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Harmonic balance analysis of pull-in range and oscillatory behavior of third-order type 2 analog PLLs

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Abstract: The most important design parameters of each phase-locked loop (PLL) are the local and global stability properties, and the pull-in range. To extend the pull-in range, engineers often use type 2 PLLs. However, the engineering design relies on approximations which prevent a full exploitation of the benefits of type 2 PLLs. Using an exact mathematical model and relying on a rigorous mathematical thinking this problem is revisited here and the stability and pull-in properties of the third-order type 2 analog PLLs are determined. Both the local and global stability conditions are derived. As a new idea, the harmonic balance method is used to derive the global stability conditions. That approach offers an extra advantage, the birth of unwanted oscillations can be also predicted. As a verification it is shown that the sufficient conditions of global stability derived by the harmonic balance method proposed here and the well-known direct Lyapunov approach coincide with each other, moreover, the harmonic balance predicts the birth of oscillations in the gap between the local and global stability conditions. Finally, an example when the conditions for local and global stability coincide, is considered.

Keywords: Phase-locked loop, third-order PLL, type 2 PLL, nonlinear analysis, harmonic balance method, describing function, global stability, birth of oscillations, hold-in range, pull-in range, lock-in range, Egan conjecture.

1. INTRODUCTION

Synchronization of signals is a fundamental problem in many applications from satellite navigation (Kaplan and Hegarty, 2017), wireless communications (Du and Swamy, 2010; Rouphael, 2014; Best et al., 2016), optical communication (Cho, 2006; Ho, 2005; Helaluddin, 2008; Rosenkranz and Schaefer, 2016), power inverter synchronization (Zhong and Hornik, 2012, pp. 361–366), to clock signal generation (Sakamoto and Nakao, 2006), and in many other applications (Best, 2018). The most frequently used solution is offered by the phase-locked loops (PLLs), which enable to synchronize the output signal of a voltage-controlled oscillator with a reference signal (Gardner, 1966; Viterbi, 1966; Shakhgil'dyan and Lyakhovkin, 1966; Best, 2007).

The most important PLL characteristics are the hold-in, pull-in and lock-in ranges. The hold-in range corresponds to the existence of a locked state, the pull-in range guarantees acquisition for any frequency errors, and the lock-in range defines fast-locking conditions (see, e.g. (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016) for rigorous mathematical definitions). While the hold-in range can be studied by the Routh-Hurwitz criterion and the Nyquist plot, the analysis of global stability and the estimation of the pull-in and lock-in ranges is a challenging task.

It is known that the second-order type 2 PLL has infinite pull-in range, i.e., the corresponding system of ODEs is globally stable for any value of frequency error [for rigorous proof refer to (Aleksandrov et al., 2016)]. Exploiting this observation, a loop filter including an ideal integrator, i.e., a pole at $s = 0$ is frequently used in higher-order loops to extend the PLL pull-in range (Gardner, 2005; Best, 2018).

This paper derives both the local and global stability conditions of a third-order type 2 Analog Phase-locked Loop (APLL). Section 2 explains the operation principle of two-phase APLL studied here. Section 3 uses the Routh-Hurwitz criterion to determine the APLL local stability conditions. The harmonic balance method is used in Section 4 to estimate that domain of APLL parameters over which the global stability is achieved. Finally, an example of the lock-in range estimation is considered.

2. MATHEMATICAL MODEL OF PLL

An ideal analog multiplier is used in the APLLs to implement the phase detector (PD). However, the analog multipliers produce an unwanted sum-frequency periodic

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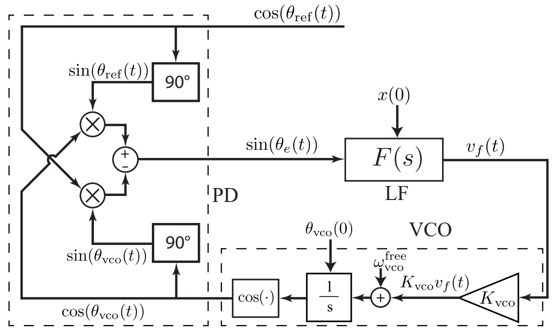

Fig. 1. Block diagram of a two-phase APLL.

component at their output in addition to the desired low-frequency error signal. The unwanted periodic PD output generating a harmful FM in the APLL output is neglected in the vast majority of APLL analyses [see, e.g., (Gardner, 2005; Kolumbán, 2005; Best, 2007)], making the results questionable. In the built APLLs extra low-pass filtering is used to suppress the effect of sum-frequency PD output.

The unwanted sum-frequency PD output is fully eliminated in the two-phase PLLs (Emura et al., 2000; Best et al., 2014; Bianchi et al., 2016) by using a special PD configurations. As shown in Fig. 1, the block diagram of two-phase PD includes multipliers, phase shifters and an adder to eliminate the sum-frequency output. From a mathematical point of view, the two-phase PD is an analog multiplier where only the desired low-frequency error signal appears at the PD output. Note, the baseband APLL model used everywhere in the literature gives an accurate model only for the two-phase APLL while it is only an approximation of the operation of other APLLs built with different PDs including analog multipliers, EX-OR gates, etc.

The unified baseband model of APLLs (Gardner, 2005; Kolumbán, 2005; Best, 2007) is depicted in Fig. 2 where $\theta_{\rm ref}(t)$ and $\theta_{\rm vco}(t)$ denote the phases of reference and voltage-controlled oscillator (VCO) signals, respectively, $\theta_e(t) = \theta_{\rm ref}(t) - \theta_{\rm vco}(t)$ defines the phase error, $K_{\rm vco} > 0$ is the VCO gain, $\omega_{\rm vco}^{\rm free}$ denotes VCO free-running frequency and the integrator $\frac{1}{s}$ describes the VCO transfer function where the initial state is given by $\theta_{\rm vco}(0)$. The APLL acquisition properties are studied here, therefore, we assume that $\dot{\theta}_{\rm ref} \equiv \omega_{\rm ref}$.

In this paper we consider a second-order loop filter (LF) with the initial state $x(0) = \big(x_1(0), x_2(0)\big)$ and the following transfer function

$$F(s) = K_F \frac{(1+s\tau_{z1})(1+s\tau_{z2})}{s(1+s\tau_p)}$$

where $K_F > 0$, $\tau_{z1} > 0$, $\tau_{z2} > 0$, $\tau_p > 0$. The parameters of the loop filter are set in such a way that $\tau_p \neq \tau_{z1}$, $\tau_p \neq \tau_{z2}$. The loop filter is driven by the PD output, i.e., the error signal, and its output $v_f(t)$ controls the instantaneous frequency of VCO circuit. Note, the APLL under study includes two integrators, one of them is implemented by the loop filter while the other is introduced by the VCO transfer function. Therefore, the APLL studied here is a *type 2 PLL* (Gardner, 2005, p.12).

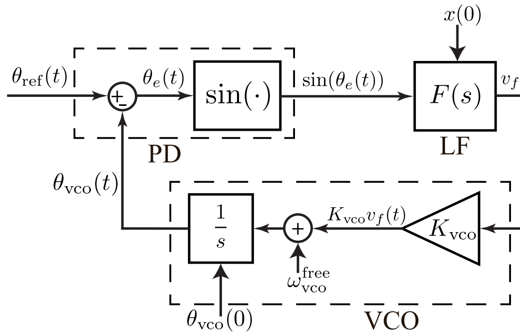

Fig. 2. Baseband model of APLLs.

The behavior of APLL baseband model in the state space is described by a third-order nonlinear ODE:

$$\dot{y} = Py + q\sin(r^T y), \tag{1}$$

$$y(t) = \left(x_1(t) - \frac{\omega_e^{\rm free}}{K_F K_{\rm vco}},\ x_2(t),\ \theta_e(t)\right)^T$$

where

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{\tau_p} & 0 \\ -K_F K_{\rm vco} & -K_F K_{\rm vco} & 0 \end{pmatrix},$$

$$r^T = (0,\ 0,\ 1),\ q = \begin{pmatrix} 1 \\ \frac{(\tau_{z1}-\tau_p)(\tau_p-\tau_{z2})}{\tau_p^2} \\ -K_{\rm vco}K_F\frac{\tau_{z1}\tau_{z2}}{\tau_p} \end{pmatrix},$$

$\big(x_1(t), x_2(t)\big) \in \mathbb{R}^2$ gives LF state and $\omega_e^{\rm free}$ is a frequency error $\omega_e^{\rm free} = \omega_{\rm ref} - \omega_{\rm vco}^{\rm free}$.

3. LOCAL STABILITY ANALYSIS

The engineering definition of the hold-in range, corresponding to the local stability analysis, is the following. Suppose the phase-locked conditions [1] have been achieved in the PLL. Now vary the reference frequency slowly and the VCO frequency will follow it. The hold-in frequency $\omega_h = |\omega_{\rm ref} - \omega_{\rm vco}^{\rm free}|$ is determined by the lower and upper values of $\omega_{\rm ref}$ for which the phase-lock is lost. The hold-in range represents the maximum static tracking range and is determined by the saturation characteristics of the nonlinear loop elements in the PLL (Gardner, 2005; Kolumbán, 2005; Best, 2007).

A strict mathematical definition for the hold-in range has been published in (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016).

Definition 1. The hold-in range (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016). A *hold-in range* is the largest interval of frequency errors $|\omega_e^{\rm free}| \in [0, \omega_h)$ where the phase-locked conditions are maintained; ω_h is called a *hold-in frequency.*

To perform the local stability analysis, the equilibrium points of system (1) have to be determined. Then the

[1] Under phase-locked conditions the phase error $\theta_e(t)$ is constant and a steady state is considered to be stable: if the APLL is in phase-lock, after applying a small perturbation to the state of the system it returns to its previous state.

stability of these equilibria can be evaluated by the Routh-Hurwitz criterion. The conclusion is that equilibria $(0, 0, \pi + 2\pi n)$, $n \in \mathbb{Z}$ are unstable and the equilibria $(0, 0, 2\pi n)$, $n \in \mathbb{Z}$ are asymptotically stable if and only if

$$K_F K_{\text{vco}} \tau_{z1} \tau_{z2} (\tau_{z1} + \tau_{z2}) > \tau_p - \tau_{z1} - \tau_{z2}. \quad (2)$$

Condition (2) has been determined using open-loop transfer function $G(s)$ of the linearized ($\sin\theta_e \approx \theta_e$) baseband model:

$$G(s) = K_{\text{vco}} \frac{F(s)}{s} = K_{\text{vco}} K_F \frac{(1 + s\tau_{z1})(1 + s\tau_{z2})}{s^2(1 + s\tau_p)}. \quad (3)$$

During the stability analysis the characteristic equation has been determined first then the Routh-Hurwitz criterion has been applied. Since system (1) does not depend on the frequency error ω_e^{free}, it does not appear explicitly in stability conditions. Thus, in the considered third-order type 2 APLL the hold-in range is infinite [2] $[0, \omega_h) = [0, +\infty)$ when condition (2) is fulfilled, otherwise $\omega_h = 0$.

4. GLOBAL STABILITY ANALYSIS VIA THE HARMONIC BALANCE METHOD

The birth of periodic or chaotic oscillations implies the loss of global stability in the PLL. Although the birth of chaotic signals in third-order analog PLL has been already shown for the case when all equilibria are unstable in the circuit [see, e.g., (Kolumbán and Vizvari, 1995; Kolumban et al., 1997)], the situation studied here is completely different from that case, because in the case studied here a locally stable equilibrium exists in the circuit but in addition to the locally stable equilibrium another attractor may be also present which develops a periodic orbit in the PLL. The harmonic balance method (Krylov and Bogolyubov, 1947; Khalil, 2002) is widely used for the study of periodic oscillations. Although the harmonic balance method relies on approximation [see, e.g., (Tsypkin, 1984; Leonov and Kuznetsov, 2013)] it can be used to predict the birth of oscillation and, consequently, to check the global stability of phase-lock loop [see, e.g., (Rey, 1960; Shakhgil'dyan and Lyakhovkin, 1966; Homayoun and Razavi, 2016)].

To analyze the periodic solutions using the harmonic balance method we introduce the coefficient of harmonic linearization k and rewrite system (1) in the form:

$$\dot{y} = P_0 y + q\varphi(r^T y) \quad (4)$$

where

$$P_0 = P + kqr^T, \quad \varphi(\sigma) = \sin\sigma - k\sigma.$$

The coefficient k is chosen in such a way that the matrix P_0 has a pair of imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$); values of k and ω_0 are found from the equation [3]

$$1 + kG(i\omega_0) = 0. \quad (5)$$

[2] From an engineering point of view, the hold-in range is limited by saturation. In general, the PD, loop filter and amplifier and VCO can get into saturation. Here the integrator implemented by the loop filter assures that the steady state phase error θ_e is always zero (mod 2π). Therefore, the PD never can get into saturation. To simplify the problem we assume in the rest of the contribution that neither the amplifier/loop filter, nor the VCO can get into saturation.

[3] Formula (5) can be obtained by consideration the following matrix: $(a_{11} = 1, a_{12} = r^T; a_{21} = -kq, a_{22} = P - sI)$. By Schur complement, we get $\det(P - sI + kqr^T) = (1 + kG(s))\det(P - sI)$ where $G(s)$ is transfer function (3). Since $P + kqr^T = P_0$, substituting $s = i\omega_0$ one gets formula (5).

Thus, ω_0 and k can be found from the following equations:

$$\text{Im}G(i\omega_0) = 0, \quad k = -\big(\text{Re}G(i\omega_0)\big)^{-1}.$$

For system (1) such k and ω_0 exist if and only if

$$\tau_p - \tau_{z1} - \tau_{z2} > 0 \quad (6)$$

and have the form

$$\omega_0 = \sqrt{\frac{\tau_p - \tau_{z1} - \tau_{z2}}{\tau_{z1}\tau_p\tau_{z2}}} > 0, \quad (7)$$

$$k = \frac{\tau_p - \tau_{z1} - \tau_{z2}}{K_F K_{\text{vco}} \tau_{z1}\tau_{z2}(\tau_{z1} + \tau_{z2})} > 0. \quad (8)$$

To determine the amplitude of the periodic solution by the harmonic balance method, we represent solutions of system (4) in the Cauchy form:

$$y(t) = e^{P_0 t} y(0) + \int_0^t e^{P_0 t} e^{-P_0 \tau} q\varphi\big(r^T y(\tau)\big) d\tau. \quad (9)$$

There exists an invertible matrix S such that the similar matrix $J = S^{-1} P_0 S$ has the Jordan form:

$$J = \begin{pmatrix} 0 & -\omega_0 & \mathbb{O} \\ \omega_0 & 0 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \lambda_3 \end{pmatrix}$$

where $\lambda_3 < 0$ is the third eigenvalue of P_0. Hence,

$$e^{P_0 t} = S e^{Jt} S^{-1} = S \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} S^{-1}.$$

Then we multiply solution (9) by the vector $(1\ 0\ 0)\, S^{-1}$ and put $t = \frac{2\pi}{\omega_0}$:

$$\begin{aligned} &(1\ 0\ 0) \int_0^{\frac{2\pi}{\omega_0}} e^{-J\tau} S^{-1} q\varphi\big(r^T y(\tau)\big) d\tau = \\ &= (1\ 0\ 0)\, S^{-1} \Big(y(\frac{2\pi}{\omega_0}) - y(0)\Big). \end{aligned} \quad (10)$$

Further, equality (10) allows one to obtain formulae for the describing function, which are used in the harmonic balance method for the determination of the amplitude of periodic solution — cycle of the first or the second kind.

4.1 Cycles of the first kind

Let us assume first that a periodic, almost sinusoidal solution develops in the PLL:

$$y(t) = y(t + \frac{2\pi}{\omega_0}) \quad \forall t \geq 0, \quad (11)$$

$$r^T y(t) = \theta_e(t) \approx a_0 \sin(\omega_0 t) \quad (12)$$

where a_0 and $\omega_0 > 0$ are the amplitude and the frequency of oscillations. Periodic solution (11) is referred to as a cycle of the first kind.

Substituting (11) and (12) into (10), we can determine a_0 from the relation

$$\Phi_1(a_0) = 0 \quad (13)$$

where

$$\Phi_1(a) = \int_0^{\frac{2\pi}{\omega_0}} \varphi\big(a\sin(\omega_0\tau)\big)\sin(\omega_0\tau) d\tau$$

is a describing function.

Evaluating (13) for $\varphi(\sigma) = \sin\sigma - k\sigma$, we get

$$2\frac{J_1(a)}{a} = k \quad (14)$$

where $J_1(a)$ is the Bessel function of the first kind $[J_n(a) = \frac{1}{\pi}\int\limits_0^{\pi}\cos(nt - a\sin t)dt]$. The amplitude a_0 can be found from equation (14) numerically. The boundedness of the left-hand side of (14) implies the existence condition of the amplitude a_0: $k < 1$. Taking into account formula (8), we get that the existence condition of a_0 coincides with the condition of local stability (2). Thus, the harmonic balance method predicts a cycle of the first kind in system (1) if and only if conditions (2) and (6) are fulfilled, i.e., in addition to the desired phase-locked conditions a periodic solution exists in the PLL according to the harmonic balance method.

4.2 Cycles of the second kind

In this section we assume that a phase modulated sinusoidal signal develops in the PLL. Suppose that system (1) has a periodic solution

$$y(t) = y(t + \frac{2\pi}{\omega_0}) - (0,\ 0,\ 2\pi)^T \quad \forall t \geq 0 \tag{15}$$

which can be expressed in the form of a phase modulated sinusoidal signal

$$r^T y(t) = \theta_e(t) \approx \omega_0 t + a_0 \sin(\omega_0 t). \tag{16}$$

Again, the harmonic balance method will be used to find the parameters of the periodic solution.

Substituting (15) and (16) into (10), we can obtain an equation for the determination of the amplitude a_0:

$$\Phi_2(a_0) = k\frac{2\pi}{\omega_0} \tag{17}$$

where $\Phi_2(a) = \int_0^{\frac{2\pi}{\omega_0}} \varphi\big(\omega_0\tau + a\sin(\omega_0\tau)\big)\sin(\omega_0\tau)d\tau$.

Evaluating (17) for $\varphi(\sigma) = \sin\sigma - k\sigma$, we get the equation $J_0(a) - J_2(a) = ka$ which gives the amplitude a_0 of a cycle of the second kind. Note, in this case a_0 exists for any value of k. Thus, the harmonic balance method predicts cycles of the second kind in system (1) if and only if condition (6) is fulfilled.

4.3 The pull-in range estimation

The engineering definition of pull-in range assumes that the phase-locked conditions do not exist at the start of the pull-in process. The pull-in frequency $\omega_p = |\omega_{\text{ref}} - \omega_{\text{vco}}^{\text{free}}|$ is the maximum initial frequency difference between the reference and VCO free-running frequencies both in positive and negative directions, for which the PLL eventually achieves the phase-locked conditions (Kolumbán, 2005).

A strict mathematical definition for the pull-in range has been published in (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016).

Definition 2. Pull-in range (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016). A *pull-in range* is the largest interval of frequency errors $|\omega_e^{\text{free}}| \in [0, \omega_p)$ for which the phase-locked conditions are acquired for arbitrary initial state; ω_p is called a *pull-in frequency.*

System (1) does not depend on the frequency error ω_e^{free}, therefore, both the hold-in and pull-in ranges are either infinite or empty. The physical explanation is that the PLL studied here is a type 2 feedback system and if a type 2 system can reach the phase-locked conditions then its hold-in and pull-in ranges are infinite. Hence, the problem of the pull-in range estimation turns to the problem of estimation of those values of $K_{\text{vco}}, K_F, \tau_{z1}, \tau_{z2}, \tau_p$ for which system (1) is globally stable.

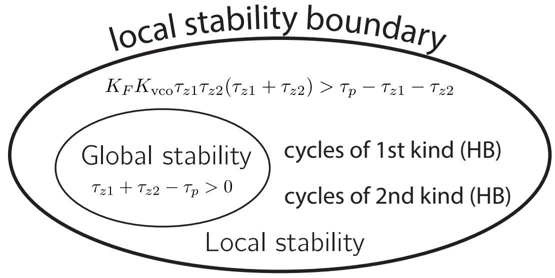

Fig. 3. Stability regions observed in the third-order type 2 APLLs. Note, the region of local stability [PLL hold-in range, given by conditions (2)] is much larger than the region of global stability [PLL pull-in range, given by conditions (18)]. The gap between the two regions is filled by the periodic steady state solutions determined by the harmonic balance method.

The harmonic balance method has been used in Sections 4.1, 4.2 to check the presence of harmonic oscillation. According to the harmonic balance, periodic solution does not develop in steady state if the following condition is met:

$$\tau_{z1} + \tau_{z2} - \tau_p > 0. \tag{18}$$

Remark that the same estimate (18) can be obtained by a Lyapunov function of the Lurie-Postnikov form and a modification[4] of the direct Lyapunov method for the cylindrical phase space (Kuznetsov et al., 2019).

Comparing the derived conditions of local and global stability [(2) and (18), respectively], we conclude that the domain of parameters satisfying (2) is wider than the domain of parameters satisfying (18). As shown in Fig. 3, a gap exists between these domains where the harmonic balance method predicts the existence of periodic oscillations referred to as the cycles of the first and the second kind above. Therefore, a parameter region can be observed where the PLL after reaching phase-lock maintains the phase-lock conditions (hold-in region) but cannot reach phase-lock when it is started from arbitrary initial conditions (pull-in range). According to our terminology, the pull-in range is empty in this region. Hence, the Egan conjecture on the infiniteness of pull-in range of type 2 PLLs (Egan, 2007, p.59, p.138, p.161) requires further study and additional clarifications.

4.4 Example

Although the harmonic balance method relies on approximation, it may provide the necessary and sufficient con-

[4] Note that in the cylindrical phase space, the classical Barbashin-Krasovsky theorem and the LaSalle invariance principle cannot be used together with a Lyapunov function $V(x, \theta_e)$ of "quadratic part plus integral of nonlinearity" form, because in those theorems the Lyapunov function $V(x, \theta_e)$ must be radially unbounded while in the cylindrical phase space $V(0, \theta_e) \not\to +\infty$ as $|\theta_e| \to +\infty$ [see (Gelig et al., 1978; Leonov and Kuznetsov, 2014; Abramovitch, 1988; Kuznetsov et al., 2020)].

ditions of global stability and the birth of oscillation. Consider the case $\tau_{z2} = \tau_p = 0$:

$$F(s) = K_F \frac{1+s\tau_{z1}}{s} = \frac{1+s\tau_2}{s\tau_1} \quad (19)$$

where $\tau_1 = \frac{1}{K_F} > 0,\ \tau_2 = \tau_{z1} > 0$. Substituting $\tau_{z2} = \tau_p = 0$ and $\tau_{z1} = \tau_2 > 0$ into (6), we can see that the parameters ω_0 and k do not exist for the corresponding second-order type 2 PLL model with $\tau_1 > 0, \tau_2 > 0$. Hence, in this case the harmonic balance method provides global stability of the model and shows that both the hold-in and pull-in ranges are infinite. This result can be verified by the direct Lyapunov method [see (Aleksandrov et al., 2016)].

However, please recognize that the existence of global stability says nothing about the pull-in time which can be extremely long. To reduce the long acquisition time acquisition aids are frequently used or the lock-in concept is exploited (Gardner, 2005; Kolumbán, 2005; Best, 2007).

In engineering, the lock-in range is defined in the following manner: Consider a PLL which is in phase-lock and then change the reference frequency suddenly in an abrupt manner according to a unit step function. The lock-in range $\omega_l = |\omega_{\text{ref}} - \omega_{\text{vco}}^{\text{free}}|$ is the frequency range over which the PLL re-establishes the phase-locked conditions without *cycle slipping* [5]. The frequency range over which the transient decays without cycle slipping is referred to a lock-in range.

For a rigorous mathematical definition on the lock-in range refer to (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016).

Definition 3. Lock-in range (Kuznetsov et al., 2015; Leonov et al., 2015; Best et al., 2016). A *lock-in range* is the largest interval of frequency errors from the pull-in range: $|\omega_e^{\text{free}}| \in [0, \omega_l) \subset [0, \omega_p)$ such that the APLL being in a steady state, after any abrupt change of ω_e^{free} within the interval acquires a steady state without cycle slipping; ω_l is called a *lock-in frequency.*

The considered second-order type 2 PLL, built with a loop-filter transfer function defined by (19), has an infinite pull-in range and a finite lock-in range. Figure 4 compares the lock-in ranges determined by the numerical computation [6] (black curve) and calculated by the approximative equations (red curve) used in engineering design: $\omega_l \approx \frac{K_{\text{vco}}\tau_2}{\tau_1}$ [see, e.g., (Gardner, 2005, p.187) where $K_d = 1,\ K_o = K_{\text{vco}}$, (Kolumbán, 2005, p.3748) where $K_d = 1,\ K_v = K_{\text{vco}}$, and (Best, 2007, p.67) where $\omega_l \approx 2\zeta\omega_n$, $\omega_n = \sqrt{K_{\text{vco}}/\tau_1}$, $\zeta = \omega_n\tau_2/2$]. The two results are similar to each other but the error between them shows that the results known from the literature have to be updated.

[5] It is said that *cycle slipping* occurs if $\sup_{t>0} |\theta_e(0) - \theta_e(t)| \geq 2\pi$.
[6] For the considered model the boundary value ω_l is determined by such an abrupt change of ω_e^{free} that the corresponding trajectory tends to the nearest unstable equilibrium, i.e., $\lim_{t\to+\infty} |\theta_e(0) - \theta_e(t)| = \pi$. For a larger frequency error $|\omega_e^{\text{free}}|$ the limit is $\lim_{t\to+\infty} |\theta_e(0) - \theta_e(t)| = 2\pi$ and cycle slipping occurs. Since the lock-in range is defined as a half-open interval, the boundary value $\omega_e^{\text{free}} = \omega_l$ is not included in it. Corresponding behaviour is not observed in practice, because it is unstable and disturbed by noise.

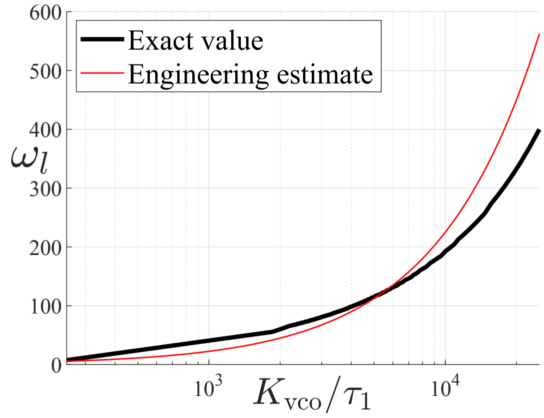

Fig. 4. Lock-in range comparison. Parameter $\tau_2 = 0.0225$.

5. CONCLUSION

As is known, the second-order type 2 PLLs have infinite hold-in and pull-in ranges, however these PLLs are not often found in practice (Gardner, 2005). To improve the transient characteristics of PLLs, high-frequency poles are often added and the corresponding high-order PLLs are considered (Abramovitch, 1988; Craninckx and Steyaert, 1998; De Muer and Steyaert, 2003; Gardner, 2005).

Exploiting the harmonic balance method the hold-in and pull-in ranges of third-order type 2 analog PLLs have been determined here. The exact rigorous mathematical analysis revealed that in general, this PLL can be characterized by three distinct operation regions: (i) the lock-in range which is always smaller than the pull-in range, (ii) the pull-in range and (iii) a periodic range. In the periodic range at least two attractors coexist in the PLL, the desired phase-lock and a periodic solution. By using the harmonic balance method we have shown that two kinds harmonic oscillations can be observed in the periodic range: a sinusoidal waveform referred to a cycle of the first kind and a phase modulated sinusoidal waveform referred to a cycle of the second kind. The theory derived here gives a better estimation of the lock-in and pull-in regions which are among the most important PLL design parameters.

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