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Set-Based State Estimation: A Polytopic Approach

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Abstract: This paper is concerned with guaranteed parameter estimation for discrete-time nonlinear systems subject to bounded uncertainties. The proposed approach is based on polytopic set parameterizations. Similar to other estimation and filtering approaches, the presented algorithm is based on two operations, propagation of the polytopic uncertainty through the dynamics and an update operation using the measurement. Both the propagation and the update steps are based on set-operations that use a parameterized lifted outer approximations of the polytopes. The performance of the approach is illustrated by applying it to the double integrator system, where the presented polytopic parameterization approach leads to accurate and rigorous parameter estimates.

Keywords: Parameter and state estimation, Bounded error identification, Nonlinear system identification.

1. INTRODUCTION

Industry strives to tackle the uncertainty present in the day-to-day operations to avoid losses and to improve production performance. The uncertainty stems from mismatched models (so-called process uncertainties) and from the inaccuracies of sensors (noises in the measurement output). The principle of robustness is used in control theory for establishing insensitivity to disturbance variation. The robustness can be reached by guaranteeing that the implemented controller steers the plant to obey the production and safety constraints. One of the ways to approach the problem is by an identification procedure that infers the values of state variables of the plant and disturbances from the available measurement outputs. A well-founded identification process under a robust control strategy will avoid constraint violation and, at the same time, will enhance the performance of the closed-loop system (Nagy and Braatz, 2003).

There are two general approaches to identification, a probabilistic and a deterministic approach. In the probabilistic approach, one requires strong assumptions on the statistical distributions of measurement noises and disturbances. This is the case for methods such as the Kalman Filter, also known as the optimal linear quadratic estimator. Deterministic approaches, usually formulated as set-based estimation (SSE) problems in the context of bounded-error estimation, do not require any consideration about the probability distribution of the measurement noise as well as in the process uncertainties (Blanchini and Miani, 2008). Algorithms based on different set representations (or parameterizations) have been proposed. Examples include ellipsoids (Bertsekas and Rhodes, 1971; Merhy et al., 2019), parallelotopes (Chisci et al., 1996; Valero and Paulen, 2019), and zonotopes (Combastel, 2003; Althoff et al., 2010), as well as combinations of these—e.g., ellipsoids and zonotopes (Chabane et al., 2014).

When deciding which set parameterization to use, it is important to manage the complexity of the set-representation while maintaining an acceptable level of accuracy. Both parallelotopes and ellipsoids have a fixed complexity for a given dimension while the complexity of zonotopes can be arbitrary. In terms of accuracy, the aforementioned parameterizations can be conservative for some of the operations involved in typical set-based estimation algorithms. For example, in the context of linear systems, these operations are: (1) affine transformations, (2) Minkowski sums, and (3) intersections. Parallelotopes and ellipsoids are closed under (1), but not under (2) and (3); while zonotopes are closed under affine transformations as well as Minkowski sums—albeit at moderate increase in complexity of the set—but not under intersections.

Convex polytopes have also been proposed in the context of set-based estimation (Kuntzevich and Lychak, 1992; Kaibel and Pfetsch, 2003). Indeed, polytopes are closed under (1), (2), and (3). However, this comes (in general) at the cost of an increase in the complexity of the resulting set. For example, the (Minkowski) sum of two polytopes has an exponential complexity in the number of its hyperplanes. Recently, Scott et al. (2016) introduced a novel parameterization for general convex polytopes, called constrained zonotopes. This parameterization combines the characteristics of polytopes with zonotopes to provide efficient set operations and algorithms for polytopic state estimation (Rego et al., 2020).

In this paper, we present an alternative method to state estimation using convex polyhedra. Our approach is based on a lifted representation for convex polyhedra, as introduced in Houska (2011). These lifted representations can be interpreted as parametric set-valued functions, defined over a suitable parameter domain—in this case matrices of fixed size with non-negative entries—and whose image is a convex polyhedron of fixed complexity in the number of facets. Lifted representations are tight in the sense that by taking their intersection over the parameter domain one can represent arbitrarily complex polyhedra. Moreover, a lifted polyhedral representation for a given parameter can be used to conservatively approximate a given polyhedron with one of reduced complexity. It is this characteristic that is used in order to formulate the set-based estimation problem as a nonlinear optimization problem, where one searches over the optimization problem over the set of parameters of a lifted representation of a polyhedron containing the states that are consistent with the measurements, in order to minimize some performance criterion.

The organization of the paper is as follows. The next section presents the notation and basic definitions needed. Section 3 introduces polytopic set operations. Section 4 presents the set-based state estimation. Also the NLP problem to perform the SSE is stated. Section 5 illustrates the case study of a double integrator.

1.1 Notation

Throughout this paper we use the following notation

$$\mathcal{P}(G,h) = \{x \in \mathbb{R}^n \mid Gx \le h\} \subset \mathbb{R}^n, \tag{1}$$

to denote polyhedra with shape matrix $G \in \mathbb{R}^{m\times n}$ and size $h \in \mathbb{R}^m$. Recall that bounded polyhedra is called polytope. For two given sets $X, Y \subseteq \mathbb{R}^n$ we use the notation

$$X \oplus Y = \{x + y \mid x \in X,\, y \in Y\}, \tag{2}$$

$$X \ominus Y = \{x - y \mid x \in X,\, y \in Y\}, \tag{3}$$

to denote their Minkowski sum and difference.

2. STATE ESTIMATION FOR UNCERTAIN SYSTEMS

This section introduces uncertain nonlinear systems and an associated set-membership state estimation problem.

2.1 Uncertain Nonlinear Discrete-Time Systems

This paper is concerned with nonlinear discrete-time systems of the form

$$x_{k+1} = f(x_k, w_k) \tag{4}$$

$$y_k = Cx_k + v_k\ , \tag{5}$$

where $x_k \in \mathbb{R}^n$ denotes the state of the system and $y_k \in \mathbb{R}^{n_y}$ a measurable output at time $k \in \mathbb{N}$. Here, the initial value $x_0 \in X_0$ is unknown, but we assume that the compact set $X_0 \subseteq \mathbb{R}^n$ is given. Similarly, the process noise sequence w and the measurement noise sequence v are assumed to be unknown but bounded,

$$\forall k \in \mathbb{N}, \quad w_k \in \mathbb{W} \quad \text{and} \quad v_k \in \mathbb{V}\ , \tag{6}$$

for given compact sets $\mathbb{W} \subseteq \mathbb{R}^{n_w}$ and $\mathbb{V} \subseteq \mathbb{R}^{n_v}$. Throughout this paper the right-hand side function f of the discrete-time system and the output function are assumed to be continuous.

Remark 1. In order to keep the technical developments in this paper as readable as possible, we do not distinguish explicity between states and parameters. Here, we recall that parameters can be regarded as constant states, which satisfy the trivial recursion $p_{k+1} = p_k$. In this sense, it is sufficient to analyze nonlinear systems of the form (4), although the structure of the function f needs to be exploited by numerical methods, if trivial constant recursions for parameters are stacked. Additionally, it is mentioned here that the developments in this paper can also be generalized for nonlinear output function, $y_k = h(x_k, v_k)$. Such nonlinearities can be reformulated, too, by regarding y_k as a (trivial) auxiliary state, stacking h to f, and v to w. In this sense, we may assume without loss of generality that the output function is linear.

2.2 Set-Membership Estimation

One of the main goals of this paper is to compute bounds on the set of system trajectories that are consistent with the measured outputs. In order to formulate this goal in mathematical terms, we use the notation

$$H_k = \{\ x \in \mathbb{R}^n \mid Cx - y_k \in \mathbb{V}\ \} \tag{7}$$

to denote the set of states that are consistent with the current measurement. By using this notation, the recursion for the set of consistent states takes the form

$$X_k = F(X_{k-1}, \mathbb{W}) \cap H_k \tag{8}$$

$$\text{with} \quad F(X_{k-1}, \mathbb{W}) = \left\{ f(x_{k-1}, w) \middle| \begin{array}{l} x_{k-1} \in X_{k-1} \\ w \in \mathbb{W} \end{array} \right\} \tag{9}$$

for all $k \ge 1$. Notice that an accurate computation of the exact set recursion (8) is in high dimensional state-spaces typically impossible, because high dimensional sets are difficult store and propagate through nonlinear functions. Therefore, the focus of this paper is on the construction of outer approximations of the sets X_k.

2.3 Decomposition of Linear and Nonlinear Dependencies

In order to develop practical methods for propagating enclosures of the set sequence X in (8), it is helpful to introduce a decomposition of the right-hand side into linear and nonlinear parts, such that

$$f(\underbrace{q+z}_{x}, w) = f(q,0) + A(q)z + B(q)w + \eta(q,z,w) \tag{10}$$

for all $x, z \in \mathbb{R}^{n_x}$ and all $w \in \mathbb{W}$. Here, the matrices $A(q) \in \mathbb{R}^{n_x \times n_x}$ and $B(q) \in \mathbb{R}^{n_x \times n_w}$ may, in the most general case, depend on x. Notice that such a decomposition is always possible (even if f is not-differentiable), as long as we define the (continuous) function η_f as the gap between the nonlinear model and its linear surrogate. For the special case that f and h are differentiable, we can set

$$A = \frac{\partial f}{\partial x}(x,0) \ \text{ and } \ B = \frac{\partial f}{\partial w}(x,0)\ .$$

Next, our system can be written in the form

$$q_{k+1} = f(q_k, 0) \tag{11}$$

$$z_{k+1} = A(q_k)z_k + \omega_k \tag{12}$$

with $\omega_k = B(q_k)w_k + \eta_f(q_k, z_k, w_k)$. Notice that this decomposition has the advantage that the discrete-time recursion for the uncertainty affected state z is linear. The nonlinear terms can now be bounded as

$$\Omega_k(q_k, Z_k) \supseteq \left\{ B(q_k)w_k + \eta_f(q_k, z_k, w_k) \;\middle|\; \begin{array}{l} z_k \in Z_k \\ w_k \in \mathbb{W} \end{array} \right\}, \tag{13}$$

i.e., such that $\omega_k \in \Omega_k(q_k, Z_k)$. The set-valued function Ω_k is called a nonlinearity bounder, which can be constructed in dependence on the set Z_k in which the current state z_k is currently known to be. Notice that this construction is such that $X_k = \{q_k\} \oplus Z_k$, where the sets Z_k satisfy the recursions

$$Z_k = [A(q_{k-1})Z_{k-1} \oplus \Omega_{k-1}(q_{k-1}, Z_{k-1})] \cap [H_k \ominus \{q_k\}] \tag{14}$$

The following sections focus on the construction of outer approximations of the sets Z_k.

Remark 2. Notice that details on how to construct nonlinearity bounders Ω_k can be found in (Villanueva et al., 2015) in a slightly different context, but the corresponding methods can be applied for state estimation problems, too.

3. POLYTOPIC SET-ARITHMETICS

This section briefly reviews computational methods from the field of polytopic set arithmetic. The related set operations will later be used to implement a polytopic guaranteed state estimation algorithm. Here, we first recall that polytopes are closed under intersection, linear transformations as well as Minkowski sums (see Blanchini and Miani (2008)). This means that, the set operations in (14) can, at least in principle, be implemented by using polytopes. However, because these operations also increase the complexity of the resulting polytopes, one needs to implement facet reduction operations in order to not run out of memory and to keep the computational time within reasonable bounds. The following proposition summarizes practical procedures for implementing such facet-reduction steps.

Proposition 3. Let $\mathcal{P}_1(G_1, h_1)$ and $\mathcal{P}_2(G_2, h_2)$ be polytopes with given pairs $(G_1, h_1) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ as well as $(G_2, h_2) \in \mathbb{R}^{p\times n} \times \mathbb{R}^p$. Then the following relations hold.

(a) The intersection of the polytopes is given by
$$\mathcal{P}_1(G_1, h_1) \cap \mathcal{P}_2(G_2, h_2) = \mathcal{P}_3\left(\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)$$
(b) If $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, then
$$A\mathcal{P}_1(G_1, h_1) = \mathcal{P}_3\left(G_1 A^{-1}, h_1\right) .$$
(c) If $\Lambda \in \mathbb{R}^{\ell\times m}$ denotes any non-negative matrix, $\Lambda \geq 0$, then the inclusion
$$\mathcal{P}_1(G_1, h_1) \subseteq \mathcal{P}_3(\Lambda G_1, \Lambda h_1) .$$
holds.
(d) The Minkowski sum of two polytopes can be bounded by the polytopic enclosure
$$\mathcal{P}_1(G_1, h_1) \oplus \mathcal{P}_2(G_2, h_2) \subseteq \mathcal{P}_3(MG_1, Mh_1 + Nh_2) .$$
This inclusion holds for any non-negative matrices $M \in \mathbb{R}_+^{\ell\times m}$ $N \in \mathbb{R}_+^{\ell\times p}$ that satisfy $MG_1 = NG_2$.

Proof. A complete proof of the first two statements of this proposition can be found in (Blanchini and Miani, 2008), as well as (Kolmanovsky and Gilbert, 1998) and (Kerrigan, 2001). Moreover, a proof of Statement (c) can be found in (Houska, 2011). Finally, the last statement of this proposition is obtained as a consequence of (c) and the properties of the Minkowski sum and support functions. In order to prove this property, it assumes that the resulting facet matrix is known, G. Thus, the constant vector is obtained by the maximization of x_1, x_2 in front of every row of the facet matrix (c). This maximization problem can be formulated as,

$$\begin{aligned} \max_{x_1, x_2} \; & [c \quad c]^\intercal \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t. } & G_1 x_1 \leq h_1 \\ & G_2 x_2 \leq h_2 \end{aligned} \tag{15}$$

The Lagrangian ($L(x_1, x_2, \Lambda_1, \Lambda_2)$) of this problem is given by,

$$= [c \quad c]^\intercal \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \Lambda_1^\intercal (G_1 x_1 - h_1) + \Lambda_2^\intercal (G_2 x_2 - h_2) \tag{16}$$

After ordering in a proper way,

$$= \Lambda_1^\intercal G_1 x_1 + \Lambda_2^\intercal G_2 x_2 + [c \quad c]^\intercal \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \Lambda_1^\intercal h_1 - \Lambda_2^\intercal h_2 \tag{17}$$

$$= \underbrace{[\Lambda_1^\intercal G_1 \;\; \Lambda_2^\intercal G_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [c \quad c]^\intercal \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} - \Lambda_1^\intercal h_1 - \Lambda_2^\intercal h_2 \tag{18}$$

Now, applying the conjugate function definition into the under-brace part.

$$\begin{aligned} \max_{\Lambda_1, \Lambda_2} \; & -\Lambda_1^\intercal h_1 - \Lambda_2^\intercal h_2 \\ \text{s.t. } & \Lambda_1^\intercal G_1 = \Lambda_2^\intercal G_2 \\ & \Lambda_1 \geq 0, \; \Lambda_2 \geq 0 \end{aligned} \tag{19}$$

Finally, it is possible to write dual optimization problem

$$\begin{aligned} \min_{\Lambda_1, \Lambda_2} \; & \Lambda_1^\intercal h_1 + \Lambda_2^\intercal h_2 \\ \text{s.t. } & \Lambda_1^\intercal G_1 = \Lambda_2^\intercal G_2 \\ & \Lambda_1 \geq 0, \; \Lambda_2 \geq 0 \end{aligned} \tag{20}$$

Remark 4. It is obvious that this conditions holds for every row of G what it will demonstrate our postulate. It is important to notice that these are necessary but do not sufficient condition to compute the Minkowski sum. However, it is still possible to compute the Minkowski sum using good guess for G.

The following uses the statements of the above proposition to construct guaranteed parameter estimation methods for nonlinear dynamic systems.

4. GUARANTEED STATE ESTIMATION FOR NONLINEAR SYSTEMS USING POLYTOPES

Guaranteed state estimation procedures can typically be divided into two steps: a propagation and an update step (Bertsekas and Rhodes, 1971). This is in analogy to Kalman filters, but instead of propagating mean values and variances, rigorous bounds on the states are considered, recalling that the recursion of the sets Z_k is given by (14). In order to be able to construct polytopic outer approximations of these sets, the following assumption is introduced.

Assumption 5. The nonlinearity and noise bounding functions Ω as well as the measurement noise sets and the initial state constraint set are bounded by known polytopes,

$$\Omega(q_k, \mathcal{P}(G,h)) \subseteq \mathcal{P}(G_\omega(G,h), h_\omega(G,h)) \quad (21)$$

$$\mathbb{V} \subseteq \mathcal{P}(G_v, h_v) \quad (22)$$

$$X_0 \subseteq \{q_0\} \oplus \mathcal{P}(G_0, h_0) \quad (23)$$

with given parametric matrix-vector pairs

$$(G_\omega(G,h), h_\omega(G,h)) \in \mathbb{R}^{p\times n_w} \times \mathbb{R}^{m_w} ,$$

$(G_v, h_v) \in \mathbb{R}^{q\times n_v} \times \mathbb{R}^{m_v}$, and $(G_0, h_0) \in \mathbb{R}^{\ell\times n} \times \mathbb{R}^{\ell}$. At this point, it needs to be mentioned that G_ω and h_ω are, in the most general case, functions of the shape parameters G and h of the input polytope, as the nonlinearity bounds depend on the current state bounds.

Next, the main idea is to use the polytopic arithmetic rules from Proposition (3) to recursively construct polytopic outer approximations of the form

$$\mathcal{P}(G_k, h_k) \supseteq Z_k ,$$

where the propagation and update step take the form

$$\mathcal{P}\left(\widehat{G}_{k+1}, \widehat{h}_{k+1}\right) \supseteq A(q_k)\mathcal{P}(G_k,h_k) \oplus \mathcal{P}(G_\omega(G_k,h_k), h_\omega(G_k,h_k)) \quad (24)$$

$$\mathcal{P}\left(G_{k+1}, h_{k+1}\right) \supseteq \mathcal{P}\left(\widehat{G}_{k+1}, \widehat{h}_{k+1}\right) \cap \left[H_{k+1} \ominus \{q_{k+1}\}\right] . \quad (25)$$

The following theorem outlines a method for implementing the above propagation and update steps such that a recursive application yields polytopic enclosures of the sets Z_k.

Theorem 6. Let Assumption 5 hold, let the matrix $A(q_k)$ be invertible, and let $(G_k, h_k) \in \mathbb{R}^{\ell\times n} \times \mathbb{R}^{\ell}$ be such that $Z_k \subseteq \mathcal{P}(G_k, h_k)$. If there exist non-negative matrices $M_k \in \mathbb{R}_+^{m\times\ell}$, $N_k \in \mathbb{R}_+^{m\times p}$, $\Lambda_k \in \mathbb{R}_+^{\ell\times(m+q)}$ and pairs $(G_{k+1}, h_{k+1}) \in \mathbb{R}^{\ell\times n} \times \mathbb{R}^{\ell}$, which satisfy

$$G_{k+1} = \Lambda_k \begin{pmatrix} M_k G_k A(q_k)^{-1} \\ G_v C \end{pmatrix}, \quad (26)$$

$$h_{k+1} = \Lambda_k \begin{pmatrix} M_k h_k + N_k h_\omega(G_k, h_k) \\ h_v + G_v(Cq_{k+1} - y_{k+1}) \end{pmatrix} \quad (27)$$

$$N_k G_\omega(G_k, h_k) = M_k G_k A(q_k)^{-1} , \quad (28)$$

then we have $\mathcal{P}(G_k, h_k) \supseteq Z_k$ for all $k \in \mathbb{N}$.

Proof. Because we assume that the current inclusion $\mathcal{P}(G_k, h_k) \supseteq Z_k$ holds in the k-th recursion step, we can bound the result of the Minkowski sum in (24). Here, we apply the second and the last statement of Proposition (a) to find that (24) holds if

$$\exists M_k \in \mathbb{R}_+^{m\times\ell}, N_k \in \mathbb{R}_+^{m\times p} : \left\{ \begin{aligned} \widehat{G}_{k+1} &= M_k G_k A(q_k)^{-1}, \\ \widehat{h}_{k+1} &= M_k h_k + N_k h_\omega(G_k, h_k), \\ M_k G_k A(q_k)^{-1} &= N_k G_\omega(G_k, h_k) \end{aligned} \right. \quad (29)$$

Next, we recall that the incoming measurement at time $k+1$ satisfies

$$y_{k+1} - Cq_{k+1} - Cz_{k+1} \in \mathbb{V} \subseteq \mathcal{P}(G_\nu, h_\nu) .$$

By writing out the definition of the latter polytope and resorting terms, we find that this inclusion implies that

$$H_{k+1} \ominus \{q_{k+1}\} \subseteq \mathcal{P}(G_\nu C, h_\nu + G_\nu(Cq_{k+1} - y_{k+1})) .$$

Notice that this representation of the set $H_{k+1} \ominus \{q_{k+1}\}$ allows us to combine the first and the third statement of Propositions 3 to show that (25) holds if

$$\exists \Lambda \in \mathbb{R}_+^{\ell\times(m+q)} : \quad G_{k+1} = \Lambda \begin{pmatrix} \widehat{G}_{k+1} \\ G_\nu C \end{pmatrix}, \quad \text{and} \quad h_{k+1} = \Lambda \begin{pmatrix} \widehat{h}_{k+1} \\ h_\nu + G_\nu(Cq_{k+1} - y_{k+1}) \end{pmatrix} . \quad (30)$$

Next, the statement of the theorem follows by substituting (29) in (30). □

In order to apply the above theorem for constructing enclosures one needs to select consistent non-negative matrices M_k, N_k, and Λ_k. In order to develop strategies for implementing such a selection, we introduce the shorthand notation

$$\mathbb{G}_k(G_k, h_k) = \left\{ (G_{k+1}, h_{k+1}) \;\middle|\; \begin{array}{l} \exists M_k \in \mathbb{R}_+^{m\times\ell},\, N_k \in \mathbb{R}_+^{m\times p}, \\ \Lambda_k \in \mathbb{R}_+^{\ell\times(m+q)} : \\ \text{Equations (26)–(28) hold} \end{array} \right\} \quad (31)$$

for the set of enclosure parameters that satisfy the conditions from Theorem 6. One way to do implement the enclosure selection is then obtained by computing a minimizer of

$$(G_{k+1}, h_{k+1}) = \underset{G,h,a,b}{\operatorname{argmin}} \sum_{j=1}^{\ell} \|a - b_j\|_2^2 \quad \text{s.t.} \left\{ \begin{array}{l} \forall j \in \{1, \ldots, \ell\}, \\ G_j b_j = h_j \\ Ga \leq h \\ (G,h) \in \mathbb{G}_k \end{array} \right. \quad (32)$$

such that the sum of the distances of the auxiliary point $a \in \mathcal{P}(G_{k+1}, h_{k+1})$ to the facets of $\mathcal{P}(G_{k+1}, h_{k+1})$ is minimal.

Remark 7. It can be shown that the statements from Proposition (a) are tight in the sense that the over-approximation error of the enclosures of the Minkowski sum and facet reduction can be made arbitrarily small if ℓ is sufficiently large (see Blanchini and Miani (2008) and Houska (2011)). Moreover, if f is twice continuously differentiable, one can construct nonlinearity bounders Ω that contract as

$$\operatorname{diam}\left(\mathcal{P}(G_\omega(G,h), h_\omega(G,h))\right) \leq \quad (33)$$

$$\mathbf{O}(\operatorname{diam}(\mathbb{W})) + \mathbf{O}(\operatorname{diam}(\mathcal{P}(G,h))^2) , \quad (34)$$

which is a consequence of Taylor's theorem (Villanueva et al., 2015). By combining these two known result, it turns out that the over-estimation error of the enclosures that are generated by a recursive application of (32) can be bounded by a term of order

$$\mathbf{o}\left(\ell^{-1}\right) + \mathbf{O}(\operatorname{diam}(\mathbb{W})) + \mathbf{O}(\operatorname{diam}(\mathcal{P}(G_0, h_0))) .$$

Thus, the approximation error of the presented procedure can be shown to be small if the diameter of the uncertainty sets is sufficiently small while the number of facets, ℓ, is sufficiently large. This statement also implies that the proposed set-based estimator is stable for large ℓ and small uncertainty sets for $k \to \infty$ under mild assumptions on the observabilty of the original nonlinear system (see Ljung (1999) and Villanueva et al. (2015) for details).

5. CASE STUDY

The double integrator system is studied in the context of bounded-error (set-membership) SSE using presented state-estimation algorithm. The process is given as in (4) with

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ B := \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C := (1\ 0),\ E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{35}$$

where the matrix E depicts the relation between the uncertainties ω and the states, the states x_1 and x_2 represent the position and the velocity of an object, respectively. Both states are subject to bounded uncertainties ± 1. The input u depicts object's acceleration at time k. It is determined by a discrete LQR controller with $Q = I$ and $R = 1$, saturated at the control bounds $u \in [-1, 1]$. The initial state vector is $x_0 := (20, 10)^\intercal$. The measurement matrix $C := [1\ 0]$ depicts the measurement of the position at a sampling rate of 1 time unit and the uniformly distributed measurement error is bounded in ± 1. For the simulations, the initial polytopic set $\mathcal{P}_0$ is selected such that it includes the true state. The initial polytope is represented as

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 32 \\ 11 \\ -15 \\ -6 \end{pmatrix}. \tag{36}$$

5.1 Implementation Details

The proposed algorithm is implemented in Matlab using BARON as a global solver and *fmincon*, IPOPT (Wächter and Biegler, 2006) interfaced through OPTI toolbox (Currie and Wilson, 2012) and YALMIP (Löfberg, 2004). Results were graphing using MPT toolbox (Herceg et al., 2013). We use the global solver to identify a feasible point of (32). Local solvers are used afterwards to improve this solution. The process and measurement noises are simulated as random numbers with uniform distributions.

5.2 Results

Figure 1 shows the reachable set obtained in each iteration. As can be seen, the initial polytope (i.e., a box) contains the initial state, $(20, 10)^\intercal$. At the first time step, the polytope is considerably reduced due to the difference between the size of the initial polytope and magnitude of the measurement noise. Our computational experience shows that it is not always possible to find the global solution. This gives rise to the non-uniform sizes of the obtained polytopes and certain over-approximations. Our future work will involve a development of sophisticated initialization strategies to obtain consistent state estimation bounds.

Figure 2 shows the true states against the bounds (extremal vertices of the polytopic estimates) and the point-prediction of the states (the Chebyshev center of the polytope). We can notice a favorable evolution of the point-prediction towards the true state values. We can also see that despite a large process noise, the estimation procedure is able to maintain the estimation bounds within almost constant range. On some occasions there are jumps occurring in the bounds. These jumps are explained by the inability to identify the global solution of the problem (32).

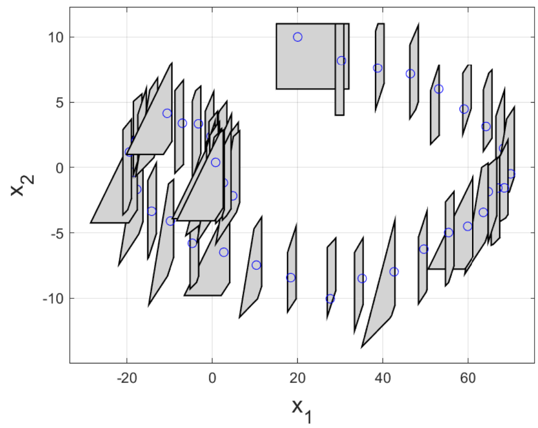

Fig. 1. Polytopic estimates found for 50 time steps.

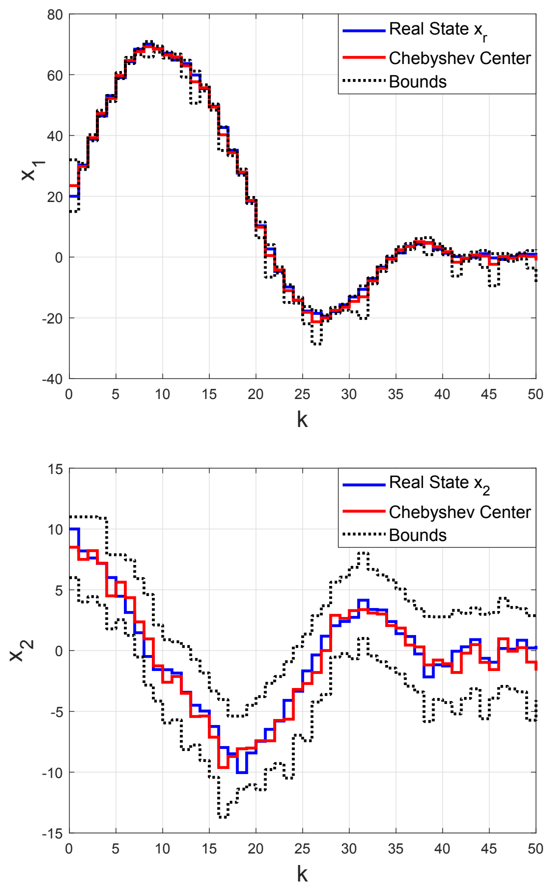

Fig. 2. Evolution of states, their bounds and a Chebyshev center of the bounding polytope.

6. CONCLUSIONS

A new method of set-membership set-based state estimation using polytopes is proposed. The estimation method solves only a single NLP to propagate, update, and reduce the polytope in every iteration. The approach was tested using the double integrator. It was demonstrated that the obtained estimates are consistent and valid. The computational complexity of the proposed approach is considerable and good initialization procedures are needed. The future work will focus on developing a sophisticated tuning method, as well as, testing the method against other set-membership estimation algorithms such as constrained zonotopes.

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REFERENCES

Althoff, M., Stursberg, O., and Buss, M. (2010). Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes. *Nonlinear analysis: hybrid systems*, 4(2), 233–249.

Bertsekas, D. and Rhodes, I. (1971). Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, 16(2), 117–128.

Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control*. Springer.

Chabane, S.B., Maniu, C.S., Alamo, T., Camacho, E.F., and Dumur, D. (2014). Improved set-membership estimation approach based on zonotopes and ellipsoids. In *2014 European Control Conference (ECC)*, 993–998. IEEE.

Chisci, L., Garulli, A., and Zappa, G. (1996). Recursive state bounding by parallelotopes. *Automatica*, 32(7), 1049–1055.

Combastel, C. (2003). A state bounding observer based on zonotopes. In *2003 European Control Conference (ECC)*, 2589–2594. IEEE.

Currie, J. and Wilson, D.I. (2012). OPTI: Lowering the Barrier Between Open Source Optimizers and the Industrial MATLAB User. In N. Sahinidis and J. Pinto (eds.), *Foundations of Computer-Aided Process Operations*. Savannah, Georgia, USA.

Herceg, M., Kvasnica, M., Jones, C., and Morari, M. (2013). Multi-Parametric Toolbox 3.0. In *Proc. of the European Control Conference*, 502–510. Zürich, Switzerland. `http://control.ee.ethz.ch/ mpt`.

Houska, B. (2011). *Robust optimization of dynamic systems*. Ph.D. thesis, PhD thesis, Katholieke Universiteit Leuven.

Kaibel, V. and Pfetsch, M.E. (2003). Some algorithmic problems in polytope theory. In *Algebra, geometry and software systems*, 23–47. Springer.

Kerrigan, E.C. (2001). *Robust constraint satisfaction: Invariant sets and predictive control*. Ph.D. thesis, Citeseer.

Kolmanovsky, I. and Gilbert, E.G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical problems in engineering*, 4(4), 317–367.

Kuntzevich, V. and Lychak, M. (1992). *Guaranteed estimates, adaptation and robustness in control systems*, volume 250. Springer.

Ljung, L. (1999). *System Identification - Theory for the User*. Prentice Hall, New Jersey, 2nd ed edition.

Löfberg, J. (2004). Yalmip : A toolbox for modeling and optimization in matlab. In *In Proceedings of the CACSD Conference*. Taipei, Taiwan.

Merhy, D., Maniu, C.S., Alamo, T., and Camacho, E.F. (2019). Ellipsoidal set-membership state estimation for descriptor systems. In *2019 23rd International Conference on System Theory, Control and Computing (ICSTCC)*, 1–6. IEEE.

Nagy, Z.K. and Braatz, R.D. (2003). Robust nonlinear model predictive control of batch processes. *AIChE Journal*, 49(7), 1776–1786.

Rego, B.S., Raffo, G.V., Scott, J.K., and Raimondo, D.M. (2020). Guaranteed methods based on constrained zonotopes for set-valued state estimation of nonlinear discrete-time systems. *Automatica*, 111, 108614.

Scott, J.K., Raimondo, D.M., Marseglia, G.R., and Braatz, R.D. (2016). Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69, 126–136.

Valero, C. and Paulen, R. (2019). Effective recursive set-membership state estimation for robust linear mpc. In *Preprints, 12th IFAC Symposium on Dynamics and Control of Process Systems, including Biosystems Florianópolis - SC, Brazil, April 23-26, 2019*, 480–485.

Villanueva, M.E., Houska, B., and Chachuat, B. (2015). Unified framework for the propagation of continuous-time enclosures for parametric nonlinear odes. *Journal of Global Optimization*, 62(3), 575–613.

Wächter, A. and Biegler, L.T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming*, 106(1), 25–57.