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State-feedback control for continuous-time LPV systems with polynomial vector fields ⋆

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Abstract: This paper is concerned with the design of state-feedback controllers for Linear Parameter Varying (LPV) polynomial continuous-time systems. The vector field presents polynomial dependence on the states. Two synthesis conditions are proposed, the first one considers arbitrary rates of variation in the time-varying parameters, while the second one provides LPV controllers that are constructed based on a smoothed approximation of the time-varying parameter. The sum of squares matrix decomposition is employed to solve the proposed conditions. The $\mathcal{L}_2$ gain is also considered to give a robustness measure of the proposed controllers. The results are illustrated with examples from the literature.

Keywords: Stabilization, time-varying systems, polynomial systems, sum of squares.

1. INTRODUCTION

Linear parameter varying (LPV) systems can be defined as dynamic linear systems whose state space representation depends on an external time-varying parameter (Mohammadpour and Scherer, 2012). The range of applications is very wide, one may find the use of LPV formulations to deal with automotive systems (Fialho and Balas, 2002), gas-turbine models (Wu and Prajna, 2005) and flight control (Pfifer et al., 2015), for instance. In the last decades, several conditions for analysis and synthesis have been addressed with the use of Lyapunov theory and Linear Matrix Inequalities (LMIs). Because of its relative simplicity, and the existence of several computational tools dedicated to solving convex optimization problems, the use of LMIs quickly became popular (Boyd et al., 1994).

The well known quadratic stability (Horisberger and Belanger, 1976; Barmish, 1985) was one of the first approaches employed to study LPV systems. This method consists of the use of a Lyapunov function with a constant matrix that is employed to guarantee robust stability for the entire domain of the LPV system. Afterward, several works developing less conservative LMI conditions were proposed. The search for Lyapunov functions that also depend on the time-varying parameter was addressed in (Chesi et al., 2007). In (Trofino and de Souza, 2001) the concept of bi-quadratic stability was introduced, the method is based on a quadratic Lyapunov function on the states and on the time-varying parameter simultaneously.

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In (Montagner and Peres, 2003) and (Geromel and Colaneri, 2006), LMI conditions to certificate the asymptotic stability for LPV continuous-time systems are proposed. The time-varying parameters are supposed to have known bounds and their time derivative are modeled in a way that they belong to a convex polytopic domain, leading to less conservative results when compared to the ones obtained with the quadratic approach. On the other hand, the computational effort to solve these types of problems becomes greater as the size of the problem increase, as discussed in (Mozelli and Adriano, 2019).

In (Briat, 2015), conditions to certificate the asymptotic stability and control design for LPV systems with piecewise constant parameters under constant and minimum-dwell-time are investigated. Subsequently, in (Briat and Khammash, 2017), stability analysis of LPV systems with piecewise differentiable parameters is studied. The proposed conditions are based on reformulating the LPV system as an equivalent hybrid one, that contains information about the dynamics of the state of the system and the parameter trajectories. The obtained conditions generalize the quadratic and robust stability approaches in one single formulation. To obtain tractable finite-dimensional conditions, the sum of squares (SOS) relaxations are used.

Although LPV systems are extremely important and useful, their reliability decreases by the time they get far from their linearization point. In this context, polynomial systems can achieve better results when modeling some physical processes (Valmórbida et al., 2013). By using Lyapunov theory and relaxations based on the sum of squares decomposition, new conditions for stability analysis and synthesis for this class of system have been studied (Prajna et al., 2005). In (Ebenbauer and Allgower, 2006), analysis and synthesis conditions concerned with state-feedback and the synchronization problem for polynomial continuous-time systems are exhibited. The solutions are based on the use of dissipation inequalities and SOS formulation. In (Prajna et al., 2004), the SOS decomposition was employed to provide convex conditions for the computation of stabilizing controllers for nonlinear systems. The method was based on the search for rational polynomials. It is also important to mention that in (Valmórbida et al., 2013) the existence of saturated actuators has been considered in the design of polynomial control laws for polynomial systems.

Working with LPV continuous-time systems may not be an easy task due to the derivative of the parameter. Some of the approaches, for example, make use of the maximum magnitude of the time derivative to provide stability certificates or design. Such values can be, sometimes, hard to obtain. In this context, one of the contributions of this work is based on the fact that even not knowing the time derivative limits, LPV controllers can be designed. Besides, if those functions are not continuous, the biggest part of the approaches on the literature can not be applied since such derivatives are not bounded. The one proposed in this paper works fine even in that case.

A great part of existing results in the control literature considers the presence of time-varying parameters and the existence of polynomial dependence in the vector field, separately. This paper intends to diminish this gap considering the design of state-feedback stabilizing controllers for LPV continuous-time systems with polynomial dependency on states. Employing the SOS decomposition, another contribution of this paper is designing controllers for state polynomial systems with time-varying parameters. Robust and LPV controllers can be calculated with the proposed methods.

The notation is standard. I and 0 denote identity matrix and null matrix of proper dimension, respectively. The transpose of a matrix U is represented by U^T. $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices. $\mathrm{He}(M) = M + M^T$. If $f(x)$ is SOS, one may say that $f(x) \in \Sigma[x]$. A matrix of degree $[0 : q]$ is composed by polynomial entries of degree $0, 1, \ldots, q$ on the treated variable.

2. PRELIMINARIES

2.1 Problem formulation

Consider the following polynomial LPV system

$$\dot{x} = A(\alpha(t), x)x + B(\alpha(t), x)u \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input and $\alpha(t)$ is the vector of time-varying parameters belonging to the unit simplex. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n_u}$, depend polynomially on the state vector x and linearly on $\alpha(t)$. The matrices in (1) can be generically represented as [1]

$$W(\alpha, x) = \sum_{i=1}^{N} \alpha_i W_i(x), \quad \alpha \in \Lambda_N, \quad (2)$$

[1] To simplify the notation the dependence on t will be omitted for the time-varying parameters.

where $W_i(x)$, $i = 1, \ldots, N$, are the vertices of the polytope and Λ_N is the unit simplex:

$$\Lambda_N = \Big\{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, N\Big\}. \quad (3)$$

The goal is to provide a state-feedback control law $u = K(\alpha, x)x$ such that the closed-loop system

$$\dot{x} = \tilde{A}(\alpha, x)x, \quad (4)$$

with $\tilde{A} = A(\alpha, x) + B(\alpha, x)K(\alpha, x)$ is asymptotically stable.

2.2 Time-varying parameter

As proposed by Cherifi et al. (2019), if a function is at least piecewise continuous on intervals $[t - \beta, t]$, then its mean values can be written as

$$\xi(t) = \beta^{-1} \int_{t-\beta}^{t} \alpha(\tau)d\tau, \quad \forall t > 0, \quad (5)$$

where $\xi(t)$ holds the convex sum properties. Hence, one can write

$$\dot{\xi}(t) = \beta^{-1}(\alpha(t) - \alpha(t - \beta)). \quad (6)$$

The benefits of writing the derivative terms in this manner is that all $\dot{\xi}(t)$ are always finite and bounded for all $t \in [-\beta, +\infty]$. Besides, for small finite and strictly positive values of β, $\xi(t)$ can be seen as a smoothed approximation of $\alpha(t)$, which may lead to smoothed state-feedback gains for the closed-loop system.

2.3 Sum of Squares

This paper makes use of the sum of squares (SOS) decomposition to certify the non-negativity of the constraints arised from the Lyapunov theory. According to Wu and Prajna (2005), a multivariate polynomial $F(x_1, \ldots, x_n)$ of degree $2d$ is SOS if exist polynomials $f_1(x), \ldots, f_m(x)$, such that

$$F(x) = \sum_{i=1}^{m} f_i^2(x), \quad (7)$$

and each f_i has degree lower or equal to d. It is clear that (7) is equal or greater than 0, allowing the SOS decomposition to be used as a non-negativity certificate. If there exists a SOS decomposition for $F(x)$, then it can be written as

$$F(x) = z^T Q z, \quad (8)$$

where z is a vector of monomials with degree up to d in x and Q is a constant positive semi-definite matrix, which can be decomposed as $Q = V^T V$. To illustrate how a SOS decomposition can be found, consider the polynomial

$$F(x_1, x_2) = 4x_1^4 + 4x_1^3x_2 - 7x_1^2x_2^2 - 2x_1x_2^3 + 10x_2^4.$$

Note that $2d = 4$, so the vector z can be written as a function of monomials of degree $d = 2$, yielding

$$F(x_1, x_2) = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -5 \\ 2 & 3 & -1 \\ -5 & -1 & 10 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}.$$

By employing a Cholesky factorization one has $Q = V^T V$ with

$$V = \begin{bmatrix} 2 & 1 & -2.5 \\ 0 & \sqrt{2} & 0.75\sqrt{2} \\ 0 & 0 & \sqrt{2.625} \end{bmatrix}.$$

With the matrix V, all $f_i(x)$ of (7) can be calculated as

$$\begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \\ f_3(x_1,x_2) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2.5 \\ 0 & \sqrt{2} & 0.75\sqrt{2} \\ 0 & 0 & \sqrt{2.625} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix},$$

which leads to

$$F(x_1,x_2) = f_1^2(x_1,x_2) + f_2^2(x_1,x_2) + f_3^2(x_1,x_2),$$

where $f_1(x_1,x_2) = 2x_1^2 + x_1x_2 - 2.5x_2^2$,
$f_2(x_1,x_2) = \sqrt{2}x_1x_2 + 0.75\sqrt{2}x_2^2$,
$f_3(x_1,x_2) = \sqrt{2.625}x_2^2$.

3. MAIN RESULTS

3.1 Polynomial state-feedback control law

The first result is based on the existence of a polynomial matrix $K(x)$ that assures the stability of the closed-loop system (4). This is the case to be considered when there is no information about the time-varying parameter α.

Theorem 1. If there exist a constant matrix $P = P^T > 0$, and a polynomial matrix $Z(x)$ such that

$$-\mathrm{He}(A_i(x)P + B_i(x)Z(x)) - \epsilon I \in \Sigma[x], \qquad (9)$$

$i = 1,\ldots,N$, then the LPV polynomial system (4) is asymptotically stable and the polynomial state-feedback controller is given by

$$K(x) = Z(x)P^{-1}.$$

Proof. By replacing $Z(x) = K(x)P$ in (9), one can write

$$-\mathrm{He}([A_i(x) + B_i(x)K(x)]P) - \epsilon I \in \Sigma[x], \qquad (10)$$

$i = 1,\ldots,N$. Multiplying (10) by α_i, $i = 1,\ldots,N$, and summing up, one has

$$-\mathrm{He}([A(\alpha,x) + B(\alpha,x)K(x)]P) - \epsilon I \in \Sigma[x],$$

which leads to

$$\tilde{A}(\alpha,x)P + P\tilde{A}(\alpha,x)^T < 0. \qquad (11)$$

Multiplying (11) on the left by x^TP^{-1}, and on the right by its transpose, one has

$$x^TP^{-1}\tilde{A}(\alpha,x)x + x^T\tilde{A}(\alpha,x)^TP^{-1}x < 0. \qquad (12)$$

Replacing (4) in (12) yields

$$\dot{x}^TP^{-1}x + x^TP^{-1}\dot{x} < 0,$$

or equivalently, $\dot{V}(x) < 0$ with

$$V(x) = x^TP^{-1}x.$$

Moreover, note that $V(x)$ is positive definite, since $P > 0$ in Theorem 1.

3.2 LPV Polynomial state-feedback control law

In this case, the search is for an LPV polynomial control law $u = K(\xi,x)x$, where ξ is a filtered time-varying parameter.

Theorem 2. If there exist matrices $P_i = P_i^T > 0$, $i = 1,\ldots,N$, and polynomial matrices $Z_i(x)$, $i = 1,\ldots,N$, such that

$$-\mathrm{He}(A_j(x)P_i + B_j(x)Z_i(x)) + \frac{1}{\beta}(P_j - P_k) - \epsilon I \in \Sigma[x], \qquad (13)$$

$i,j,k = 1,\ldots,N$, then the LPV polynomial system (4) is asymptotically stable and the LPV polynomial state-feedback controller is given by

$$K(\xi,x) = Z(\xi,x)P(\xi)^{-1}.$$

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Proof. Multiplying (13) by ξ_i, $i = 1,\ldots,N$, summing up and replacing $Z(\xi,x) = K(\xi,x)P(\xi)$ one has

$$-\mathrm{He}(A_j(x)P(\xi) + B_j(x)K(\xi,x)P(\xi)) + \frac{1}{\beta}(P_j - P_k) - \epsilon I \in \Sigma[x]. \qquad (14)$$

Multiplying (14) by α_j, $j = 1,\ldots,N$, summing up and applying the same procedure with ξ_k, $k = 1,\ldots,N$, and summing up, yields

$$-\mathrm{He}([A(\alpha,x) + B(\alpha,x)K(\xi,x)]P(\xi)) + \frac{1}{\beta}(P(\alpha) - P(\alpha(t-\beta))) > 0. \qquad (15)$$

By employing (6), it is possible to write

$$\mathrm{He}([A(\alpha,x) + B(\alpha,x)K(\xi,x)]P(\xi)) - \dot{P}(\xi) < 0,$$

which leads to

$$\tilde{A}(\alpha,x)P(\xi) + P(\xi)\tilde{A}(\alpha,x)^T - \dot{P}(\xi) < 0. \qquad (16)$$

Multiplying (16) on the left by $x^TP(\xi)^{-1}$ and on the right by its transpose, and using the fact that $\dot{P}(\xi) = -P(\xi)\dot{P}(\xi)^{-1}P(\xi)$, one has

$$x^T\left(P(\xi)^{-1}\tilde{A}(\alpha,x) + \tilde{A}(\alpha,x)^TP(\xi)^{-1} + \dot{P}(\xi)^{-1}\right)x < 0, \qquad (17)$$

Replacing (4) in (17) yields

$$\dot{x}^TP(\xi)^{-1}x + x^TP(\xi)^{-1}\dot{x} + x^T\dot{P}(\xi)^{-1}x < 0,$$

or equivalently, $\dot{V}(x) < 0$ with $V(x) = x^TP(\xi)^{-1}x$. Moreover, note that $V(x)$ is positive definite, since $P > 0$ in Theorem 2.

3.3 $\mathcal{L}_2$-gain

To evaluate the $\mathcal{L}_2$-gain associated with the state-feedback controller, consider the following polynomial LPV system

$$\begin{aligned} \dot{x} &= \tilde{A}(\alpha,x)x + B_w(\alpha,x)w \\ y &= C(\alpha,x)x + D_w(\alpha,x)w \end{aligned} \qquad (18)$$

where $w \in \mathcal{R}^{n_w}$ is the input disturbance, $y \in \mathcal{R}^{n_y}$ is the measured output and $\tilde{A}(\alpha,x)$ is the closed-loop matrix from (4).

Theorem 3. If there exist a constant matrix $P = P^T > 0$ and a polynomial matrix $Z(x)$ such that

$$-M_i - \epsilon I \in \Sigma[x], \qquad (19)$$

$i = 1,\ldots,N$, with

$$M_i = \begin{bmatrix} \mathrm{He}(A_i(x)P + B_i(x)Z(x)) & B_{wi}(x) & PC_i^T(x) \\ B_{wi}^T(x) & -\gamma^2 I & D_{wi}^T(x) \\ C_i(x)P & D_{wi}(x) & -I \end{bmatrix}, \qquad (20)$$

then, the closed-loop system (18) is asymptotically stable with a bound to the $\mathcal{L}_2$-gain given by γ. Moreover, the state-feedback control gain is given by

$$K(x) = Z(x)P^{-1}.$$

Proof. By replacing $Z(x) = K(x)P$ in (20), yields

$$M_i = \begin{bmatrix} \mathrm{He}([A_i(x) + B_i(x)K(x)]P) & B_{wi}(x) & PC_i^T(x) \\ B_{wi}^T(x) & -\gamma^2 I & D_{wi}^T(x) \\ C_i(x)P & D_{wi}(x) & -I \end{bmatrix},$$

Multiplying (19) by α_i, $i = 1, \ldots, N$, and summing up, one can write

$$\begin{bmatrix} \mathrm{He}(\tilde{A}(\alpha,x)P) & B_w(\alpha,x) & PC^T(\alpha,x) \\ {B_w}^T(\alpha,x) & -\gamma^2 I & {D_w}^T(\alpha,x) \\ C(\alpha,x)P & D_w(\alpha,x) & -I \end{bmatrix} < 0.$$

By applying a congruence transformation with

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

yields

$$\begin{bmatrix} \mathrm{He}(P^{-1}\tilde{A}(\alpha,x)) & B_w(\alpha,x) & PC^T(\alpha,x) \\ {B_w}^T(\alpha,x) & -\gamma^2 I & {D_w}^T(\alpha,x) \\ C(\alpha,x)P & D_w(\alpha,x) & -I \end{bmatrix} < 0.$$

By means of Schur Complement and considering the system described as in (18), one can compute the $\mathcal{L}_2$-gain condition as

$$\dot{x}^T P^{-1} x + x^T P^{-1} \dot{x} + y^T y - \gamma^2 w^T w < 0,$$

where the Lyapunov function is $V(x) = x^T P^{-1} x > 0$.

In the sequel the $\mathcal{L}_2$ gain is established by using the filtered Lyapunov function.

Theorem 4. If there exist matrices $P_i = P_i^T > 0$ and $Z_i(x)$ such that

$$-W_{i,j,k} - \epsilon I \in \Sigma[x],$$

$i, j, k = 1, \ldots, N$, with

$$W_{i,j,k} = \begin{bmatrix} \mathcal{T} & B_{wj}(x) & P_i C_j(x)^T \\ B_{wj}^T(x) & -\gamma^2 I & D_{wj}^T(x) \\ C_j(x)P_i & D_{wj}(x) & -I \end{bmatrix} \tag{21}$$

and

$$\mathcal{T} = \mathrm{He}(A_j(x)P_i + B_j(x)Z_i(x)) - \frac{1}{\beta}(P_j - P_k),$$

then, the closed-loop system (18) is asymptotically stable with a bound to the $\mathcal{L}_2$-gain given by γ. Moreover, the state-feedback control gain is given by

$$K(\xi, x) = Z(\xi, x)P(\xi)^{-1}.$$

Proof. Multiplying (21) by ξ_i, $i = 1, \ldots, N$, summing up and replacing $Z(\xi, x) = K(\xi, x)P(\xi)$ one has

$$W_{j,k} = \begin{bmatrix} \mathcal{T} & B_{wj}(x) & P(\xi)C_j(x)^T \\ B_{wj}^T(x) & -\gamma^2 I & D_{wj}^T(x) \\ C_j(x)P(\xi) & D_{wj}(x) & -I \end{bmatrix} < 0 \tag{22}$$

with

$$\mathcal{T} = \mathrm{He}(A_j(x) + B_j(x)K(\xi,x)P(\xi)) - \frac{1}{\beta}(P_j - P_k).$$

Multiplying (22) by α_j, $j = 1, \ldots, N$, summing up and applying the same procedure with ξ_k, $k = 1, \ldots, N$, and summing up, yields

$$\begin{bmatrix} \mathcal{Q} & B_w(\alpha,x) & P(\xi)C^T(\alpha,x) \\ {B_w}^T(\alpha,x) & -\gamma^2 I & {D_w}^T(\alpha,x) \\ C(\alpha,x)P(\xi) & D_w(\alpha,x) & -I \end{bmatrix} < 0,$$

with $\mathcal{Q} = \tilde{A}(\alpha,x)P(\xi) + P(\xi)\tilde{A}(\alpha,x)^T - \dot{P}(\xi)$. By applying a congruence transformation with

$$\begin{bmatrix} P(\xi)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and using $\dot{P}(\xi) = -P(\xi)\dot{P}(\xi)^{-1}P(\xi)$ it is possible to write

$$\begin{bmatrix} \mathcal{R} & P(\xi)^{-1}B_w(\alpha,x) & C^T(\alpha,x) \\ {B_w}^T(\alpha,x)P(\xi)^{-1} & -\gamma^2 I & {D_w}^T(\alpha,x) \\ C(\alpha,x) & D_w(\alpha,x) & -I \end{bmatrix} < 0,$$

where $\mathcal{R} = P(\xi)^{-1}\tilde{A}(\alpha,x) + \tilde{A}(\alpha,x)^T P(\xi)^{-1} + \dot{P}(\xi)^{-1}$. By means of the Schur complement, it is possible to get the $\mathcal{L}_2$-gain condition as

$$\dot{V}(x) + y^T y - \gamma^2 w^T w < 0, \tag{23}$$

with $V(x) = x^T P(\xi)^{-1} x$. Note that $P(\xi)$ is positive definite in Theorem 4, concluding the proof.

4. NUMERICAL EXAMPLES

This section illustrates the results for the design of stabilizing controllers for LPV polynomial continuous-time systems. The effects of the filtered time-varying parameter $\xi(t)$ in the control design will be explored. The routines were implemented in MATLAB, version 8.2.0.701 (R2013b) using SOSTOOLS (Papachristodoulou et al., 2013) and SeDuMi (Sturm, 1999).

Example 1 Consider the following system, adapted from Prajna et al. (2004),

$$\dot{x} = \begin{bmatrix} -x_1^2 + x_1 & 1 \\ \theta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad 0 \leq \theta \leq 4, \tag{24}$$

where θ is the varying parameter, but it is not available online. By using Theorem 1, the state-feedback controller designed to guarantee the asymptotic stability of the closed-loop system is $K(x) = [\, k_1 \;\; k_2 \,]$, where

$$k_1 = -2.964x_1^2 + 0.9868x_1 - 7.905,$$

$$k_2 = -1.586x_1^2 + 0.2698x_1 - 3.122.$$

Consider now that the parameter is available, hence it can be used to compute the state-feedback gain, as well its smoothed approximation. The controller designed with Theorem 2 is

$$K(\xi, x) = (\xi_1 Z_1(x) + \xi_2 Z_2(x))(\xi_1 P_1 + \xi_2 P_2)^{-1},$$

where $Z_1(x) = [z_a \quad z_b]$ and $Z_2(x) = [z_c \quad z_d]$ with

$$z_a = -0.3275x_1^2 + 0.2408x_1 - 1.185,$$

$$z_b = -0.4872x_1^2 - 0.03752x_1 - 0.2317,$$

$$z_c = -0.2838x_1^2 + 0.2049x_1 - 1.111,$$

$$z_d = -0.4898x_1^2 - 0.05427x_1 - 0.2652,$$

$$P_1 = \begin{bmatrix} 0.2502 & -0.3412 \\ -0.3412 & 0.9918 \end{bmatrix}, P_2 = \begin{bmatrix} 0.25 & -0.2955 \\ -0.2955 & 0.8706 \end{bmatrix}.$$

The controller was designed for $\beta = 0.1$, which leads to ξ, a smoothed approximation of α. The influence of β is depicted in Figure 1.

As expected, the value of β also impacts on the values of the state-feedback gains. As shown in Figure 2, the highest β used provides the smoothest behavior for k_1 and k_2, where $K(\xi, x) = [k_1 \quad k_2]$. However, it is also possible to notice that the largest magnitude are assumed by k_1 and k_2 when $\beta = 0.5$. It means that more energy is used to stabilize the system. Consider that the control energy spent is computed by the function

$$\mathcal{U} = \int_0^\infty u^T(t)u(t)dt, \tag{25}$$

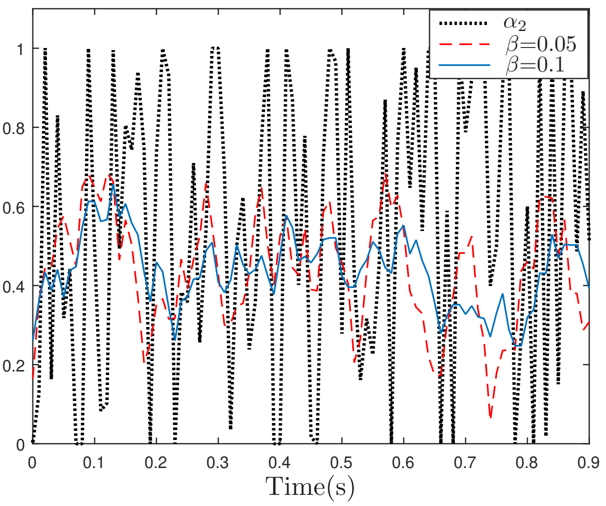

Fig. 1. Smoothed approximation of α_2 for different β.

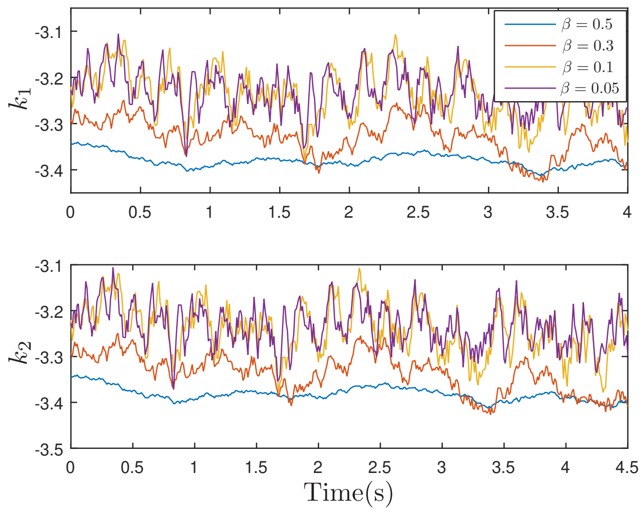

Fig. 2. State-feedback gains variation for different β.

where $u(t)$ is the control signal. The values of $\mathcal{U}$ for the employed filtering values (β) are presented in Table 1. It can be seen that higher values of β spent more control energy, which is expected since a smooth behavior of the control signal is obtained.

Table 1. Control energy for different values of β.

β	0.05	0.1	0.3	0.5
$\mathcal{U}$	1.1958	1.2493	1.5758	1.8388

Example 2 Consider the following polynomial LPV system, adapted from Tanaka et al. (2009), with matrices

$$A(\theta, x) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & -1 + \theta \end{bmatrix},$$

$$B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_w = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T, \; D_w = 0,$$

$0 \le \theta \le 4$. The goal is designing a state-feedback controller that minimizes the $\mathcal{L}_2$-gain under two different circumstances: on the first one, the parameter is not available online and on the second the parameter is available all the time, so it can be used to design the controller. In both cases, it is desired that the stabilization works for arbitrary rates of variation on the time-varying parameter. For the first case, Theorem 3 should be used, the state-feedback gain is designed with a matrix $Z(x)$ with degree $[0 : 2]$. The $\mathcal{L}_2$-gain is given by $\gamma = 2.0133$ and 16 variables were employed to solve the problem. If θ is known through time, Theorem 4 can be used. The $\mathcal{L}_2$-gain is given by $\gamma = 2.0339$ and 31 variables were used. The controller was designed for $\beta = 0.05$ and matrices $Z_i(x)$ with degree $[0 : 2]$.

Example 3 Consider the nonlinear Lorenz chaotic system with polynomial dynamic, adapted from Rakhshan et al. (2018)

$$A(\theta, x) = \begin{bmatrix} -a & a & 0 \\ \theta & -1 & -x_1 \\ x_2 & 0 & -b \end{bmatrix}, \; C(x) = \begin{bmatrix} 0.1 \\ 0 \\ x_1 + x_2 \end{bmatrix}^T$$

$$B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; B_w(\theta) = \begin{bmatrix} 1 \\ 0 \\ 0.1\theta \end{bmatrix}, \; D_w = 0, \; a = 10, \; b = \frac{8}{3}.$$

The time-varying parameter is $\theta \in [30, k]$ and, according to Alam and Ahmed (2017), in the absence of control, the system is chaotic when $\theta \in [27.9, \; 99.6]$. The results, provided by Theorems 3 and 4, are presented in Table 2. Both approaches are formulated with matrices Z of degree $[0 : 4]$ on the states x_1 and x_2 and $[0]$ on x_3. One may see that when k grows, the $\mathcal{L}_2$ gain also increases. Although Theorem 4 design LPV polynomial stabilizing controllers it presents greater values of γ when compared with Theorem 3. This is because the controller provided by Theorem 4 depends upon a smoothed approximation of the time-varying parameter. Theorems 3 and 4 used 52 and 103 variables to solve the problem, respectively.

Table 2. $\mathcal{L}_2$-Gain γ when considering Theorem 3 and Theorem 4 for different values of k.

k	Theorem 3	Theorem 4 ($\beta = 0.05$)
35	24.98	27.71
40	26.00	30.62
45	28.76	32.06

By using Theorem 1, it is possible design a state-feedback controller considering a matrix Z with degree $g_Z = [0 : 4]$ on the states. The parameter variation and the phase portrait of the open-loop (chaotic behavior) and closed-loop (considering four different initial conditions) system are depicted in Figure 3.

5. CONCLUSION

New conditions for the synthesis of state-feedback controllers for LPV polynomial continuous-time systems are presented. Two approaches, based on the sum of squares decomposition, have been developed: the first one is based on a fixed robust state-feedback gain, providing stabilization for arbitrary fast variations of the time-varying parameter. The second one is an LPV controller, making use of a smoothed approximation of the real time-varying parameter and providing stabilization not only for arbitrary fast variation but also for piecewise continuous behavior of the time-varying parameters. Both conditions

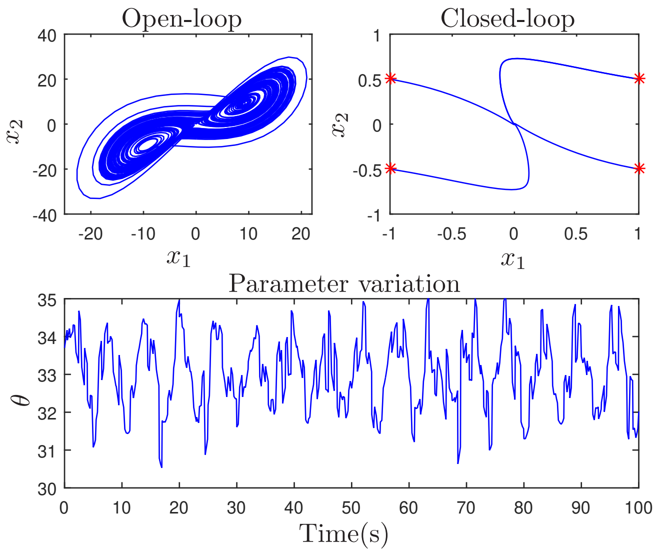

Fig. 3. Response to initial condition and θ behavior.

are extended to compute controllers that minimize the $\mathcal{L}_2$-gain from the input disturbance to the output of the system. It is important to emphasize the fact that the control strategies proposed in this article can handle the existence of time-varying parameters and polynomial dependency in the vector field simultaneously. Future works include the use of Lyapunov matrices that depend polynomially on the state vector x.

REFERENCES

Alam, S. and Ahmed, P. (2017). Several chaotic analysis of Lorenz system. *European Scientific Journal*, 13(9), 438–455.

Barmish, B.R. (1985). Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *Journal of Optimization Theory and Applications*, 46(4), 399–408.

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, Philadelphia, PA.

Briat, C. (2015). Stability analysis and control of a class of LPV systems with piecewise constant parameters. *System & Control Letters*, 82, 10–17.

Briat, C. and Khammash, M. (2017). Stability analysis of LPV systems with piecewise differentiable parameters. *IFAC-PapersOnline*, 50(1), 7554–7559.

Cherifi, A., Guelton, K., Arcese, L., and Leite, V.J.S. (2019). Global non-quadratic D-stabilization of Takagi–Sugeno systems with piecewise continuous membership functions. *Applied Mathematics and Computation*, 351(1), 23–36.

Chesi, G., Garulli, A., Tesi, A., and Vicino, A. (2007). Robust stability of time-varying polytopic systems via parameter-dependent homogeneous Lyapunov functions. *Automatica*, 43(2), 309–316.

Ebenbauer, C. and Allgower, F. (2006). Analysis and design of polynomial control systems using dissipation inequalities and sum of squares. *Computers & Chemical Engineering*, 30(10-12), 1590–1602.

Fialho, I. and Balas, G. (2002). Road adaptive active suspension design using linear parameter-varying gain-scheduling. *IEEE Transactions on Control Systems Technology*, 10(1), 43–54.

Geromel, J.C. and Colaneri, P. (2006). Robust stability of time varying polytopic systems. *Systems & Control Letters*, 55(1), 81–85.

Horisberger, H.P. and Belanger, P.R. (1976). Regulators for linear, time invariant plants with uncertain parameters. *IEEE Transactions on Automatic Control*, 21, 705–708.

Mohammadpour, J. and Scherer, W.C. (2012). *Control of Linear Parameter Varying Systems with Applications.* (Springer Science+Business Media, New York.

Montagner, V.F. and Peres, P.L.D. (2003). A new LMI condition for the robust stability of linear time-varying systems. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, 6133–6138. Maui, HI, USA.

Mozelli, L.A. and Adriano, R.L.S. (2019). On computational issues for stability analysis of LPV systems using parameter-dependent Lyapunov functions and LMIs. *International Journal of Robust and Nonlinear Control*, 29(10), 3267–3277.

Papachristodoulou, A., Anderson, J., Valmorbida, G., Prajna, S., Seiler, P., and Parrilo, P.A. (2013). *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*. `http://arxiv.org/abs/1310.4716`. Available from `http://www.eng.ox.ac.uk/control/sostools`.

Pfifer, H., Moreno, P.C., and Theis, J. (2015). Linear parameter varying techniques applied to aeroservoelastic aircraft: In memory of Gary Balas. *IFAC-PapersOnline*, 48(26), 103–108.

Prajna, S., Papachristodoulou, A., Seiler, P., and Parrilo, P.A. (2005). SOSTOOLS and its control applications. In D. Henrion and A. Garulli (eds.), *Positive Polynomials in Control*, volume 312 of *Lecture Notes in Control and Information Sciences*, 273–292. Springer-Verlag, Berlin.

Prajna, S., Parrilo, P.A., and Rantzer, A. (2004). Nonlinear control synthesis by convex optimization. *IEEE Transactions on Automatic Control*, 49(2), 310–314.

Rakhshan, M., Vafamand, N., Mardani, M.M., Khooban, M.H., and Dragičević, T. (2018). Polynomial control design for polynomial systems: A non-iterative sum of squares approach. *Transactions of the Institute of Measurement and Control*, 41(7), 1993–2004.

Sturm, J.F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1–4), 625–653. `http://sedumi.ie.lehigh.edu/`.

Tanaka, K., Yoshida, H., Ohtake, H., and Wang, H.O. (2009). A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 17(4), 911–922.

Trofino, A. and de Souza, C.E. (2001). Biquadratic stability of uncertain linear systems. *IEEE Transactions on Automatic Control*, 46(8), 1303–1307.

Valmórbida, G., Tarbouriech, S., and Garcia, G. (2013). Design of polynomial control laws for polynomial systems subject to actuator saturation. *IEEE Transactions on Automatic Control*, 58(7), 1758–1770.

Wu, F. and Prajna, S. (2005). SOS-based solution approach to polynomial LPV system analysis and synthesis problems. *International Journal of Control*, 78(8), 600–611.