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Synchronization of Heterogeneous Dynamical Networks via Phase Analysis ⋆

Dan Wang * **Wei Chen** ** **Li Qiu** *

* *Department of Electronic & Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China (e-mail: dwangah@connect.ust.hk, eeqiu@ust.hk)*
** *Department of Mechanics and Engineering Science & Beijing Innovation Center for Engineering Science and Advanced Technology, Peking University, Beijing, China (e-mail: w.chen@pku.edu.cn)*

Abstract: This paper is concerned with the synchronization of heterogeneous agents interacting over a dynamical network, where the edge dynamics are heterogeneous, modeling the nonuniform communication environment between the agents. Novel synchronization conditions are obtained from a phasic perspective by utilizing a newly formulated small phase theorem. These conditions have lower conservatism compared to gain-based conditions and generalize positive real and negative imaginary type conditions. They scale well with the size of the network and reveal the trade-off between the phases of node dynamics and edge dynamics.

1. INTRODUCTION

1.1 Background

Networks with dynamical nodes and static edges have been the focus of many studies over a long period of time. See Lin (1974); Moylan and Hill (1978); Araposthatis et al. (1981); Olfati-Saber et al. (2007), to name just a few. More recently, general networks containing both node and edge dynamics have attracted considerable attention (Bürger et al., 2014; Khong et al., 2016; Nepusz and Vicsek, 2012; Pates and Vinnicombe, 2017; Wang et al., 2017) due to an increasing awareness that edge dynamics are often equally important to node dynamics in the study of complex networks.

Among many important problems in dynamical networks, synchronization appears to have gained a lot of popularity in the past decades. Generally speaking, synchorinization seeks to make the outputs of multiple agents converge to a common trajectory over time (Li et al., 2010; Wieland et al., 2011; Khong et al., 2016). An important special case of synchronization is consensus, which is also known as the agreement or rendezvous problem; see, for example, Jadbabaie et al. (2003); Lin et al. (2006); Olfati-Saber et al. (2007); Ren and Beard (2008); Lestas and Vinnicombe (2010); Trentelman et al. (2013). The synchronization problem is ubiquitous in various engineering applications such at power systems and multi-robot systems. In most applications, both agent and edge dynamics are multivariable systems.

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It is widely recognized that the synchronization problem can be transformed into a feedback stability problem on a disagreement subspace (Ren and Beard, 2008), to which numerous approaches can be applied. For instance, Li et al. (2010) proposed an observer-based synchronization protocol for homogeneous LTI agents over a directed network. The authors obtained a distributed synchronization condition by casting the problem into a simultaneous stabilization problem. In Wieland et al. (2011), an internal model requirement has been proposed for output synchronization of heterogeneous LTI agents over a time-varing directed network.

As one of the most used results in stability analysis, the small gain theorem appears as a natural tool in studying the synchronization problem. However, applying the small gain theorem could lead to undesirable conservativeness, especially when the gains of the agents are very large or subject to a big amount of uncertainties. Researchers have been seeking alternatives beyond the small gain (Lestas and Vinnicombe, 2010; Khong et al., 2016). In particular, there have been conditions for synchronization based on positive realness or negative imaginariness of the dynamics (Bürger et al., 2014; Fujimori et al., 2011; Wang et al., 2015), which effectively bring in a phasic perspective into the analysis of synchronization. One motivation of studying the synchronization problem from phasic perspective comes from applications, in particular from power networks and unmanned system networks. One can observe that individual agents in these networks, such as generators, loads, UAVs, and UGVs, often have similar phase properties regardless of their physical sizes. In recent work Chen et al. (2019), the authors studied phase bounded systems by virtue of a newly defined phase response of MIMO LTI systems. In addition, a small phase theorem was devised therein, which serves as a counterpart to the small gain theorem. It would be very interesting to see what these new developments in MIMO phase have to offer in solving the synchronization problem.

In this paper, the synchronization of networks with both node and edge dynamics will be considered from a phasic perspective. We will show that in studying the synchronization problem, understanding the phases of the agents and edges is not only closer to our physical intuition but also helps the solution tremendously. We assume semi-stable agents and extend the small phase theorem in Chen et al. (2019) to accommodate such systems. Synchronization conditions will be explored when heterogeneous agents interact over heterogenous edges, which model the nonuniform communication dynamics between agents.

The rest of this paper is organized as follows. Some preliminaries are introduced in the remaining of this section. The synchronization problem is formulated in Section 2. Matrix phases and their properties are introduced in Section 3, followed by the frequecncy-wise semi-sectorial systems and the small phase theorem in Section 4. Section 5 presents synchronization conditions in terms of phases of node dynamics and edge dynamics. A numerical example is simulated in Section 6. Section 7 concludes this paper with some remarks.

Notation: Denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Denote by $\mathbb{C}^+$ and $\overline{\mathbb{C}}^+$ the open and closed right-half plane, respectively. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. We use $\mathbf{1}$ to denote the vector with all its entries equal to 1, while the size of the vector is to be understood from the context. The symbol diag$\{\cdot\}$ denotes the diagonal operation.

1.2 Preliminaries

Consider an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ which consists of a set of nodes $\mathcal{V} = \{1, \ldots, n\}$ and a set of edges $\mathcal{E} = \{e_1, \ldots, e_l\}$. We use $e_k = (i, j)$ to denote the edge connecting node i and node j. A weighted graph is one with each of its edges associated with a weight. Let the weight of edge (i, j) be a_{ij}. Then, the Laplacian matrix $L = [l_{ij}]$ of such a weighted graph is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j=1, j\neq i}^{n} a_{ij}, & i = j. \end{cases} \tag{1}$$

Conventionally, a_{ij} are taken to be positive real numbers, making L an $n \times n$ constant matrix. Since all the rows of L sum up to 0, L has a zero eigenvalue with a corresponding eigenvector being $\mathbf{1}_n$. Moreover, when the graph is connected, 0 is a simple eigenvalue and L is an irreducible matrix, i.e., L is not similar via a permutation to a block diagonal matrix (Horn and Johnson, 1990). See Merris (1994) for a survey on Laplacian matrices.

In later developments of this paper, we will also encounter weighted graphs with stable dynamical weights $A_{ij}(s)$, which in general can be multivariable transfer functions. Let the dimension of $A_{ij}(s)$ be $m \times m$. Then, the Laplacian matrix as in (1) becomes a dynamical Laplacian $L(s)$ of dimension $nm \times nm$.

There is a useful factorization of $L(s)$. For a graph $\mathbb{G}$, we can assign an arbitrary direction to each edge, i.e., for each edge $e_k \in \mathcal{E}$, denote one endpoint as the head and the other as the tail. Then, the incidence matrix $E \in \mathbb{R}^{n \times l}$ is defined as

$$[E]_{ik} = \begin{cases} 1, & \text{if } i \text{ is the head node of } e_k, \\ -1, & \text{if } i \text{ is the tail node of } e_k, \\ 0, & \text{otherwise.} \end{cases}$$

An important property of the incidence matrix is $E'\mathbf{1} = 0$. Let $W(s) = \text{diag}\{W_1(s), W_2(s), \ldots, W_l(s)\}$ denote a block diagonal transfer matrix with diagonal blocks given by the dynamic edge weights, i.e., $W_k(s) = A_{ij}(s)$, for $e_k = (i, j)$. Then, the dynamical Laplacian $L(s)$ can be factorized as

$$L(s) = (E \otimes I_m) W(s) (E' \otimes I_m).$$

Note that $L(j\omega)$ has m zero eigenvalues with corresponding eigenvectors being $\mathbf{1}_n \otimes x$ for all $\omega \in \overline{\mathbb{R}}$, where $x \in \mathbb{C}^m$ is an arbitrary nonzero vector. If $\mathbb{G}$ is connected and $W(j\omega)$ is nonsingular, then $L(j\omega)$ has exactly m zero eigenvalues.

2. PROBLEM FORMULATION

Suppose there are n agents. Each agent is a dynamical system, whose dynamics are given by

$$y_i(s) = P_i(s) u_i(s) \tag{2}$$

where $P_i(s)$ is an $m \times m$ semi-stable LTI system in the sense that its poles are all in the closed left half plane, $u_i(t), y_i(t) \in \mathbb{R}^m$ represent the input and output of agent i, respectively. Assume that all the agents share exactly the same poles on the imaginary axis that generate some common persistent modes. Denote by $j\Omega^+ = \{0, j\omega_1, \ldots, j\omega_q\}$ the common poles on the nonnegative imaginary axis, where $\omega_1, \ldots, \omega_q$ are distinct positive frequencies. Then, by symmetry, $\{-j\omega_1, \ldots, -j\omega_q\}$ are also poles of all the agents. We assume that the residue matrix at each of them is nonsingular. If $j\omega_0 \in j\Omega^+$ is a pole of any element of P_i, then it is also assumed that $j\omega_0$ is a simple pole. Apart from these poles, the agents do not have any other poles on the imaginary axis. The agents can be significantly heterogeneous. They can have different magnitudes and phases. What we mean by magnitudes and phases of a MIMO system will be clarified later. They may have different orders.

The agents exchange information with their neighbors over an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ through the following synchronization protocol:

$$u_i(s) = \sum_{(i,j)=e_k \in \mathcal{E}} W_k(s)(y_j(s) - y_i(s)), \quad i \in \mathcal{V}, \tag{3}$$

where $W_k(s)$ is an $m \times m$ stable transfer matrix. The edge dynamics $W_k(s)$ are nonuniform, modeling communication imperfections in each interconnection edge. We assume that the graph is connected.

The multi-agent system (2)-(3) is said to achieve synchronization if $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{V}, i \neq j$. In particular, if all agents have only one common pole at the origin, synchronization simply means consensus. If all agents have common pairs of complex poles on the imaginary axis, synchronization means synchronous oscillation.

Let

$$\begin{aligned} P(s) &= \text{diag}\{P_1(s), \ldots, P_n(s)\}, \\ u(t) &= [u_1(t)' \ \cdots \ u_n(t)']', \\ y(t) &= [y_1(t)' \ \cdots \ y_n(t)']'. \end{aligned}$$

Such synchronization dynamics can be casted into a feedback block diagram (Wang et al., 2017), as shown in Fig. 1, where E is the incidence matrix of graph $\mathbb{G}$.

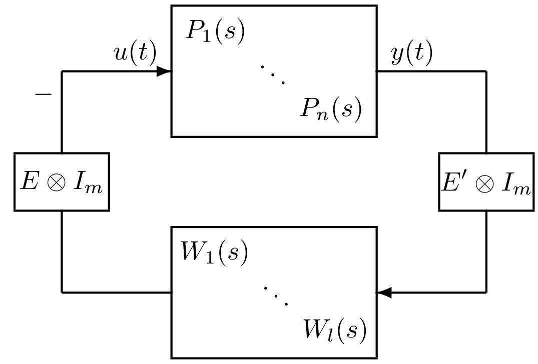


Figure 1. The block diagram of synchronization problem.

It is widely recognized that synchronization can be transformed to a feedback stability problem on a disagreement subspace (Ren and Beard, 2008). Clearly, applying the small gain theorem with diagonal multiplier will yield certain sufficient conditions for synchronization. The issue is that such conditions may be rather conservative when the gains of the agents vary to a large extent. There have also been conditions based on positive realness or negative imaginariness of the agents, which effectively bring in a phasic perspective into the analysis of synchronization (Bürger et al., 2014; Fujimori et al., 2011; Wang et al., 2015). Roughly speaking, positive realness and negative imaginariness can be considered as qualitative descriptions of phase information. Recently, a more delicate phase analysis of LTI MIMO systems has been conducted in Chen et al. (2019), opening up the quantitative studies on system phases. In particular, a small phase theorem has been formulated, complementary to the small gain theorem.

The purpose of this paper is to add to the understandings of synchronization by utilizing the newest development of phase analysis. We will explore synchronization conditions from the phasic perspective.

3. PHASES OF COMPLEX MATRICES

The numerical range, also called field of values, of a matrix $C \in \mathbb{C}^{n\times n}$ is defined as $W(C) = \{x^*Cx : x \in \mathbb{C}^n, \|x\| = 1\}$, which, as a subset of $\mathbb{C}$, is compact and convex, and contains the spectrum of C (Horn and Johnson, 1991).

If 0 is not in the interior of $W(C)$, then $W(C)$ is contained in a closed half complex plane due to its convexity. In this case, C is said to be a semi-sectorial matrix. Furthermore, if $0 \notin W(C)$, then C is said to be sectorial. A semi-sectorial matrix is possibly singular.

Lemma 1. (Furtado and Johnson (2003)). Let $C \in \mathbb{C}^{n\times n}$ be a singular semi-sectorial matrix. Then there exists a unitary matrix U such that

$$U^*CU = \begin{bmatrix} 0_{n-r} & \\ & \hat{C} \end{bmatrix}, \tag{4}$$

where $r = \operatorname{rank}(C)$ and $\hat{C}$ is nonsingular semi-sectorial.

For a semi-sectorial C with rank r, we define r phases of C, denoted to be $\overline{\phi}(C) = \phi_1(C) \geq \cdots \geq \phi_r(C) = \underline{\phi}(C)$, as

$$\begin{aligned} \phi_i(C) &= \sup_{\mathcal{M}:\dim\mathcal{M}=i} \; \inf_{x\in\mathcal{M}, x^*Cx\neq 0} \angle x^*Cx, \\ &= \inf_{\mathcal{N}:\dim\mathcal{N}=n-i+1} \; \sup_{x\in\mathcal{N}, x^*Cx\neq 0} \angle x^*Cx, \end{aligned} \tag{5}$$

such that $\overline{\phi}(C) - \underline{\phi}(C) \leq \pi$ and $\gamma(C) = \frac{\overline{\phi}(C)+\underline{\phi}(C)}{2}$, called the phase center, lies in $(-\pi, \pi]$. Define $\phi(C) = [\phi_1(C) \ \cdots \ \phi_r(C)]$. As in the scalar case, we do not define the phases of a zero matrix. For notational convenience, let $\overline{\phi}(0_n) = -\infty$, $\underline{\phi}(0_n) = +\infty$.

A graphic interpretation of the matrix phases is illustrated in Fig. 2. The two angles from the positive real axis to each of the two supporting rays of $W(C)$ are $\overline{\phi}(C)$ and $\underline{\phi}(C)$ respectively. The other phases of C lie in between.

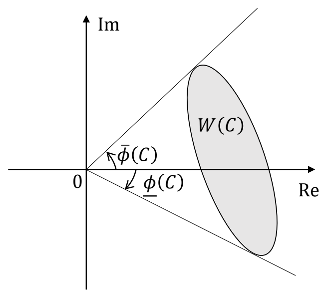


Figure 2. Geometric interpretation of $\overline{\phi}(C)$ and $\underline{\phi}(C)$.

The matrix phases defined above have a number of nice properties, see (Wang et al., 2020) for more details. Here we briefly introduce several of them which will play important roles in later developments.

The phases of a principle submatrix of a semi-sectorial matrix satisfy the following property.

Lemma 2. (Furtado and Johnson (2003)). Let $C \in \mathbb{C}^{n\times n}$ be a nonzero semi-sectorial matrix and $\tilde{C}$ be a nonzero principal submatrix of C. Then $\tilde{C}$ is semi-sectorial and

$$\underline{\phi}(C) \leq \underline{\phi}(\tilde{C}) \leq \overline{\phi}(\tilde{C}) \leq \overline{\phi}(C).$$

The next lemma involves the product of a semi-sectorial matrix and a sectorial matrix.

Lemma 3. Let $A, B \in \mathbb{C}^{n\times n}$ be semi-sectorial and sectorial matrices, respectively, and $\lambda(AB)$ be a nonzero eigenvalue of AB. If $\angle\lambda(AB)$ takes value in $(\gamma(A) + \gamma(B) - \pi, \gamma(A) + \gamma(B) + \pi]$, then

$$\underline{\phi}(A) + \underline{\phi}(B) \leq \angle\lambda(AB) \leq \overline{\phi}(A) + \overline{\phi}(B).$$

Another important property concerns the phases of the Kronecker product of two semi-sectorial matrices.

Lemma 4. Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ be nonzero semi-sectorial matrices. If $\overline{\phi}(A) + \overline{\phi}(B) - \underline{\phi}(A) - \underline{\phi}(B) \leq \pi$, then $A \otimes B$ is semi-sectorial and its phases are given by $\phi_i(A) + \phi_j(B), 1 \leq i \leq \operatorname{rank}(A), 1 \leq j \leq \operatorname{rank}(B)$.

4. SMALL PHASE THEOREM

In this section, we will study the feedback stability of a class of semi-stable systems: the frequency-wise semi-sectorial systems.

4.1 Frequency-wise semi-sectorial systems

Definition 1. Let G be an $m \times m$ real rational proper system. It is said to be *frequency-wise semi-sectorial* if the following conditions are satisfied:
(1) G has no pole in $\mathbb{C}^+$.
(2) $G(s)$ is semi-sectorial for all $s \in \mathbb{C}^+$.

Let $j\Omega^+$ be the set of poles of G on the nonnegative imaginary-axis. In this paper, we assume, for the sake of simplicity, that each pole $j\omega_0 \in j\Omega^+$ of any element of G is at most a simple pole. In this case, the reason for the term "frequency-wise semi-sectorial" can be seen from the following lemma.

Lemma 5. Let G be an $m \times m$ real rational proper system. Then G is frequency-wise semi-sectorial if and only if
(i) G has no pole in $\mathbb{C}^+$.
(ii) $G(j\omega)$ is semi-sectorial for all $\omega \in [0, \infty]\backslash\Omega^+$.
(iii) For each pole $j\omega_0 \in j\Omega^+$, the residue matrix $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)G(s)$ satisfies $[\underline{\phi}(K_0), \overline{\phi}(K_0)] \subset (-\pi, \pi)$.

We also define the frequency-wise sectorial systems.

Definition 2. Let G be an $m \times m$ real rational proper stable system, i.e., $G \in \mathcal{RH}_\infty^{m\times m}$. Then G is *frequency-wise sectorial* if $G(j\omega)$ is sectorial for all $\omega \in [0, \infty]$.

We define

$$\overline{\phi}(G) = \sup_{\omega\in[0,+\infty]\backslash\Omega^+} \overline{\phi}(G(j\omega)),$$
$$\underline{\phi}(G) = \inf_{\omega\in[0,+\infty]\backslash\Omega^+} \underline{\phi}(G(j\omega))$$

as the maximum and minimum phase of $G(s)$ over the entire positive frequency, respectively. Moreover, the $\mathcal{H}_\infty$ phase sector, also called Φ_∞ sector, of G is defined to be

$$\Phi_\infty(G) = [\underline{\phi}(G), \overline{\phi}(G)],$$

which serves as the counterpart to the $\mathcal{H}_\infty$ norm of G.

Note that frequency-wise semi-sectorial systems generalize the well-known notion of positive real systems (Anderson and Vongpanitlerd, 1973; Kottenstette et al., 2014). A real rational proper transfer matrix G is positive real if it is analytic in $\mathbb{C}^+$ and $G^*(s) + G(s) \geq 0$ for all $s \in \mathbb{C}^+$. Furthermore, G is strongly positive real if it is analytic and bounded in $\mathbb{C}^+$ and $G^*(s) + G(s) > 0$ for all $s \in \overline{\mathbb{C}}^+$. In the language of phase, G is positive real if it is frequency-wise semi-sectorial and $\Phi_\infty(G) \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and G is strongly positive real if it is frequency-wise sectorial and $\Phi_\infty(G) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$.

4.2 Small phase theorem

Suppose G and H are $m \times m$ real rational proper transfer function matrices. The feedback interconnection of G and H, as depicted in Fig. 3, is said to be stable if the Gang of Four matrix

$$G\#H = \begin{bmatrix} (I + HG)^{-1} & (I + HG)^{-1}H \\ G(I + HG)^{-1} & G(I + HG)^{-1}H \end{bmatrix}$$

is proper and stable, i.e., $G\#H \in \mathcal{RH}_\infty^{2m\times 2m}$.

Theorem 1. (Small phase theorem). Let G be a frequency-wise semi-sectorial system with $j\Omega^+$ the set of poles on the nonnegative imaginary axis and H be a frequency-wise sectorial system. Then $G\#H$ is stable if

$$\begin{aligned} \overline{\phi}(G(j\omega)) + \overline{\phi}(H(j\omega)) &< \pi, \\ \underline{\phi}(G(j\omega)) + \underline{\phi}(H(j\omega)) &> -\pi \end{aligned} \tag{6}$$

for all $\omega \in [0, \infty]\backslash\Omega^+$.

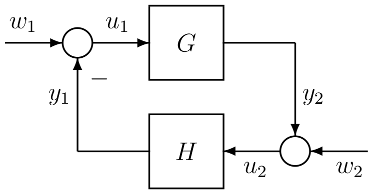


Figure 3. A standard feedback system.

The small phase theorem generalizes the passivity theorem (Liu and Yao, 2016; Vidyasagar, 1981), which states that the feedback system shown in Fig. 3 is stable if G is positive real and H is strongly positive real.

Note that the small phase theorem is necessary in the following sense. Let $G \in \mathcal{RH}_\infty^{m\times m}$ be a frequency-wise sectorial system, $h \in \mathcal{RH}_\infty$ be a scalar strongly positive real transfer function, and define a cone of systems

$$\begin{aligned} \mathfrak{C}(h) = \{H \in \mathcal{RH}_\infty^{m\times m} :\ & \overline{\phi}(H(j\omega)) < \pi/2 + \angle h(j\omega), \\ & \underline{\phi}(H(j\omega)) > -\pi/2 - \angle h(j\omega) \text{ for all } \omega \in [0, \infty]\}. \end{aligned}$$

Then, $G\#H$ is stable for all $H \in \mathfrak{C}(h)$ if and only if $\overline{\phi}(G(j\omega)) \leq \pi/2 - \angle h(j\omega)$ and $\underline{\phi}(G(j\omega)) \geq -\pi/2 + \angle h(j\omega)$ for all $\omega \in [0, \infty]$.

5. MAIN RESULTS

It is widely known that a synchronization problem can be transformed to a stability problem. By exploiting the above small phase theorem, we have the following result.

Theorem 2. Suppose $P(s)$ is frequency-wise semi-sectorial and $W(s)$ is frequency-wise sectorial. Then the multi-agent system (2)-(3) achieves synchronization if

$$\begin{aligned} \max_i \overline{\phi}(P_i(j\omega)) + \max_k \overline{\phi}(W_k(j\omega)) &< \pi, \\ \min_i \underline{\phi}(P_i(j\omega)) + \min_k \underline{\phi}(W_k(j\omega)) &> -\pi \end{aligned}$$

for all $\omega \in [0, \infty]\backslash\Omega^+$.

The above theorem guarantees the output synchronization of a heterogeneous multi-agent system with nonuniform dynamical edges by imposing only local phase conditions. Importantly, these conditions are independent of the network topology. Such a result would generalize positive real and negative imaginary type conditions. Specifically, there is no requirement that $P_i(j\omega)$ be positive real or negative imaginary across all frequencies; they may be, for instance, positive real at some frequencies and negative imaginary at others.

Note that the conditions in Theorem 2 scale well with the size of the network. In particular, when a new agent joins the network or a new communication link is established, the information about the new entry simply needs to be contrasted with the outcome of the phase analysis previously conducted for the original network with n agents and l links. In other words, re-performing a centralized phase analysis involving all nodes is not necessary.

6. SIMULATIONS

Now we consider an undirected network with four nodes and five edges, as shown in Fig. 4. Each agent is a 2×2 frequency-wise semi-sectorial system with imaginary-axis poles $\{0, \pm j1\}$. Each edge is a 2×2 frequency-wise sectorial system. The transfer matrices of the agents are given in equation (7) and those of edges are given in the following.

$$W_1 = \begin{bmatrix} \frac{0.4s^2+1.3s+0.9}{s^2+7s+10} & \frac{0.5s^2+1.8s+1.3}{s^2+7s+10} \\ \frac{0.5s^2+1.8s+1.3}{s^2+7s+10} & \frac{0.7s^2+3.2s+2.5}{s^2+7s+10} \end{bmatrix},$$

$$W_2 = \begin{bmatrix} \frac{2.7s^2+11s+7}{s^2+9s+18} & \frac{1.1s^2+4.9s+3}{s^2+9s+18} \\ \frac{1.1s^2+4.9s+3}{s^2+9s+18} & \frac{0.6s^2+3.4s+2}{s^2+9s+18} \end{bmatrix},$$

$$W_3 = \begin{bmatrix} \frac{2.6s^2+4.5s+1.9}{2s^2+12s+10} & \frac{2s^2+3.8s+1.6}{2s^2+12s+10} \\ \frac{2s^2+3.8s+1.6}{2s^2+12s+10} & \frac{2.1s^2+6s+2.7}{2s^2+12s+10} \end{bmatrix},$$

$$W_4 = \begin{bmatrix} \frac{0.6s^2+3.8s+3.3}{2s^2+17s+30} & \frac{0.8s^2+4.9s+4.1}{2s^2+17s+30} \\ \frac{0.8s^2+4.9s+4.1}{2s^2+17s+30} & \frac{2.7s^2+27.2s+25}{2s^2+17s+30} \end{bmatrix},$$

$$W_5 = \begin{bmatrix} \frac{1.7s^2+15.8s+18}{2s^2+27s+70} & \frac{2s^2+20.1s+20.4}{2s^2+27s+70} \\ \frac{2s^2+20.1s+20.4}{2s^2+27s+70} & \frac{2.7s^2+27.2s+25}{2s^2+27s+70} \end{bmatrix}.$$

The phase response plots of the agents P and the edges W are shown in Fig. 5 and Fig. 6, respectively. The maximum and minimum phases of the agent dynamics and edge dynamics at each frequency can be read from these figures. It can be easily verified that the conditions in Theorem 2 are satisfied. Therefore, the agents achieve synchronization, as shown in Fig. 7.

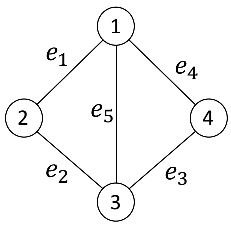


Figure 4. An example network.

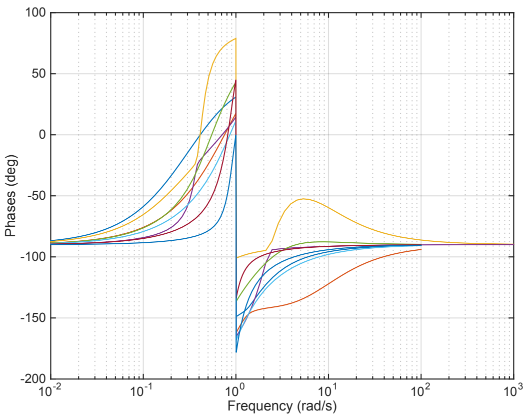


Figure 5. Phase response plots of the agents.

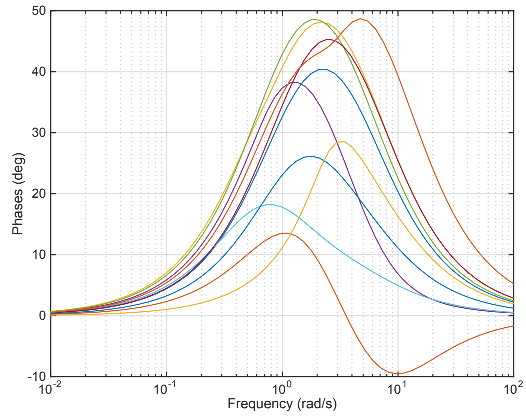
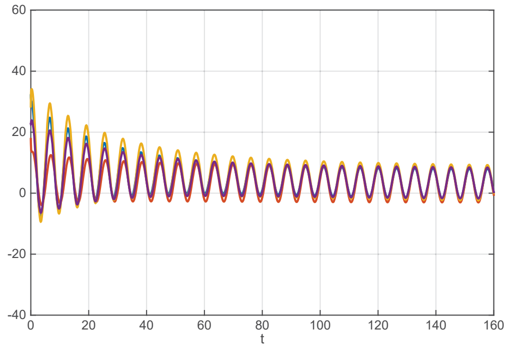
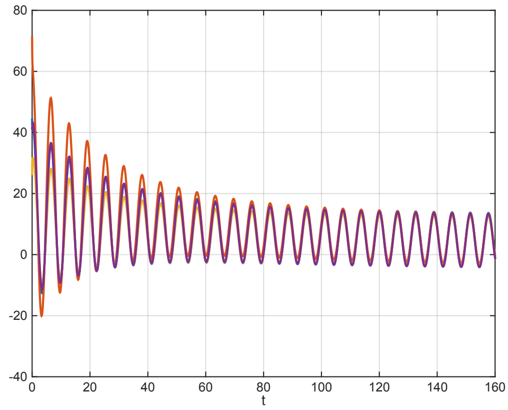


Figure 6. Phase response plots of the edges.



(a) Trajectories of the first outputs of the agents.



(b) Trajectories of the second outputs of the agents.

Figure 7. Trajectories of the outputs of the agents.

7. CONCLUSIONS

In this paper, we have studied synchronization of heterogeneous agents with nonuniform communication dynamics via the small phase theorem. We have obtained synchronization conditions with good scalability, which also reveal the trade-off between the phases of node dynamics and edge dynamics.

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\begin{aligned}
P_1 &= \begin{bmatrix} \frac{1.3s^3+3.9s^2+3.9s+1.3}{s^4+2s^3+s^2+2s} & \frac{1.7s^3+5.2s^2+5s+1.7}{s^4+2s^3+s^2+2s} \\ \frac{1.7s^3+5.2s^2+5s+1.7}{s^4+2s^3+s^2+2s} & \frac{2.4s^3+8.5s^2+7.5s+2.4}{s^4+2s^3+s^2+2s} \end{bmatrix}, \\
P_2 &= \begin{bmatrix} \frac{0.4s^4+2.1s^3+4.8s^2+4.3s+1}{s^5+11s^4+25s^3+11s^2+24s} & \frac{0.9s^4+6.6s^3+15.2s^2+7.7s+2.6}{s^5+11s^4+25s^3+11s^2+24s} \\ \frac{0.9s^4+6.6s^3+15.2s^2+7.7s+2.6}{s^5+11s^4+25s^3+11s^2+24s} & \frac{3.6s^4+35.4s^3+82.2s^2+24s+13}{s^5+11s^4+25s^3+11s^2+24s} \end{bmatrix}, \\
P_3 &= \begin{bmatrix} \frac{1.7s^4+11.1s^3+20.7s^2+15.7s+5}{s^5+5.2s^4+5.8s^3+5.2s^2+4.8s} & \frac{1.6s^4+10.3s^3+19.1s^2+14s+4.4}{s^5+5.2s^4+5.8s^3+5.2s^2+4.8s} \\ \frac{1.6s^4+10.3s^3+19.1s^2+14s+4.4}{s^5+5.2s^4+5.8s^3+5.2s^2+4.8s} & \frac{1.6s^4+10s^3+18.3s^2+12.9s+4}{s^5+5.2s^4+5.8s^3+5.2s^2+4.8s} \end{bmatrix}, \\
P_4 &= \begin{bmatrix} \frac{1.1s^2+0.7s+0.8}{s^3+s} & \frac{2s^2+1.2s+1.4}{s^3+s} \\ \frac{2s^2+1.2s+1.4}{s^3+s} & \frac{3.8s^2+2.1s+2.6}{s^3+s} \end{bmatrix}.
\end{aligned} \tag{7}
$$

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