Designing Fuzzy Descriptor Observer with Unmeasured Premise Variables for Head-Two-Arms-Trunk System

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Abstract: Using the technique of unknown input observer, this paper aims at estimating internal variables of people living with a complete spinal cord injury (SCI). The goal is to provide a better understanding on the sitting control strategy of SCI people. The observer design is based on a Head-Two-Arms-Trunk (H2AT) model, belonging to a class of nonlinear descriptor systems. For observer design, this model is represented in a specific Takagi-Sugeno (TS) fuzzy form with nonlinear consequents. In contrast to previous fuzzy estimation results based on conventional TS fuzzy modeling, the new TS formulation allows separating all unmeasured premise variables in the nonlinear consequent parts. This contributes to reduce the computational burden of the observer design and the structural complexity of the designed fuzzy observer. In particular, the new formulation enables a more effective way to deal with unmeasured premise variables. Using Lyapunov stability theorem, sufficient conditions to design the unknown input observer are derived in the form of linear matrix inequalities, conveniently solved by convex optimization techniques. Simulation results demonstrate the effectiveness of the proposed observer design.

Keywords: Takagi-Sugeno fuzzy systems, unknown input observer, unmeasured premise variables, descriptor fuzzy systems, spinal cord injury, sitting control strategy.

1. INTRODUCTION

People living with a complete spinal cord injury (SCI) lose all sensibility and mobility below their injury level. Due to the absence of muscle auxiliary, their limbs and head are regarded as compensatory strategies to maintain equilibrium when they are trained in rehabilitation Grangeon et al. (2012). Therefore, sitting control is one of the most crucial goals of rehabilitation for paraplegic patients.

Several models have been introduced to study the sitting control stability of people living with SCI. Most of the existing works have been based on linked-segments models, which can be linear Reeves et al. (2009) or nonlinear Tanaka et al. (2010). These models are based on an active torque at the lumbar joint to stabilize the upper body with head, arms and trunk represented by one rigid segment Vette et al. (2010). This assumption is not applicable to people with a complete thoracic SCI due to the lack of trunk muscle activity. To overcome this drawback, the Head-Two-Arms-Trunk (H2AT) model, taking into account the actions of the upper limbs and the head, was proposed in Blandeau (2018). It has been shown that the H2AT model is more suitable to understand how paraplegic patients are able to maintain seated when disturbed. Based on the advantages of the H2AT model, we discuss a new approach to estimate *simultaneously* the internal variable of SCI people and their effort in sitting control situations. However, being the form of a nonlinear descriptor system with unmeasured nonlinearities, this biomechanical H2AT model represents a major challenge in designing an effective estimation algorithm.

This paper presents a Takagi-Sugeno (TS) fuzzy-model-based approach Tanaka and Wang (2004); Nguyen et al. (2019b) to design an unknown input (UI) observer for H2AT system. With the help of the sector nonlinearity approach Tanaka and Wang (2004), the nonlinear dynamics of the H2AT model is exactly represented in a TS fuzzy descriptor form. Then, a fuzzy-model-based observer approach can be investigated for the estimation of unmeasured variables and unknown input via direct Lyapunov stability method. However, in contrast to fuzzy descriptor observer design with measured premise variables Guelton et al. (2008); Nguyen et al. (2019a), the TS descriptor form of the H2AT model presents the well-known challenge in dealing with *unmeasured* premise variables, which still remains an open research topic of TS fuzzy observer design Nguyen et al. (2020). Exploiting the Lipschitz property of the nonlinear membership functions has been commonly adopted to deal with this technical difficulty Bergsten et al. (2002); Lendek et al. (2011). However, Lipschitz-based approaches usually lead to conservative design results Lendek et al. (2011). Several promising approaches have been proposed to avoid the use of the function Lipschitz property, including quasi input-to-state stability approach, dynamic extension based approach, and so on. Based on the differential mean value theorem, TS fuzzy observer design in presence of unmeasured premise variables has been addressed in Ichalal et al. (2011); Nguyen et al. (2020).

This paper presents a new approach to deal with unmeasured premise variables in fuzzy unknown input observer design for H2AT model. To this end, the H2AT model is exactly represented in a new TS fuzzy form with nonlinear consequents, which will be called here N-TS fuzzy model Dong et al. (2010); Coutinho et al. (2020). This allows *isolating* all unmeasured premise variables in the nonlinear consequent parts which enables a more effective way to exploit the differential mean value theorem for fuzzy observer design then existing design approaches Lendek et al. (2011); Ichalal et al. (2011). Moreover, without requiring any modeling simplification, the new N-TS fuzzy formulation allows to reduce significantly the complexity of not only the offline computational burden but also the observer structure compared to the previous work in Blandeau (2018). This is particularly interesting for real-time estimation of internal variables of people living with SCI. Using Lyapunov stability theorem, sufficient conditions are derived in the form of linear matrix inequalities to design the fuzzy UI observer. Then, the estimation algorithm is recast as a convex optimization problem which can be effectively solved with available numerical solvers.

The paper is organized as follows. Section 2 presents the modeling of the biomechanical H2AT system. Section 3 formulates the UI observer design problem for the H2AT system. The LMI-based observer design conditions are derived in Section 4. The numerical illustrations are presented in Section 5. Finally, Section 6 provides some concluding remarks and future works.

Notation. For real symmetric matrices P, A and B, $P > 0$ (respectively $P \geq 0$) denotes a positive definite (respectively positive semidefinite) matrix, and $A > B$ (respectively $A \geq B$) means that $A - B > 0$ (respectively $(A - B \geq 0)$. I denotes an identity matrix with appropriate dimensions. For a symmetric matrix X, we denote $\mathrm{He}X = X + X^\top$. Let x, y be two elements in $\mathbb{R}^n$, the convex hull of the set $\{x, y\}$ is defined by $\mathrm{co}(x, y)$, with

$$\mathrm{co}(x, y) = \{\lambda x + (1 + \lambda)y, \quad \lambda \in [0, 1]\}.$$

The scalar functions $v_i(z)$ of any argument z, for $i \in \{1, 2, \ldots, r\}$, verify the convex sum property if $v_i(z) \geq 0$ and $\sum_{i=1}^r v_i(z) = 1$. For such scalar functions, the following notations are adopted for brevity:

$$U_v = \sum_{i=1}^r v_i(z_k)U_i, \qquad U_v^{-1} = \left(\sum_{i=1}^r v_i(z_k)U_i\right)^{-1} \qquad (1)$$
$$U_{v^+} = \sum_{i=1}^r v_i(z_{k+1})U_i, \quad V_{vv^+} = \sum_{i=1}^r \sum_{j=1}^r v_i(z_k)v_j(z_{k+1})V_{ij}$$

where z_k denotes the value of the signal z taken at the k-instant, and U_i and V_{ij}, for $i, j \in \{1, 2, \ldots, r\}$, are matrices of appropriate dimensions.

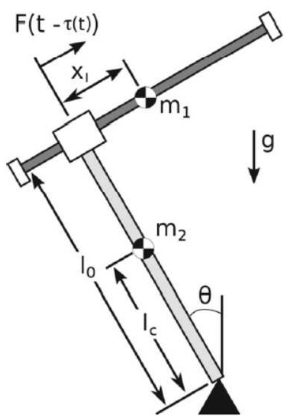

Fig. 1. Schematic of the H2AT model taken from Blandeau (2018).

2. HEAD-TWO-ARMS-TRUNK SYSTEM MODELING

The Head-Two-Arms-Trunk (H2AT) model is an extended version of the planar inverted pendulum consisting of two rods Blandeau (2018). This model has been introduced to understand how paraplegic patients are able to maintain seated under external disturbances. The schematic of the H2AT model is shown in Fig. 1. The system parameters are given in Table 1, which correspond to a 80 kg male subject.

Table 1. Parameter values of H2AT model.

	Description	Value
m_1	Mass of the upper segment	16.1 [kg]
m_2	Mass of the trunk	26.64 [kg]
l_0	Length of the trunk	477 [mm]
l_c	Length of the mass center of the trunk	276.66 [mm]
g	Gravitational constant	9.81 [m/s^2]

Using Lagrangian formulation, the dynamical equations of H2AT model are given by

$$\begin{aligned} 0 &= m_1\ddot{x}_l - m_1 l_0\ddot{\theta} - m_1 x_l\dot{\theta}^2 + m_1 g\sin(\theta) - F(t - \tau(t)) \\ 0 &= (m_1 l_0^2 + m_2 l_c^2 + m_1 x_l^2)\ddot{\theta} - m_1 l_0\ddot{x}_l + 2m_1 x_l\dot{x}_l\dot{\theta} \\ &\quad - (m_1 l_0 + m_2 l_c)g\sin(\theta) + m_1 g x_l\cos(\theta), \end{aligned} \qquad (2)$$

where $x_l(t)$ is the position of the trunk with respect to the mass center of the upper segment, $\theta(t)$ is the angular position of the trunk, $F(t-\tau(t))$ is the controlling force. From anatomical constraints of the trunk and upper segments, we consider the following limitations of the system states:

$$\begin{aligned} x_l &\in [-0.075, 0.105] \text{ m}, & \dot{x}_l(t) &\in [-1, 1] \text{ m/s} \\ \theta &\in [-0.175, 0.349] \text{ rad}, & \dot{\theta} &\in [-0.5, 0.5] \text{ rad/s}. \end{aligned} \qquad (3)$$

Remark 1. Note that the H2AT model (2) is inherently open-loop unstable. For the purposes of observer validation, an internal control law $F(t - \tau(t))$ must be designed to stabilize the H2AT model. To reproduce a human-like control behaviors, the time-varying delay $\tau(t)$ should be taken into account in the control design. This control issue to "design" the human effort $F(t - \tau(t))$ is out of the scope of this paper, which was previously discussed in

Blandeau (2018). In this work, $d(t)$ is considered as the *unknown* human effort, *i.e.*, which must be estimated using an unknown input observer design.

Let us denote $x_1(t) = x_l(t)$, $x_2(t) = \dot{x}_l(t)$, $x_3(t) = \theta(t)$, $x_4(t) = \dot{\theta}(t)$ and $d(t) = F(t - \tau(t))$. Then, the nonlinear system (2) can be put in the following descriptor form:

$$\begin{aligned}\mathbf{E}(x(t))\dot{x}(t) &= \mathbf{A}(x(t))x(t) + Bd(t)\\ y(t) &= Cx(t),\end{aligned} \tag{4}$$

where the system state is defined as $x = [x_1\ x_2\ x_3\ x_4]^\top$. The state-space matrices of system (4) are given as follows:

$$\mathbf{E}(x(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & -m_1 l_0 \\ 0 & 0 & 1 & 0 \\ 0 & -m_1 l_0 & 0 & m_1 l_0^2 + m_2 l_c^2 + m_1 z_1(t) \end{bmatrix}$$

$$\mathbf{A}(x(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -m_1 g z_2(t) & m_1 z_4(t) \\ 0 & 0 & 0 & 1 \\ -m_1 g z_3(t) & -2m_1 z_4(t) & m_c g z_2(t) & 0 \end{bmatrix}$$

$$B = [0\ 1\ 0\ 0], \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^\top,$$

where $m_c = m_1 l_0 + m_2 l_c$. The vector of premise variables of system (4) can be defined from its four nonlinearities, *i.e.*, $z = [z_1\ z_2\ z_3\ z_4]^\top$, with $z_1 = x_l^2$, $z_2 = \frac{\sin(x_3)}{x_3}$, $z_3 = \cos(x_3)$ and $z_4 = x_1 x_4$. Note that the premise variable z_4 is not available from the system output. Note also that from the state limitations defined in (3), the nonlinear descriptor H2AT model (4) is valid within the compact set $\mathscr{C} = \{x \in \mathbb{R}^4 : \ x_1 \in [-0.075, 0.105],\ x_2 \in [-1, 1],\ x_3 \in [-0.175, 0.349],\ x_4 \in [-0.5, 0.5]\}$.

3. PROBLEM STATEMENT

This section first formulates the problem of UI observer design. Then, technical materials useful for the theoretical developments are also provided.

3.1 Takagi-Sugeno Fuzzy Modeling

A direct application of the sector nonlinear approach Tanaka and Wang (2004) to system (4) with the premise vector $z(t)$ leads to an *exact* TS fuzzy descriptor model of the form

$$\begin{aligned}\sum_{i=1}^{16} h_i(z) E_i \dot{x}(t) &= \sum_{i=1}^{16} h_i(z) A_i x(t) + Bd(t)\\ y(t) &= Cx(t)\end{aligned} \tag{5}$$

where the constant matrices A_i, E_i can be easily obtained from the maximal and minimal bounds of the premise variables. The membership functions (MFs) satisfy the following convex sum property:

$$\sum_{i=1}^{16} h_i(z) = 1, \quad h_i(z) \geq 0, \quad i \in \{1, \cdots, 16\}.$$

The details on the local state-space matrices A_i, E_i, and the MFs $h_i(z)$, $i \in \{1, \cdots, 16\}$, are not given here for brevity. Note that the premise variables z_4 cannot be measured in practice. Then, using the TS fuzzy H2AT model (5) leads to major challenges in observer design Nguyen et al. (2019a) to estimate both the system state $x(t)$ and the unknown input $d(t)$. For the continuous-time TS model (5), a solution was proposed in Blandeau (2018), without requiring any Lipschitz-like condition. The idea was to rewrite, via the mean value theorem and uncertain description, a convergence problem of the state error including the input. This paper proposes an alternative solution that can be applied to both continuous-time and discrete-time TS modeling frameworks. Here, we consider the discrete-time framework with a real-time validation perspective. To this end, we reformulate the nonlinear system (4) in the form

$$\begin{aligned}E(\chi(t))\dot{x}(t) &= Ax(t) + Bd(t) + f(y(t)) + G(\chi(t))\phi(x(t))\\ y(t) &= Cx(t),\end{aligned} \tag{6}$$

where

$$E(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & -m_1 l_0 \\ 0 & 0 & 1 & 0 \\ 0 & -m_1 l_0 & 0 & m_1 l_0^2 + m_2 l_c^2 + m_1 x_1^2 \end{bmatrix}$$

$$f(y) = \begin{bmatrix} 0 \\ -m_1 g \sin(x_3) \\ 0 \\ m_c g \sin(x_3) - m_1 g x_1 \cos(x_3) \end{bmatrix}, \ \phi(x) = \begin{bmatrix} x_4^2 \\ x_2 x_4 \end{bmatrix}$$

$$G(\chi) = \begin{bmatrix} 0 & 0 \\ m_1 x_1 & 0 \\ 0 & 0 \\ 0 & -2m_1 x_1 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\chi(t)$ denotes the vector of measured premise variables, *i.e.*, $\chi(t) = \begin{bmatrix} x_1(t) & x_1^2(t) \end{bmatrix}$. Applying the sector nonlinear approach Tanaka and Wang (2004) to system (6) with the measured premise vector $\chi(t)$, we can obtain the following exact TS fuzzy descriptor model with nonlinear consequents $f(\chi(t))$ and $\phi(x(t))$:

$$\begin{aligned}\sum_{i=1}^{4} v_i(\chi) E_i \dot{x} &= Ax + Bd + f(y) + \sum_{i=1}^{4} v_i(\chi) G_i \phi(x)\\ y &= Cx.\end{aligned} \tag{7}$$

Note that the details on the state-space matrices E_i and G_i, and the measured MFs $v_i(\chi)$, for $i \in \{1, \dots, 4\}$, can be easily obtained, and omitted for conciseness reasons.

Remark 2. Note that all existing observer design results on H2AT system Blandeau (2018) rely on the use of classical TS fuzzy modeling Tanaka and Wang (2004); Nguyen et al. (2019b). It would lead to a TS fuzzy H2AT model with 16 fuzzy rules as in (5). Recently, using some least-square based nonlinearity approximation, another TS fuzzy H2AT model with 8 fuzzy rules was obtained in Blandeau (2018) for UI observer design. By rewriting the nonlinear descriptor system (4) in the form (6), without requiring any further model approximation, we can obtain an exact N-TS fuzzy representation (8) of the H2AT system with only 4 fuzzy rules. This can reduce not only the computational burden for observer design but also the structural observer complexity for real-time implementation. Moreover, since the nonlinear consequent $f(y)$ is measured, and all measured premise variables are "isolated" in the unmeasured nonlinear consequent $\phi(x)$. This particular feature of the N-TS fuzzy model (8) enables an effective solution to design fuzzy descriptor UI observer design for H2AT system as shown in Section 4.

To ease the real-time implementation, the observer design is performed in discrete-time domain. To this end, we use the following Euler's transformation:

$$\dot{x}(t) = \frac{x_{k+1} - x_k}{s},$$

where s is the sampling time. Then, using the notations defined in (1), the following discrete-time counterpart of the continuous-time N-TS fuzzy system (7) can be obtained:

$$\begin{aligned}\sum_{i=1}^{4} v_i(\chi_k) E_i x_{k+1} &= \sum_{i=1}^{4} v_i(\chi_k)(\bar{A}_i x_k + sG_i\phi(x_k)) + sBd_k \\ &\quad + sG_i\phi(x_k) + sf(y_k) \\ y_k &= Cx_k,\end{aligned} \tag{8}$$

where $\bar{A}_i = sA + E_i$. Using the notations in (1), the N-TS fuzzy system (8) can be represented in a compact form

$$\begin{aligned}E_v x_{k+1} &= \bar{A}_v x_k + sBd_k + sf(y_k) + sG_v\phi(x_k) \\ y_k &= Cx_k.\end{aligned} \tag{9}$$

To estimate the unknown input d_k, we assume that d_k is a slowly time-varying signal whose dynamics is represented as a double integrator, namely

$$\begin{bmatrix} d_{k+1} \\ d_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} d_k \\ d_{k+1} \end{bmatrix}. \tag{10}$$

Remark 3. The dynamics of unknown input d_k is chosen based on preliminary work with experimental data Blandeau (2018). It represents an acceptable compromise between the estimation performance and the observer structure complexity.

From (9) and (10), the following extended N-TS fuzzy system can be obtained:

$$\begin{aligned}E_v^e x_{k+1}^e &= A_v^e x_k^e + T^e f(y_k) + G_v^e \phi(x_k) \\ y_k &= C^e x_k^e,\end{aligned} \tag{11}$$

where $x_k^e = \begin{bmatrix} x_k^\top & d_k & d_{k+1} \end{bmatrix}^\top$. The system matrices in (11) are given by

$$E_v^e = \begin{bmatrix} E_v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_v^e = \begin{bmatrix} \bar{A}_v & sB & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}, \quad T^e = \begin{bmatrix} sI \\ 0 \\ 0 \end{bmatrix}$$

$$C^e = [C \;\; 0 \;\; 0], \qquad G_v^e = \begin{bmatrix} sG_v^\top & 0 & 0 \end{bmatrix}^\top.$$

To simultaneously estimate the state x_k and the unknown input d_k, we consider the N-TS fuzzy observer structure

$$\begin{aligned}E_v^e \hat{x}_{k+1}^e &= A_v^e \hat{x}_k^e + T^e f(y_k) + G_v^e \phi(\hat{x}_k) + H_v^{-1} L_v (y_k - \hat{y}_k) \\ \hat{y}_k &= C^e \hat{x}_k^e,\end{aligned} \tag{12}$$

where $\hat{x}_k^e$ is the estimate of x_k^e. The observer design problem is to determine the matrices of appropriate dimensions H_v and L_v such that $\hat{x}_k^e$ asymptotically converges to x_k^e.

3.2 Technical Lemmas

The following differential mean value theorem Rudin (1964) is useful to deal with the *unmeasured* nonlinear consequent $\phi(x_k)$ in the N-TS fuzzy system (11).

Definition 1. We define by ξ_n be the canonical basis of the vectorial space $\mathbb{R}^n$ for all $n \geq 1$, *i.e.*,

$$\xi_n = \left\{ \xi_n(i) : \; \xi_n(i) = \Big[\underbrace{0, \ldots, 0, \overbrace{1}^{i^{th}}, 0, \ldots, 0}_{n \text{ components}} \Big]^\top \right\},$$

for $i = 1, 2, \ldots, n$.

Lemma 1. (Rudin (1964)). Let $\phi(x) : \mathbb{R}^n \to \mathbb{R}^q$ and $a, b \in \mathbb{R}^n$. We assume that ϕ is differentiable on $\mathrm{co}(a, b)$. Then, there are constant vectors $c_1, \ldots, c_q \in \mathrm{co}(a, b)$, $c_i \neq a$, $c_i \neq b$ for $i = 1, \ldots, q$, such that

$$\phi(a) - \phi(b) = \left(\sum_{i=1}^{q} \sum_{j=1}^{n} \xi_q(i) \xi_n^\top(j) \frac{\partial \phi_i}{\partial x_j}(c_i) \right)(a - b),$$

where x_j and ϕ_i are the j^{th} component of x and i^{th} component of ϕ, respectively.

For the observer design of N-TS fuzzy systems, multiple nested convex sums appear in the theoretical developments. The following relaxation result provides a good compromise between the numerical complexity and the design conservatism.

Lemma 2. (Tuan et al. (2001)). Let Υ_{ijl} be matrices of appropriate dimensions where $i, j, l \in \{1, 2, \ldots, r\}$. Then, the inequality

$$\Upsilon_{vvv^+} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} v_i(\chi_k) v_j(\chi_k) v_l(\chi_{k+1}) \Upsilon_{ijl} < 0, \tag{13}$$

holds if

$$\Upsilon_{iil} < 0, \quad \frac{2}{r-1}\Upsilon_{iil} + \Upsilon_{ijl} + \Upsilon_{jil} < 0, \quad i \neq j. \tag{14}$$

The following lemma is useful for the convexification procedure of the observer design.

Lemma 3. (Boyd et al. (1994)). Consider a vector $x \in \mathbb{R}^n$ and two matrices $Q = Q^\top \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{m \times n}$ such that $rank(R) < n$. The following two statements are equivalent.

1. $x^\top Q x < 0, \; \forall x \in \mathbb{R}^n \backslash \{0\}, \; Rx = 0.$
2. $\exists M \in \mathbb{R}^{n \times m}$, such that $Q + MR + R^\top M^\top < 0$.

Exploiting the above technical lemmas, sufficient conditions, expressed in terms of LMIs, to design UI observer for the H2AT system are derived in the next section.

4. LMI-BASED UNKNOWN INPUT OBSERVER DESIGN FOR HEAD-TWO-ARMS-TRUNK SYSTEM

Let us denote the estimation error $e_k = x_k^e - \hat{x}_k^e$. From (11) and (12), the error dynamics can be obtained as

$$E_v^e e_{k+1} = (A_v^e - H_v^{-1} L_v C^e) e_k + G_v^e \left(\phi(x_k) - \phi(\hat{x}_k) \right) \tag{15}$$

Applying Lemma 1 to the nonlinear function $\phi(x)$, it follows that there exists $\omega_1, \omega_2 \in \mathrm{co}(x_k, \hat{x}_k)$ such that

$$\begin{aligned}\phi(x_k) - \phi(\hat{x}_k) &= \left(\sum_{i=1}^{2} \sum_{j=1}^{4} \xi_2(i) \xi_4^\top(j) \frac{\partial \phi_i}{\partial x_j}(\omega_i) \right)(x_k - \hat{x}_k) \\ &= \Phi_\omega (x_k - \hat{x}_k)\end{aligned} \tag{16}$$

where

$$\Phi_\omega = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}(\omega_1) & \frac{\partial \phi_1}{\partial x_2}(\omega_1) & \frac{\partial \phi_1}{\partial x_3}(\omega_1) & \frac{\partial \phi_1}{\partial x_4}(\omega_1) \\ \frac{\partial \phi_2}{\partial x_1}(\omega_2) & \frac{\partial \phi_2}{\partial x_2}(\omega_2) & \frac{\partial \phi_2}{\partial x_3}(\omega_2) & \frac{\partial \phi_2}{\partial x_4}(\omega_2) \end{bmatrix}.$$

Note that we can also represent relation (16) in the form $\phi(x_k) - \phi(\hat{x}_k) = \Psi_\omega e_k$ with

$$\Psi_\omega = [\Phi_\omega \;\; 0 \;\; 0] = \begin{bmatrix} 0 & 0 & 0 & 2x_4(\omega_1) & 0 & 0 \\ 0 & x_4(\omega_2) & 0 & x_2(\omega_2) & 0 & 0 \end{bmatrix}.$$

Then, the error dynamics (15) can be rewritten as

$$E_v^e e_{k+1} = (\mathcal{A}_v^e(\Psi) - H_v^{-1} L_v C^e) e_k, \tag{17}$$

where

$$\mathcal{A}_v^e(\Psi) = A_v^e + G_v^e \Psi_\omega.$$

From the definition of the compact set $\mathscr{C}$ of the H2AT system in Section 2, we can easily compute the bounds

$$\underline{\rho}_{ij} \leq \Psi_{\omega ij} \leq \overline{\rho}_{ij},$$

where $\Psi_{\omega ij}$ be the elements of matrix Ψ_ω. This means that matrix Ψ_ω belongs to a bounded domain $\mathcal{H}_{2,6}$, whose the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{2,6}} = \left\{ \psi \in \mathbb{R}^{2\times 6} : \ \psi_{ij} \in \{\underline{\rho}_{ij}, \overline{\rho}_{ij}\} \right\}.$$

The following theorem provides LMI-based conditions to design an UI observer (12) for the H2AT system (2).

Theorem 1. The estimation error dynamics (15) is asymptotically stable if there exist symmetric positive definite matrices P_i, matrices H_i, L_i, $i \in \{1, \cdots, 4\}$ and a positive scalar μ such that condition (14) holds with

$$\Upsilon_{ijl} = \begin{bmatrix} -P_j & 0 \\ 0 & P_l \end{bmatrix} + \mathrm{He}\begin{bmatrix} \mu\Sigma_{ij} & -\mu H_j E_i^e \\ \Sigma_{ij} & -H_j E_i^e \end{bmatrix} \tag{18}$$

and

$$\begin{aligned} \mathcal{A}_i^e(\psi) &= A_i^e + G_i^e \psi, \quad \forall \psi \in \mathcal{V}_{\mathcal{H}_{2,6}} \\ \Sigma_{ij}(\psi) &= H_j \mathcal{A}_i^e(\psi) - L_j C^e. \end{aligned} \tag{19}$$

Proof. Let us consider the following nonquadratic Lyapunov function:

$$V_k = e_k^\top P_v e_k = e_k^\top \sum_{i=1}^{4} v_i(\chi_k) P_i e_k.$$

The time variation of V_k is defined as

$$\Delta V = V_{k+1} - V_k = e_{k+1}^\top P_{v+} e_{k+1} - e_k^\top P_v e_k.$$

The estimation error dynamics (17) is asymptotically stable if $\Delta V < 0$, for all $e_k \neq 0$. Note that ΔV can be represented in the form

$$\Delta V = \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}^\top \begin{bmatrix} -P_v & 0 \\ 0 & P_{v+} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} < 0. \tag{20}$$

Moreover, the error dynamics expression (17) can be rewritten as

$$\Pi \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} = 0, \tag{21}$$

where $\Pi = \begin{bmatrix} \mathcal{A}_v^e(\Psi) - H_v^{-1} L_v C^e & -E_v^e \end{bmatrix}$. Applying Finsler's lemma, inequality (20) holds under equality constraint (21) if

$$\begin{bmatrix} -P_v & 0 \\ 0 & P_{v+} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \mu H_v \\ H_v \end{bmatrix} \Pi \right) < 0. \tag{22}$$

Since the matrix $\mathcal{A}_v^e(\Psi)$ is affine in Ψ, employing the convexity principle, we deduce that (22) hold if the following condition is satisfied

$$\begin{bmatrix} -P_v & 0 \\ 0 & P_{v+} \end{bmatrix} + \mathrm{He}\begin{bmatrix} \mu\Sigma_{vv}(\psi) & -\mu H_v E_v^e \\ \Sigma_{vv}(\psi) & -H_v E_v^e \end{bmatrix} < 0, \tag{23}$$

for all $\psi \in \mathcal{V}_{\mathcal{H}_{2,6}}$, where $\Sigma_{ij}(\psi)$ is defined in (19). Note that condition (23) can be represented in the form (13) with Υ_{ijl} defined in (18). Then, using the relaxation result in Lemma 2, it is clear that (23) can be ensured by (14). Note also that the satisfaction of condition (23) leads to $\mathrm{He}(H_v E_v^e) > 0$. This guarantees the nonsingularity of matrix H_v, thus the validity of observer gain expression $H_v^{-1} L_v$. This completes the proof.

5. SIMULATION RESULTS

This section presents numerical illustrations obtained with the proposed UI observer design. Note that all LMI-based optimization procedures are performed with YALMIP toolbox using SDPT3 solver Löfberg (2004). Solving the LMI-based design condition in Theorem 1, an observer solution can be obtained with $\mu = 0.792$. For conciseness, only the decision matrices corresponding to the first subsystem of the N-TS fuzzy H2AT model are given as

$$P_1 = \begin{bmatrix} 3576.21 & -147.92 & 1712.16 & -43.80 & 4.55 & -3.41 \\ -147.92 & 10.32 & -42.56 & 2.92 & -0.79 & 0.66 \\ 1712.16 & -42.56 & 4106.54 & -138.30 & 3.79 & -2.70 \\ -43.80 & 2.92 & -138.30 & 8.79 & -0.71 & 0.58 \\ 4.55 & -0.79 & 3.79 & -0.71 & 0.26 & -0.23 \\ -3.41 & 0.66 & -2.70 & 0.58 & -0.23 & 0.20 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 2043.07 & -4.58 & 313.77 & -6.79 & 0.10 & -0.05 \\ -132.45 & 2.22 & -28.10 & 3.42 & -0.76 & 0.64 \\ 1408.03 & -8.13 & 2318.95 & -15.18 & 0.14 & -0.05 \\ -40.40 & 2.32 & -120.32 & 4.54 & -0.69 & 0.57 \\ 4.45 & -0.31 & 3.62 & -0.54 & 0.26 & -0.23 \\ -3.35 & 0.25 & -2.61 & 0.45 & -0.23 & 0.21 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 2006.85 & 290.80 \\ -16.16 & 4.32 \\ 1386.68 & 2276.79 \\ -6.90 & -12.38 \\ -0.43 & -0.60 \\ 0.38 & 0.49 \end{bmatrix}.$$

To evaluate the estimation performance of the designed UI observer, let us consider the following initial conditions for the H2AT system (11) and the N-TS fuzzy observer (12):

$$x_k = \begin{bmatrix} 0 & 0 & -0.01 & 0 \end{bmatrix}^\top, \quad \hat{x}_k^e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top.$$

Fig. 2 depicts the state estimation performance obtained with the proposed UI observer. We can see that both unmeasured states $\dot{x}_l(t)$ and $\dot{\theta}(t)$ can be effectively estimated. The corresponding unknown input human effort is shown in Fig.3. Observe also that a good estimation performance can be achieved to recover this unknown input.

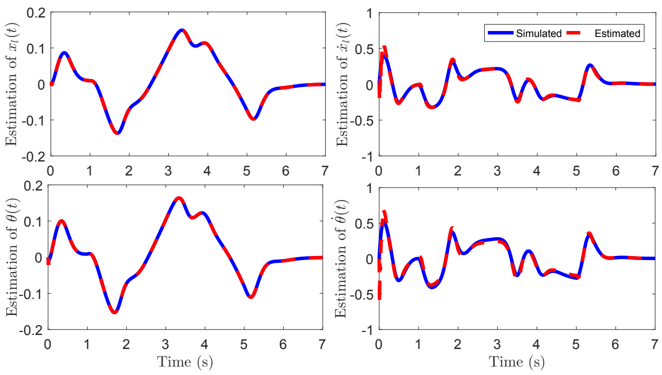

Fig. 2. Estimation of the state for H2AT system.

6. CONCLUSIONS

A new approach has been proposed to deal with unmeasured premise variables in the design of fuzzy unknown input observer for the H2AT model. For fuzzy observer design, we reformulate this nonlinear descriptor model in the form of a N-TS fuzzy system where all the unmeasured

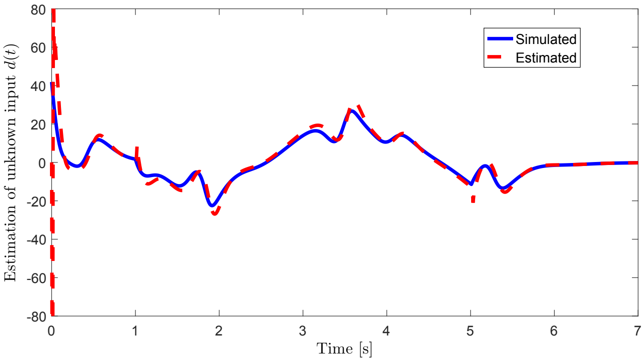

Fig. 3. Estimation of the unknown input for H2AT system.

premise variables are isolated in the consequent parts. Using differential mean value theorem and Lyapunov stability theorem, this formulation allows for an effective LMI-based framework to design fuzzy UI observer with unmeasured premise variables. Moreover, without requiring any model simplification, the N-TS fuzzy formulation contributes to reduce the offline computational burden and the complexity of observer structure. Simulation results clearly demonstrate the effectiveness of the proposed UI observer design. For future works, we apply the proposed observer design to an improved biomechanical model of SCI people, taking into account simultaneously the active, passive and neural features in the stability of the upper body, see Guerra et al. (2020). Experimental validations will be carried out with a dummy platform available in our LAMIH-CNRS laboratory.

ACKNOWLEDGEMENTS

This work is supported in part by the French Ministry of Higher Education and Research, in part by the National Center for Scientific Research (CNRS), in part by the Nord-Pas-de-Calais Region under the project ELSAT 2020, in part by the Natural Science Foundation of Ningxia Hui Autonomous Region under Grant 2018AAC03107, in part by the Fundamental Research Funds for Central Universities, North Minzu University under Grant 2018XYZDX02, 2018XYZDX03, in part by the Third Batch of Ningxia Youth Talents Supporting Program, in part by the Advanced Intelligent Perception and Control Technology Innovative Team of Ningxia.

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