity of segment $i \in \{0,1,2\}$. The same applies for U, the potential energy of the system

$$U_i = m_i \vec{g}^{\,T} \overrightarrow{TG_i} \tag{2}$$

where $\vec{g}$ is the gravity vector. Calculating and adding for each segment, the Lagrangian L of the S3S model equals to

$$\begin{aligned} L = {} & \frac{\dot{\theta}_0^2}{2}\left(\mathrm{I}_{G_0} + m_0 l_{G_0}^2 + m_1 l_0^{\,2} + m_2 l_0^{\,2}\right) + \frac{\dot{\theta}_1^2}{2}\left(\mathrm{I}_{G_1} + m_1 l_{G_1}^{\,2} + m_2 l_1^{\,2}\right) \\ & + \frac{\dot{\theta}_2^2}{2}\left(\mathrm{I}_{G_2} + m_2 l_{G_2}^{\,2}\right) - \dot{\theta}_0\dot{\theta}_1\left(m_1 l_0 l_{G_1} + m_2 l_0 l_1\right)\cos(q_1) \\ & + \dot{\theta}_0\dot{\theta}_2 m_2 l_0 l_{G_2}\cos(q_2 + q_1) + \dot{\theta}_1\dot{\theta}_2 m_2 l_1 l_{G_2}\cos(q_2) \\ & - m_2 g l_{G_2}\cos(\theta_2) - \left(m_0 g l_{G_0} + m_1 g l_0 + m_2 g l_0\right)\cos(\theta_0) \\ & - \left(m_1 g l_{G_1} + m_2 g l_1\right)\cos(\theta_1) \end{aligned} \tag{3}$$

Considering no control at the lumbar level, the dynamics is as $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} - \frac{\partial L}{\partial q_0} = 0$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = M_S$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = M_E$. Then, the nonlinear system is obtained by

$$E\big(q(t)\big)\dot{x}(t) = A\big(x(t)\big)x(t) + Bu(t) \tag{4}$$

with $x = \begin{bmatrix} q^T & \dot{q}^T \end{bmatrix}^T = \begin{bmatrix} q_0 & q_1 & q_2 & \dot{q}_0 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$ and $E(q) = \begin{bmatrix} I_3 & 0_3 \\ 0_3 & \bar{E}(q) \end{bmatrix}$, $A(x) = \begin{bmatrix} 0_3 & I_3 \\ S(q) & \bar{A}(x) \end{bmatrix}$, $B = \begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix}$.

The control input $u(t) = \begin{bmatrix} T_T(t) & T_S(t) & T_E(t) \end{bmatrix}^T$ represents the torques at points T, S and E. The elements of the state-space matrices in (5) are given in (Blandeau, 2018, Chapter 3). Following previous works, the continuous model (4) is express in the discrete framework (Blandeau et al., 2017) to hold on experimental data. The classical Euler's method is used, i.e., $\dot{x}(t) \approx \frac{x_{k+1} - x_k}{s}$, with s the sampling time. As a result, system (4) becomes

$$E(q_k)x_{k+1} = A_d(x_k)x_k + B_d u_k \tag{6}$$

with $x_k = \begin{bmatrix} q_{0_k} & q_{1_k} & q_{2_k} & \dot{q}_{0_k} & \dot{q}_{1_k} & \dot{q}_{2_k} \end{bmatrix}^T$, $B_d = s \times B$. Note that $A_d(x_k) = \begin{bmatrix} I_3 & sI_3 \\ sS(q_k) & s\bar{A}(x_k) + \bar{E}(q_k) \end{bmatrix}$. When there is no possible ambiguity, the index k is omitted and x^+ replaces x_{k+1}. Therefore, (6) writes

$$\begin{bmatrix} I_3 & 0 \\ 0 & \bar{E}(q) \end{bmatrix} x^+ = \begin{bmatrix} I_3 & sI_3 \\ sS(q) & s\bar{A}(x) + \bar{E}(q) \end{bmatrix} x + \begin{bmatrix} 0 \\ sI_3 \end{bmatrix} u \tag{7}$$

From (6), the goal is to derive a stabilizing control law that can reproduce the control of a PMR in a sitting situation.

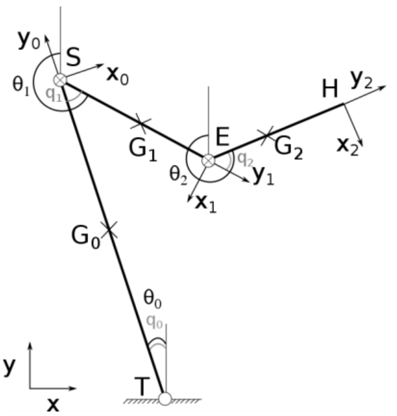

Fig. 1. S3S Model where the joint T is free whereas joints S and E are active.

2.2 Position of the problem

For a person without injuries, model (6) has 3 torque inputs (at points S, E and T) and 3 outputs $y_k = q_k \in \mathbb{R}^3$. Since $E(q_k)$ is regular for all q_k, consider a classical description:

$$x_{k+1} = E^{-1}(q_k)A_d(x_k)x_k + E^{-1}(q_k)B_d u_k$$

or equivalently

$$x^+ = \begin{bmatrix} I_3 & sI_3 \\ s\bar{E}^{-1}(q)S(q) & s\bar{E}^{-1}(q)\bar{A}(x) + I_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ s\bar{E}^{-1}(q)I_3 \end{bmatrix} u \tag{8}$$

From which a full state feedback linearization applies directly using the nonlinear control:

$$u = -\begin{bmatrix} s^{-1}\bar{E}(x)L_1 + S(q) & s^{-1}\bar{E}(x)(L_2 + I_3) + \bar{A}(x) \end{bmatrix} x \tag{9}$$

to get the linear closed-loop model: $x_{k+1}=\begin{bmatrix} I_3 & sI_3 \\ L_1 & L_2 \end{bmatrix} x_k$ with free matrices L_1 and L_2.

Considering now sensorimotor impairments means that the first and most important control, i.e. torque at Trunk T, is not available renders the problem under actuated.

3. MAIN PROBLEM

From the "ideal" control law (9), consider the following law:

$$u=-u_{cancel}-u_l \tag{10}$$

$$u_{cancel}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S(q) & \bar{A}(x)+s^{-1}\bar{E}(q) \end{bmatrix} x(k) \tag{11}$$

$$u_l=s^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} L_1(x) & L_2(x) \end{bmatrix} x \tag{12}$$

The part u_{cancel} in (11) follows the perfect cancellation of the 2 last rows of the open-loop, whereas u_l represents the only possible controls for the PRM, i.e. Shoulder S and Elbow E, Trunk T being unavailable. From (10) and (8) the closed-loop writes:

$$E(q)x^{+}=\left(\Sigma(x)-\Gamma\times L(x)\right)x \tag{13}$$

with $L(x)\in\mathbb{R}^{2\times 6}$. Considering the notation in brackets as $[A]_i$ corresponds to the i^{th} row of matrix A:

$$\Sigma(x)=\begin{bmatrix} I_3 & sI_3 \\ \left[sS(q)\right]_1 & \left[s\bar{A}(x)+\bar{E}(q)\right]_1 \\ 0_{2\times 3} & 0_{2\times 3} \end{bmatrix},\ \Gamma=\begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix} \tag{14}$$

3.1 Takagi-Sugeno framework

Considering a compact set of the state variables, therein the possible movements of the PRM, a T-S model represents *exactly* the nonlinear model (6) in convex sums of linear systems interconnected with nonlinear functions issued from the nonlinearities of (6). The methodology uses the so-called sector nonlinearity approach (SNA) (Tanaka et al., 2001). Consider a bounded nonlinearity $z_j(\cdot)\in\left[\underline{z}_j,\bar{z}_j\right]$ possibly depending on state and input variables. A convex basis of functions, i.e., $\eta_0^j(.)\geq 0$, $\eta_1^j(.)\geq 0$ and $\eta_0^j(.)+\eta_1^j(.)=1$ exist such that: $z_j=\underline{z}_j\eta_0^j(z_j)+\bar{z}_j\eta_1^j(z_j)$, we consider

$$\eta_0^j(.)=\frac{\bar{z}_j-z_j(.)}{\bar{z}_j-\underline{z}_j},\ \eta_1^j(.)=1-\eta_0^j(.),\ j\in\{1,2,...,r\} \tag{15}$$

r represents the number of nonlinear terms wher the SNA should be applied. Each vertex will correspond to a combination of the weighting functions $\eta_0^j(\cdot)$ product, i.e. $h_i(z)=\prod_{j=1}^{r}\eta_i^j(z_j)$. Of course, the convex property of the weighting functions $\eta_0^j(\cdot)$ remains for the membership functions $w_i(z)$, i.e. $h_i(z)\geq 0$ and $\sum_{i=1}^{m}h_i(z)=1$. Therefore the model (13) is written as (Tanaka et al., 2001):

$$E_z x_{k+1}=\left(\Sigma_z-\Gamma\times L(x_k)\right)x_k \tag{16}$$

with $E_z=\sum_{i=1}^{m}h_i(z)E_i$ and $\Sigma_z=\sum_{i=1}^{m}h_i(z)\Sigma_{d_i}$ where matrix E_i and Σ_{d_i} are constant matrices Hence (16) represents exactly (13) in a defined compact set. The compact set is directly issued from anatomical constraints of the trunk and upper segments and corresponds to (in ° and °/s):

$$\Omega_x=\begin{Bmatrix} -5\leq q_0\leq 5 & \|\dot{q}_0\|\leq 29 \\ -20\leq q_1\leq 60\ , & \|\dot{q}_1\|\leq 57 \\ -10\leq q_2\leq 45 & \|\dot{q}_2\|\leq 57 \end{Bmatrix} \tag{17}$$

The open-loop unstable behavior of the system is obvious due to the absence of control at the lumbar level; any movement of the UL will produce a rotation of the trunk segment that will destabilize the system and lead to a fall. Therefore, a stabilization step is required prior to designing any observer. The goal is to define the state feedback gains $L(x)$ that stabilizes (13). The "best" option, in the sense of limiting both the number of LMI constraints (avoiding double sums) and the number of slack variables (see discussion in (Estrada-Manzo et al., 2015)) is to consider:

$$L(x)=K_{zz-}G_{z-}^{-1} \tag{18}$$

The regularity of G_{z-} will be discussed later on. Consider the Lyapunov function $V(x_k)=x_k^T P_{z-}^{-1}x_k$, its variation writes:

$$\Delta V(x_k)=\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}^T\begin{bmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1} \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}<0. \tag{19}$$

Let us rewrite (16) as:

$$\begin{bmatrix} \Sigma_z-\Gamma K_{zz-}G_{z-}^{-1} & -E_z \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}=0 \tag{20}$$

By Finsler's lemma, conditions (19)-(20) hold if

$$\begin{bmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1} \end{bmatrix}+M_{(\cdot)}\begin{bmatrix} \Sigma_z-\Gamma K_{zz-}G_{z-}^{-1} & -E_z \end{bmatrix}+(*)<0 \tag{21}$$

with $M_{(\cdot)}=\begin{bmatrix} 0 & R_{z-}^{-1} \end{bmatrix}^T$. Applying the congruence transformation with $diag\left(G_z^T,R_{z-}^{-T}\right)$ to (21) while considering $-G_{z-}-G_{z-}^T+P_{z-}\leq -G_z^T P_{z-}^{-1}G_z$, it follows that

$$\begin{bmatrix} -G_{z-}-G_{z-}^T+P_{z-} & (*) \\ \Sigma_z G_{z-}-\Gamma K_{zz-} & -E_z R_{z-}-(*)+R_{z-}^T P_z^{-1}R_{z-} \end{bmatrix}<0 \tag{22}$$

Using Schur complement, condition (23) is equivalent to

$$\begin{bmatrix} -G_{z-}-G_{z-}^T+P_{z-} & (*) & (*) \\ \Sigma_z G_{z-}-\Gamma K_{zz-} & -E_z R_{z-}-(*) & (*) \\ 0 & R_{z-} & -P_z \end{bmatrix}<0 \tag{24}$$

Remark: If inequality (24) holds then $G_{z-}+G_{z-}^T-P_{z-}>0$, thus it guarantees the regularity of G_{z-}.

This brute force way of doing will be named *LMI_pb1*, the number of nonlinearities present in (14) gives a number of vertices: $2^9 = 512$ due to:

for $\bar{E}(q)$: $\cos(q_1)$, $\cos(q_2)$, $\cos(q_1+q_2)$ for $\left[S(q)\right]_1$: $\frac{\sin(q_0)}{q_0}$, $\frac{\sin(q_0+q_1)}{q_0+q_1}$, $\frac{\sin(q_2+q_1+q_0)}{q_2+q_1+q_0}$ for $\left[\bar{A}(x)\right]_1$: $a_{11}(q_1,q_2,\dot{q}_1,\dot{q}_2)$, $a_{12}(x)$, $a_{13}(x)$	*LMI_pb1*

(25)

Details of the matrices entries are given in (Blandeau, 2018, Chapter 3).

LMI constraints: r^2, size of the LMI constraints $3n$, number of unknowns: $r \times n(2.5n + 0.5 + r \times m)$. To reduce this high number of variables that are not compatible with actual solvers, several ways are possible. A first one is to replace vertices of the polytope by uncertainty descriptions, a second to use properties on the membership functions.

3.2 Using uncertainties description

To reduce the number of vertices, a part of the nonlinearities can be handled via robustness properties as

$$E(q)x^+ = \left(\Sigma(q) + s\Delta\Sigma(x) - \Gamma \times L(x)\right)x \tag{26}$$

with $\Sigma(q) = \begin{bmatrix} I_3 & sI_3 \\ \left[sS(q)\right]_1 & \left[\bar{E}(q)\right]_1 \\ 0_{2\times3} & 0_{2\times3} \end{bmatrix}$ and using the vector notation $e_i^T = \begin{bmatrix} 0 & \dots & 0 & \underset{i^{th}\text{ position}}{1} & 0 & \dots \end{bmatrix}$:

$$\Delta\Sigma(x) = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ 0_{1\times3} & \left[\bar{A}(x)\right]_1 \\ 0_{2\times3} & 0_{2\times3} \end{bmatrix} = \begin{bmatrix} e_4 & e_4 & e_4 \end{bmatrix} \begin{bmatrix} a_{11}(x) & 0 & 0 \\ 0 & a_{12}(x) & 0 \\ 0 & 0 & a_{13}(x) \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \\ e_6^T \end{bmatrix}$$

The $a_{1i}(x)$, $i \in \{1,2,3\}$ are centered and can write: $a_{1i}(x) = \delta_i(x) f_i$ with $|\delta_i(x)| < 1$ to get

$$\Delta\Sigma(x) = H\delta(x)F,\ F = \begin{bmatrix} f_1 e_4^T \\ f_2 e_5^T \\ f_3 e_6^T \end{bmatrix}, H = \begin{bmatrix} e_4 & e_4 & e_4 \end{bmatrix} \tag{27}$$

Therefore, it consists in changing (22) as

$$\begin{bmatrix} -G_{z-} - G_{z-}^T + P_{z-} & (*) \\ \Sigma_z G_{z-} - \Gamma K_{zz-} & -E_z R_{z-} - (*) + R_{z-}^T P_z^{-1} R_{z-} \end{bmatrix} + s\begin{bmatrix} 0 & (*) \\ \Delta\Sigma(x) G_{z-} & 0 \end{bmatrix} < 0 \tag{28}$$

Using the classical square inequality property (Boyd et al., 1994) with a scalar $\mu > 0$

$$\begin{bmatrix} 0 & (*) \\ H\delta(x)FG_{z-} & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ H \end{bmatrix} \delta(x) \begin{bmatrix} FG_{z-} & 0 \end{bmatrix} + (*) \le \begin{bmatrix} \mu^{-1} G_{z-}^T F^T F G_{z-} & 0 \\ 0 & \mu H H^T \end{bmatrix} \tag{29}$$

Thus (28) holds if:

$$\begin{bmatrix} -G_{z-} - G_{z-}^T + s\mu^{-1} G_{z-}^T F^T F G_{z-} + P_{z-} & (*) \\ \Sigma_z G_{z-} - \Gamma K_{zz-} & \Phi \end{bmatrix} < 0 \tag{30}$$

and $\Phi = -E_z R_{z-} - (*) + s\mu HH^T + R_{z-}^T P_z^{-1} R_{z-}$, and via Schur complement on the first entry of (30):

$$\begin{bmatrix} -G_{z-} - G_{z-}^T + P_{z-} & (*) & (*) \\ sFG_{z-} & -s\mu I & 0 \\ \Sigma_z G_{z-} - \Gamma K_{zz-} & 0 & \Phi \end{bmatrix} < 0 \tag{31}$$

Considering that:

$$s\mu HH^T + R_{z-}^T P_z^{-1} R_{z-} = \begin{bmatrix} s\mu H & R_{z-}^T \end{bmatrix} \begin{bmatrix} s^{-1}\mu^{-1} I & 0 \\ 0 & P_z^{-1} \end{bmatrix} \begin{bmatrix} s\mu H^T \\ R_{z-} \end{bmatrix}$$

and using again the Schur complement for Φ gives the following LMI constraints problem:

$$\begin{bmatrix} -G_{z-} - G_{z-}^T + P_{z-} & (*) & (*) & 0 & 0 \\ sFG_{z-} & -s\mu I & 0 & 0 & 0 \\ \Sigma_z G_{z-} - \Gamma K_{zz-} & 0 & -E_z R_{z-} - R_{z-}^T E_z^T & (*) & (*) \\ 0 & 0 & s\mu H^T & -s\mu I & 0 \\ 0 & 0 & R_{z-} & 0 & -P_z \end{bmatrix} < 0 \tag{32}$$

On a complexity point of view, we reduced the vertices of the polytope to $2^6 = 64$ giving LMI constraints problem named *LMI_pb2*, as we only keep:

for $\bar{E}(q)$: $\cos(q_1)$, $\cos(q_2)$, $\cos(q_1+q_2)$ for $\left[S(q)\right]_1$: $\frac{\sin(q_0)}{q_0}$, $\frac{\sin(q_0+q_1)}{q_0+q_1}$, $\frac{\sin(q_2+q_1+q_0)}{q_2+q_1+q_0}$	*LMI_pb2*

3.3 Using properties on the membership functions

Note that the interior of the polytope is partially empty. Effectively, some of the nonlinearities being dependent on the same variables, not all the interior of the polytope can be reached. For instance, let us consider $\cos(q_1)$ and $\frac{\sin(q_1)}{q_1}$, with $q_1 \in [-2,2]$. The maximum error when approximating $\frac{\sin(q_1)}{q_1}$ with the membership function of $\cos(q_1)$, using the technique in (Guerra et al., 2018) is about 3.8%. Then, it is possible to reduce the number of vertices by considering

$\sin(q_1+q_0)=\begin{bmatrix}\frac{\sin(q_0)}{q_0}\cos(q_1) & \frac{\sin(q_1)}{q_1}\cos(q_0)\end{bmatrix}\begin{bmatrix}q_0\\ q_1\end{bmatrix}$. It appears that the nonlinearities to be treated include $f(x)=\frac{\sin(x)}{x}$ and $g(x)=\cos(x)$. Thus, based on the Taylor's expansion we can write: $\frac{\sin(x)}{x}=1-\frac{x^2}{6}+o(x^2)$, $\cos(x)=1-\frac{x^2}{2}+o(x^2)$.

Explications:

For $f(x)$ and $g(x)$ with $x\in[-2,2]rd$

$$\left.\frac{\sin(x)}{x}\right|_{app}=w_f(x)\times\cos(2)+(1-w_f(x))\times 1 \qquad (33)$$

with $w_f(x)=\lambda_f\frac{1-\cos(x)}{1-\cos(2)}+\lambda_0$. Optimal values via Mean Squared Error give $\lambda_0=-0.014$, $\lambda_f=0.9762$ and a maximum approximation error $|g(x)-g_{app}(x)|\le 3.8\%$ (see (Guerra et al., 2018)). At last consider the last case:

$$\sin(q_2+q_1+q_0)=\sin(q_1+q_2)\cos(q_0)+\sin(q_0)\cos(q_1+q_2)$$

$$=\begin{bmatrix}q_0 & q_1 & q_2\end{bmatrix}\begin{bmatrix}\frac{\sin(q_0)}{q_0}\cos(q_1+q_2)\\ \cos(q_0)\frac{\sin(q_1+q_2)}{q_1+q_2}\\ \cos(q_0)\frac{\sin(q_1+q_2)}{q_1+q_2}\end{bmatrix}$$

Thus to represent s_0, s_{01}, s_{02}, $\cos(q_1)$, $\cos(q_2)$ and $\cos(q_1+q_2)$, 4 nonlinearities are enough and nonlinearities such as $\frac{\sin(q_1)}{q_1}$ are treated via optimal values of λ_0 and λ_f by (33). On a complexity point of view, we reduced the vertices of the polytope to $2^7=128$ giving LMI constraints problem named *LMI_pb3*, as we only keep

for $\bar{E}(q)$ and $[S(q)]_1$: $\cos(q_0)$, $\cos(q_1)$, $\cos(q_2)$, $\cos(q_1+q_2)$ for $[\bar{A}(x)]_1$: $a_{11}(q_1,q_2,\dot{q}_1,\dot{q}_2)$, $a_{12}(x)$, $a_{13}(x)$	*LMI_pb3*

4. LMI CONSTRAINT PROBLEMS AND RESULTS

We have now, 3 LMI constraint problems and create a 4th one mixing both approaches of reducing complexity. Thus, we get *LMI_pb4*, which uses uncertainty description for $[\bar{A}(x)]_1$ and ends with $2^4=16$ vertices:

for $\bar{E}(q)$ and $[S(q)]_1$: $\cos(q_0)$, $\cos(q_1)$, $\cos(q_2)$, $\cos(q_1+q_2)$	*LMI_pb4*

Table 1 summarizes the approaches tested as well as their complexity according to the number of LMI constraints and unknowns.

Considering the special form of $E(q)=\begin{bmatrix}I_3 & 0_3\\ 0_3 & \bar{E}(q)\end{bmatrix}$ we can extend the freedom, i.e. number of slack variables, of R_{z-} without increasing the number of LMI constraints. Effectively, consider $R^1_{zz-}, R^2_{z-}\in\mathbb{R}^{3\times 6}$ such as:

$$\begin{bmatrix}I_3 & 0_3\\ 0_3 & \bar{E}_z\end{bmatrix}\begin{bmatrix}R^1_{zz-}\\ R^2_{z-}\end{bmatrix}=\begin{bmatrix}R^1_{zz-}\\ \bar{E}_zR^2_{z-}\end{bmatrix} \qquad (34)$$

The 3 first rows of (34) also depend on the delayed sample variable z_{k-1}. Of course, this way of doing can only be applied on the LMI_pbs that are compatible with the solvers. Therefore, we construct LMI_pb5 as LMI_pb4 and (34).

The feasibility of the LMI conditions were compared between the four presented problems. For one single problem (given the initial vertices) *LMI_pb2, LMI_pb3* and *LMI_pb4* took respectively around 10 minutes, 25 minutes and 4 seconds to find a control solution, *LMI_pb1* was stopped after 8 hours before having found any solution. Due to the numerical complexity, feasible solutions were found for only 62% of compact set (17) for *LMI_pb2* whereas feasible solutions were found for the whole compact set for *LMI_pb4*.

Table 1. Size, number of LMI constraints and number of unknowns for the 4 LMI problems

$n=6, m=3$	LMI	Size	Unknowns
LMI_pb1 (24) $r=512$	262 144	18	4 766 208
LMI_pb2 (32) $r=64$	4 096	30	79 680
LMI_pb3 (24) $r=128$	16 384	18	306 816
LMI_pb4 (32) $r=16$	256	30	6 096

All LMI problems were run on Matlab R2019 using YALMIP and MOSEK solver on a computer equipped with Intel® Core™ i7-5600 CPU, and a 2.6GHz processor.

Simulations were run with the two sets of control gains obtained with *LMI_pb2* and *LMI_pb4*. The simulation respected the following structure, starting with a set of non-zero initial conditions, the continuous S3S model (4) was then submitted to a 10Hz sinusoidal perturbation applied to the trunk angular velocity $\dot{q}_0$. Two parameters were studied to assess the robustness of the obtained gains: the amplitude of both initial conditions (cf. Fig 2.a.) and perturbation (cf. Fig 2.b.), i.e. both amplitude were increased until failure of the control law. The maximum initial conditions before divergence were similar for both sets of control gains, whereas using gains from *LMI_pb4* allowed to increase the perturbation amplitude by 12.5%.

5. CONCLUSIONS

Stabilization conditions for a biomechanical S3S model representing a person living with a spinal cord injury have been proposed. To this end, a variation of a double inverted pendulum has been formulated, then a control law has been designed in order to mimic a human behavior. Classic brute force technique leads to a very heavy LMI conditions to be solved. Hence, previous works using both robustness and model simplification using membership functions similarities were used to obtain a lighter LMI problem. Those LMI problems are essentials to stabilize the model in simulation and thus test the features of the future observers which will permit to better understand sitting stability for people living with a SCI. A future step for the application work would be to exploit the TS fuzzy modeling with nonlinear consequents (Coutinho et al., 2020) to significantly reduce the numerical complexity for real-time control implementation.

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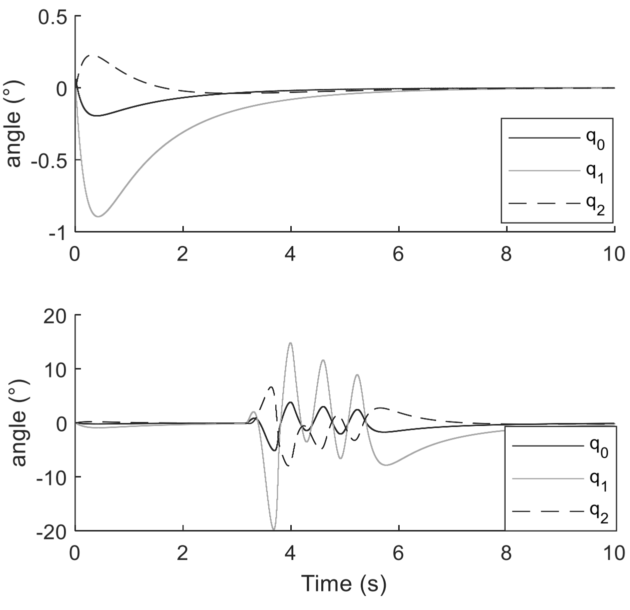

Fig. 2. Simulation results using *LMI_pb4* control gains with specific initial conditions (up) and perturbation (down).

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