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Linear Boundary Port Hamiltonian Systems defined on Lagrangian submanifolds

Bernhard Maschke * **Arjan van der Schaft** **

* *Univ Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F–69100, Villeurbanne, France (e–mail:bernhard.maschke@univ-lyon1.fr)*
** *Bernoulli Institute for Mathematics, Computer Science and AI, University of Groningen, the Netherlands (e-mail: a.j.van.der.schaft@rug.nl)*

Abstract: Recently Port Hamiltonian systems have been extended to encompass an implicit definition of the energy function of the system, by defining it in terms of a Lagrangian submanifold. In this paper, we extend the definition of Port Hamiltonian systems defined with respect to Lagrangian submanifold to a class of infinite-dimensional systems where the Lagrangian submanifold is defined by first-order differential operators. We show that this adds some port boundary variables and derive the energy balance equation. This construction is illustrated on the model of a flexible nanorod made of composite material.

Keywords: Port Hamiltonian systems, Dirac structures, Lagrangian subspaces

1. INTRODUCTION

Boundary Port Hamiltonian systems have been introduced as an extension of Hamiltonian systems of conservations laws to open systems admitting energy flow through their boundary by [van der Schaft and Maschke 2002, Duindam et al. 2009, chap.4]. The extension consists in defining a *Boundary Port Hamiltonian system* on a Dirac structure, called *Stokes-Dirac structure* and derived from an Hamiltonian differential operator, and encompassing pairs of conjugated external variables, called *port variables*. Thereby the Port Hamiltonian System is defined in an implicit way, using an image representation of the Dirac structure.

However, recently it has been suggested that Port Hamiltonian systems and their implicit representation, could be related to descriptor systems [Beattie et al. 2018]. One motivation for using descriptor representations, is that the energy function is not determined in the energy variables but in the coenergy variables, another motivation is that certain constitutive relations are non-local; for instance the elasticity relation are defined by an integral relation [Heidari and Zwart 2019]. This has led to a generalization of the definition of Port Hamiltonian systems with the more geometric perspective that the energy is no more defined by a function but rather by a Lagrangian submanifold [van der Schaft and Maschke 2018a, 2020] following a usual approach in Hamiltonian dynamics [Abraham and Marsden 1987]. Note that a similar perspective has been suggested for open Irreversible Thermodynamic systems where the state space is defined as a Legendre submanifold of the Thermodynamic Phase Space [Eberard et al. 2007, Favache et al. 2009, 2010, Ramirez et al. 2013b,a, van der Schaft and Maschke 2018c,b].

In this paper, we shall extend the definition of Port Hamiltonian systems defined on Lagrangian submanifolds from finite-dimension [van der Schaft and Maschke 2018a] to infinite-dimensional systems and introduce boundary port variables associated with the differential operators defining the Lagrangian submanifold. However, as a first step towards a general definition, we shall restrict the paper to the class of linear Hamiltonian systems defined on a 1-dimensional spatial domain.

2. BOUNDARY PORT HAMILTONIAN SYSTEMS

In this section, we shall recall the definition of Stokes-Dirac structure associated with differential Hamiltonian matrix operators of order 1 [Le Gorrec et al. 2005, Jacob and Zwart 2012] and the definition of boundary Port Hamiltonian systems defined on this structure.

2.1 Dirac structures associated with first-order Hamiltonian matrix operators

We first recall the definition of Dirac structures defined on vector spaces. Let $\mathcal{F}$ and $\mathcal{E}$ be two real vector spaces and assume that they are endowed with a non degenerated bilinear form, called *pairing* and denoted by:

$$\begin{aligned} \langle . | . \rangle : \mathcal{F} \times \mathcal{E} &\to \mathbb{R} \\ (f, e) &\mapsto \langle e | f \rangle \end{aligned} \tag{1}$$

On the product space, called bond space:

$$\mathcal{B} = \mathcal{F} \times \mathcal{E}$$

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the bilinear product leads to the definition of a symmetric bilinear form, called plus pairing as follows:

$$\begin{aligned} \ll \cdot, \cdot \gg : \mathcal{B} \times \mathcal{B} &\to \mathbb{R} \\ ((f_1,\, e_1)\,,\, (f,_2\, e_2)) &\mapsto \ll (f_1, e_1), (f_2, e_2) \gg \\ &:= \langle e_1 |\, f_2 \rangle + \langle e_2 |\, f_1 \rangle \end{aligned} \tag{2}$$

Definition 1. A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll, \gg$.

Let us recall the definition of Stokes-Dirac structures associated with *first-order Hamiltonian differential operators*

$$\mathcal{J} = \frac{\partial}{\partial z} J_1 + J_0 \tag{3}$$

where $J_1 \in \mathbb{R}^{n \times n}$ is symmetric and $J_0 \in \mathbb{R}^{n \times n}$ is skew-symmetric and the 1-dimensional spatial domain is the interval defined $Z = [a,\, b]$, $a,\, b \in \mathbb{R}$ and $a < b$.

Note that these operators naturally arize in models of physical systems, written in the form of systems of balance equations [van der Schaft and Maschke 2002, Maschke and van der Schaft 2005, Baaiu et al. 2009] and appear also in models of mechanical systems such as the Timoshenko beam model [Le Gorrec et al. 2005, Macchelli and Maschke 2009] and the formulation of fluid models [Hamroun et al. 2006].

Proposition 2. [Villegas 2007, p.156] Consider the flow space $\mathcal{F} = L_2\,(a,\, b,\, \mathbb{R}^n) \times \mathbb{R}^n$ and the effort space being is its dual $\mathcal{E} = \mathcal{F}^* \sim L_2\,(a,\, b,\, \mathbb{R}^n) \times \mathbb{R}^n$. The bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is equipped with the symmetric pairing

$$\begin{aligned} &\left\langle \left(f^1,\, f^1_\partial,\, e^1,\, e^1_\partial\right),\, \left(f^2,\, f^2_\partial,\, e^2,\, e^2_\partial\right) \right\rangle_+ = \\ &\left\langle f^1, e^2 \right\rangle_{L_2} + \left\langle f^2, e^1 \right\rangle_{L_2} - \left\langle f^1_\partial, e^2_\partial \right\rangle_{\mathbb{R}^n} - \left\langle f^2_\partial, e^1_\partial \right\rangle_{\mathbb{R}^n} \end{aligned}$$

where $\left(f^i,\, f^i_\partial,\, e^i,\, e^i_\partial\right) \in \mathcal{B} \quad i \in \{1,\, 2\}$ and $\langle , \rangle_{L_2}$ denotes the Hilbert space inner product and $\langle , \rangle_{\mathbb{R}^n}$ the Euclidean inner product.

For any Hamiltonian operator $\mathcal{J}$ defined in (3) with $k = \operatorname{rank} J_1$, there exist a symmetric matrix $S_1 \in \mathbb{R}^{k \times k}$ and a full-rank matrix $M \in \mathbb{R}^{k \times n}$, defining reduced effort variables $\tilde{e} = Me$, such that the vector subspace

$$\begin{aligned} \mathcal{D}_{\mathcal{J}} = \Bigg\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \in \mathcal{B} \Bigg| \, & e \in H^1\,(a,\, b,\, \mathbb{R}^n)\,, \\ & f = \mathcal{J} e,\, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} S_1 & -S_1 \\ I_k & I_k \end{pmatrix} \operatorname{tr}(M\, e) \Bigg\} \end{aligned} \tag{4}$$

is a Dirac structure in $\mathcal{B}$, called *Stokes-Dirac structure* associated with the operator .

Note that when the matrix J_1 is full-rank, then $M = I_n$ and $S_1 = J_1$ [Le Gorrec et al. 2005] but the general case is treated in [Villegas 2007, p.156] where the expression of S_1 and M may be found.

2.2 Boundary Port Hamiltonian System

Using these Dirac structures, one defines models of open conservative physical systems including port variables (i.e. interface variables with the environment) as port Hamiltonian systems [van der Schaft and Maschke 2002]. In the sequel, we recall their definition in the particular case when the energy is a quadratic functional and with Stokes-Dirac structure (4) associated with first-order Hamiltonian differential operators.

Definition 3. [Linear First Order Boundary Port Hamiltonian System] The Boundary Port Hamiltonian System on the state space $\mathcal{X} = L_2\,(a,\, b,\, \mathbb{R}^n)$ generated by the functional

$$H\,[x] = \int_a^b \frac{1}{2} x\,(z)^\top\, \mathcal{Q}\, x\,(z)\, dz \tag{5}$$

where $\mathcal{Q} \in L_1\,(a,\, b,\, \mathbb{R}^{n \times n})$ is a positive symmetric *real-valued* matrix operator, is defined by

$$\begin{pmatrix} \dfrac{\partial x}{\partial t} \\ f_\partial \\ \mathcal{Q}\, x\,(z) \\ e_\partial \end{pmatrix} \in \mathcal{D}_{\mathcal{J}}$$

These Boundary Port Hamiltonian systems have proven to posses remarkable properties and to lead to passivity-based control of distributed parameter systems controlled through their boundary [Jacob and Zwart 2012].

Example 4. Example of the vibrating rod

Let us briefly recall the model of an elastic body as a Boundary Port Hamiltonian System, following [van der Schaft and Maschke 2002, Maschke and van der Schaft 2005]. Denote the *displacement* of the elastic body by $u(t,z)$, the *velocity* by $v(t,z) = \frac{\partial}{\partial t} u(t,z)$, the *strain* $\epsilon(t,z) = \frac{\partial u}{\partial z}(t,z)$ and the *momentum density* $p(t,z) = (\rho A)\, v(t,z)$ where (ρA) denotes the lineic mass density (product of the section A and the the mass density ρ). The total energy of the system is given by the Hamiltonian functional

$$H\,(u,\, \epsilon,\, p) = U_g\,(u) + U_{el}\,(\epsilon) + K(p)$$

with the kinetic energy

$$K(p) = \int_a^b \frac{1}{2} \frac{1}{(\rho A\,(z))} p^2\, dz \tag{6}$$

$U_g\,(u)$ being a potential energy depending on the displacement (e.g. the gravity force: $\rho A\,(z)\; g$ or here for simplicity some linear elasticity $U_g\,(u) = \int_a^b \frac{1}{2}\; k\,(z)\;\; u\,(z)\;\; dz$) and $U_{el}\,(\epsilon)$ being the structural elastic energy density

$$U_{el}(\epsilon) = \int_a^b \frac{1}{2}\, T\,(z)\;\, \epsilon\,(z)^2\;\, dz \tag{7}$$

where $T\,(z)$ denotes the elasticity modulus. The coenergy variables are then the position-dependent potential force: $k\,(z)\;\; u\,(z)$, the structural elastic force corresponding to the elasticity law: $T\,(z)\;\, \epsilon\,(z)$ and the velocity: $\frac{p}{(\rho A(z))}$. The dynamics of the elastic rod may then be formulated as a Boundary Port Hamiltonian System generated by the Hamiltonian functional $H_0\,(u,\, \epsilon,\, p)$ with respect to the Stokes-Dirac structure associated with the operator (of the type (3))

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \partial_z \\ -1 & \partial_z & 0 \end{pmatrix}$$

This amounts to writing the Hamiltonian system consisting in the kinematic relation relating the velocity to the momentum, its derivative and the momentum balance equation

$$\partial_t \begin{pmatrix} u \\ \epsilon \\ p \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \partial_z \\ -1 & \partial_z & 0 \end{pmatrix}}_{=\mathcal{J}_0} \underbrace{\begin{pmatrix} k & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & \dfrac{1}{(\rho A)} \end{pmatrix}}_{=Q_0} \begin{pmatrix} u \\ \epsilon \\ p \end{pmatrix} \tag{8}$$

augmented with the definition of the boundary port variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \operatorname{tr} Q_0 \begin{pmatrix} u \\ \epsilon \\ p \end{pmatrix} = \operatorname{tr} \begin{pmatrix} p \\ \overline{(\rho A\,(z))} \\ T\,(z)\ \epsilon\,(z) \end{pmatrix}$$

which reduce to the stress and the velocities at the boundaries.

3. BOUNDARY PORT HAMILTONIAN SYSTEMS ON A LAGRANGIAN SUBMANIFOLD

In this section, we extend the definition of a Boundary Port Hamiltonian System of the definition 3 to a more general definition of the Hamiltonian functional in terms of a Lagrangian subspace. This definition is classically used in the definition of Hamiltonian systems [Abraham and Marsden 1987, chap. 5.3] and has been recently adapted to linear and nonlinear Port Hamiltonian systems [van der Schaft and Maschke 2018a, 2020]. In this section we shall extend the definition of Lagrangian subspace that allows to include boundary port variables.

3.1 Lagrangian subspace with boundary energy variables

Definition 5. Let $\mathcal{X}$ be a vector space and $\mathcal{X}^*$ its dual space. A *Lagrangian subspace* is a subspace $\mathcal{L} \subset \mathcal{X} \times \mathcal{X}^*$ such that

$$\mathcal{L} = \mathcal{L}^{\perp_-}$$

where $\perp_-$ denotes the orthogonal with respect to the alternate bilinear form on $\mathcal{X} \times \mathcal{X}^*$

$$\langle (x_1, e_1), (x_2, e_2) \rangle_- \doteq \langle e_1 | x_2 \rangle - \langle e_2 | x_1 \rangle$$

In analoguous way to the Dirac subspaces which characterize graphs of skew-symmetric mappings, Lagrangian subspaces characterize graphs of symmetric mappings.

Example 6. As the most simple example, take the graph of a positive symmetric matrix operator $\mathcal{Q} \in L_1\left(a,\, b,\, \mathbb{R}^{n\times n}\right)$

$$\mathcal{L} = \left\{ (x,\, e) \in L_2\left(a,\, b,\, \mathbb{R}^n\right)\ s.t. \forall z \in [a,\, b]\ e\,(z) = \mathcal{Q}\,(z)\, x\,(z) \right\}$$

It is the graph of the constitutive relation of the effort variables

$$e\,(z) = \mathcal{Q}\,(z)\, x\,(z) = \frac{\delta H}{\delta x}$$

derived from the definition of the energy H in (5) where $\frac{\delta H}{\delta x}$ denotes the column vector of the variational derivatives of H with respect to x_i, $i \in \{1, \ldots, n\}$

In the sequel, we shall show that using Lagrangian subspaces, one may enlarge the constitutive relations of the energy in in (5) to *differential* symmetric differential operators $\mathcal{Q}$. Inspired by the finite dimensional case [van der Schaft and Maschke 2018a], we shall now define a class of *Lagrangian subspaces associated with first order differential symmetric matrix operators.*

Therefore consider two first order, constant coefficient, $n \times n$ matrix differential operators:

$$\mathcal{P} = \frac{\partial}{\partial z} P_1 + P_0 \qquad \text{and} \quad \mathcal{S} = \frac{\partial}{\partial z} S_1 + S_0 \tag{9}$$

where $P_i \in \mathbb{R}^{n\times n}$ and $S_i \in \mathbb{R}^{n\times n}$ $i \in \{1, 2\}$, are real matrices.

Let us compute, using the formal adjoint $\left(\frac{\partial}{\partial z}\right)^*$

$$\begin{aligned} \mathcal{S}^*\mathcal{P} - \mathcal{P}^*\mathcal{S} &= \left(S_1^\top \left(\frac{\partial}{\partial z}^*\right) + S_0^\top\right)\left(\frac{\partial}{\partial z} P_1 + P_0\right) \\ &\quad - \left(P_1^\top \left(\frac{\partial}{\partial z}^*\right) + P_0^\top\right)\left(\frac{\partial}{\partial z} S_1 + S_0\right) \\ &= \left(S_1^\top P_1 - P_1^\top S_1\right)\left(-\frac{\partial^2}{\partial z^2}\right) + \left(S_0^\top P_0 - P_0^\top S_0\right) \\ &\quad + \left(-S_1^\top P_0 + S_0^\top P_1 + P_1^\top S_0 - P_0^\top S_1\right)\left(-\frac{\partial}{\partial z}\right) \end{aligned}$$

which leads to the following lemma.

Lemma 7. The first-order linear operators in (9) satisfy the formal symmetry condition $\mathcal{S}^*\mathcal{P} - \mathcal{P}^*\mathcal{S} = 0$ if and only if the matrices $S_1^\top P_1$ and $S_0^\top P_0$ are symmetric and $\left(S_0^\top P_1 - P_0^\top S_1\right)$ is skew-symmetric.

These symmetry conditions calculated above, are true for functions with support strictly included in $Z = [a, b]$. In general, using integration by parts, one obtains for any pair of functions $(\zeta_1,\, \zeta_2) \in H^2\,(a,\, b,\, \mathbb{R}^n)^2$,

$$\begin{aligned} &\langle \mathcal{S}\zeta_1,\, \mathcal{P}\zeta_2 \rangle_{L_2} - \langle \mathcal{S}\zeta_2,\, \mathcal{P}\zeta_1 \rangle_{L_2} \\ &= \int_a^b \mathcal{S}\zeta_1\,(z)^\top\, \mathcal{P}\zeta_2\,(z)\ dz + \int_a^b \mathcal{S}\zeta_2\,(z)^\top\, \mathcal{P}\zeta_1\,(z)\ dz \\ &= \left[\frac{\partial \zeta_2}{\partial z}^\top \left(S_1^\top P_1 - P_1^\top S_1\right)\zeta_1\right]_a^b \\ &- \int_a^b \zeta_1^\top \left(S_1^\top P_1 - P_1^\top S_1\right)\frac{\partial \zeta_2}{\partial z}\, dz + \left[\zeta_1^\top \left(S_0^\top P_1 - P_0^\top S_1\right)\zeta_2\right]_a^b \\ &- \int_a^b \frac{\partial \zeta_1}{\partial z}^\top \left(-S_1^\top P_0 + S_0^\top P_1 + P_1^\top S_0 - P_0^\top S_1\right)\zeta_2\, dz \\ &+ \int_a^b \zeta_1\,(z)^\top \left(S_0^\top P_0 - P_0^\top S_0\right)\zeta_2\,(z)\ dz \end{aligned}$$

Assuming that the operator $\mathcal{S}^*\mathcal{P}$ is formally symmetric (Lemma 7), this reduces to the boundary terms

$$\begin{aligned} &\langle \mathcal{S}\zeta_1,\, \mathcal{P}\zeta_2 \rangle_{L_2} - \langle \mathcal{S}\zeta_2,\, \mathcal{P}\zeta_1 \rangle_{L_2} \\ &= \left[\zeta_1^\top \left(S_0^\top P_1 - P_0^\top S_1\right)\zeta_2\right]_a^b \end{aligned} \tag{10}$$

which is, by lemma 7, indeed skew-symmetric. The boundary term is characterized by the *skew-symmetric* matrix

$$J_b = S_0^\top P_1 - P_0^\top S_1 \tag{11}$$

Lemma 8. Denote the rank of the matrix J_b by $2m = \operatorname{rank} J_b$ and define $r = n - 2m$. Then the matrix associated with the bilinear product in (10) is contragradient with the matrix

$$J_b = \Pi^\top \begin{pmatrix} J_m^s & 0_{2m\times r} \\ 0_{r\times 2m} & 0_{r\times r} \end{pmatrix} \Pi$$

where is $J_m^s = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}$ and $\Pi \in \mathbb{R}^{n\times n}$ is full-rank.

Then the boundary term (10) may be written

$$\left[\zeta_1^\top \left(S_0^\top P_1 - P_0^\top S_1\right)\zeta_2\right]_a^b = \left[\left(\tilde{\Pi}\,\zeta\right)^\top J_s \left(\tilde{\Pi}\,\zeta\right)\right]_a^b \tag{12}$$

where $\tilde{\Pi} = (\ I_{2m}\ 0_r\ )\,\Pi$.

Now we may define a Lagrangian subspace associated with energy functionals involving derivatives of the state variables in the following way, using the permutation matrix

$$\Sigma = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

Define the port variables

$$\begin{pmatrix} \phi_\partial \\ \psi_\partial \end{pmatrix} = \Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta\right) \tag{13}$$

then the boundary term (10) may be written in terms of the port variables

$$\left[\zeta_1^\top \left(S_0^\top P_1 - P_0^\top S_1\right)\zeta_2\right]_a^b = (\,\phi_{1\partial},\,\psi_{1\partial}\,)\, J_{2m}^s \begin{pmatrix} \phi_{2\partial} \\ \psi_{2\partial} \end{pmatrix} \tag{14}$$

Proposition 9. Consider the vector space $\mathcal{V} = L_2(a,\, b,\, \mathbb{R}^n) \times \mathbb{R}^n$, the subspace $\mathcal{L} \subset \mathcal{V} \times \mathcal{V}^*$ with $\mathcal{V}^* \sim \mathcal{V}$,

$$\mathcal{L} = \Big\{(x,\, \phi_\partial,\, e,\, \epsilon_\partial)\, \mathcal{V} \times \mathcal{V}^* \,/\, \exists \zeta \in H^1(a,\, b,\, \mathbb{R}^n)\,, \quad s.t.\ \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} \mathcal{P} \\ \mathcal{S} \end{pmatrix} \zeta;\ \begin{pmatrix} \phi_\partial \\ \psi_\partial \end{pmatrix} = \Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta\right)\Big\} \tag{15}$$

with the operators defined in (9) and satisfying the assumptions of the lemma 7, is a Lagrangian submanifold with respect to the skew-symmetric product for $(v_1, v_2) \in \mathcal{V}^2$

$$\langle v_1,\, v_2\rangle_- = \int_a^b \left(x_1^\top e_2 - x_2^\top e_1\right)\, dz - (\,\phi_{1\partial},\,\psi_{1\partial}\,)\, J_{2m}^s \begin{pmatrix} \phi_{2\partial} \\ \psi_{2\partial} \end{pmatrix} \tag{16}$$

The proof is closely analoguous to the proof for Stokes-Dirac structures presented in Le Gorrec et al. [2005].

Proof 10. The first step consists in proving the inclusion $\mathcal{L} \subset \mathcal{L}^{\perp -}$. Therefore it sufficies to evaluate (16) for any pair $(v_1, v_2) \in \mathcal{L}^2$ and use the definition of the port variables (13) and the expression of the boundary term (14) and one obtains $\langle v_1,\, v_2\rangle_- = 0$.

The second step consists in proving $\mathcal{L} \supset \mathcal{L}^{\perp -}$. Consider an element $v_2 \in \mathcal{L}^{\perp -}$ then, by definition, for any $v_1 \in \mathcal{L}$, $\langle v_1,\, v_2\rangle_- = 0$. Using (16) and the definition of the Lagrangian subspace (15), one computes

$$\langle v_1,\, v_2\rangle_- = \int_a^b \left(\left(\mathcal{P}\zeta\right)^\top e_2 - x_2^\top \left(\mathcal{S}\zeta\right)\right)\, dz - \left(\Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta\right)\right)^\top J_{2m}^s \begin{pmatrix} \phi_{2\partial} \\ \psi_{2\partial} \end{pmatrix} \tag{17}$$

Choosing an element $v_1 \in \mathcal{L}$, generating by a function $\zeta_1 \in H^1(a,\, b,\, \mathbb{R}^n)$ with support strictly included in $[a, b]$, hence $\zeta(a) = \zeta(b) = 0$, one computes

$$\langle v_1,\, v_2\rangle_- = \int_a^b \left(\left(\mathcal{P}\zeta_1\right)^\top e_2 - x_2^\top \left(\mathcal{S}\zeta_1\right)\right)\, dz = \int_a^b \left(\zeta_1^\top \left(\mathcal{P}^* e_2 - \mathcal{S}^* x_2\right)\right)\, dz$$

By surjectivity of the operators $\mathcal{P}$ and $\mathcal{S}$ and the symmetry condition on $\mathcal{S}^*\mathcal{P}$, this implies that the first constitutive equation in (15) is satisfied: there is a function $\zeta_2 \in H^1(a,\, b,\, \mathbb{R}^n)$ such that $\begin{pmatrix} x_2 \\ e_2 \end{pmatrix} = \begin{pmatrix} \mathcal{P} \\ \mathcal{S} \end{pmatrix} \zeta_2$. Hence computation (17) may be continued

$$\langle v_1,\, v_2\rangle_- = \int_a^b \left(\zeta_1^\top \left(\mathcal{P}^* e_2 - \mathcal{S}^* x_2\right)\right)\, dz - \left(\Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta_1\right)\right)^\top J_{2m}^s \left[\begin{pmatrix} \phi_{2\partial} \\ \psi_{2\partial} \end{pmatrix} - \Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta_1\right)\right] \tag{18}$$

As the matrix $J_{2m}^s$is full-rank, the condition $\langle v_1,\, v_2\rangle_- = 0$ for any ζ_1 , implies that $\begin{pmatrix} \phi_{2\partial} \\ \psi_{2\partial} \end{pmatrix} - \Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta_1\right) = 0$. This implies that the second constitutive equation in (15) is satisfied. Hence $v_2 \in \mathcal{L}$ and $\mathcal{L} \supset \mathcal{L}^{\perp -}$ is proven.

In the sequel, we shall call the Lagrangian submanifold defined in this proposition *Stokes-Lagrangian submanifold.*

3.2 Boundary Port Hamiltonian systems defined on a Stokes-Lagrangian submanifold

Using the previously defined Lagrangian submanifold, the definition 3 of linear Boundary Port Hamiltonian systems may be generalized by considering Hamiltonian (or energy) functionals defined in (5) from *real-valued* symmetric matrix operators $\mathcal{Q}$ to first-order *differential* symmetric operators.

Definition 11. [Linear First Order Boundary Port Hamiltonian System on a Lagrangian submanifold] The Boundary Port Hamiltonian System on the state space $\mathcal{X} = L_2(a,\, b,\, \mathbb{R}^n)$ with respect to the Stokes-Dirac structure defined in Proposition 3 and on the Lagrangian subspace of proposition 9 is defined by the dynamical system

$$\begin{pmatrix} \dfrac{\partial x}{\partial t} \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \in \mathcal{D}_\mathcal{J} \qquad \text{and} \qquad (x,\, \phi_\partial,\, e,\, \epsilon_\partial) \in \mathcal{L}$$

with $(x,\, \phi_\partial,\, e,\, \epsilon_\partial)\, \mathcal{V} \times \mathcal{V}^*$ where $\mathcal{V} = L_2(a,\, b,\, \mathbb{R}^n) \times \mathbb{R}^n$ and $(f_\partial,\, e_\partial) \in \mathbb{R}^{2n}$.

Using the constitutive relations of the Stokes-Dirac structure $\mathcal{D}_\mathcal{J}$ in (4) and the kernel representation (15) of the Lagrangian subspace $\mathcal{L}$, the Boundary Port Hamiltonian system of definition 11 may also be expressed in terms of the following *Differential-Algebraic system*

$$\begin{aligned} &\frac{\partial}{\partial t}\mathcal{P}\,\zeta = \mathcal{J}\mathcal{S}\,\zeta,\ , \\ &\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} J_1 & -J_1 \\ I_n & I_n \end{pmatrix} \mathrm{tr}\left(\mathcal{S}\,\zeta\right) \\ &\begin{pmatrix} \phi_\partial \\ \psi_\partial \end{pmatrix} = \Sigma \mathrm{tr}\left(\tilde{\Pi}\,\zeta\right) \end{aligned}$$

This definition encompasses the definition of energy function which depends not only on the state variable but also on its spatial derivatives as for instance the vibrating string written on the classical symplectic space of displacement and momenta.

It allows also to consider non-local constitutive relation of the energy functional as it had been considered in

[Dorfman 1993] and that will be illustrated on the example of a model of nanorod [Heidari and Zwart 2019] in the next section.

3.3 Energy balance equation

The Hamiltonian of the Boundary Port Hamiltonian system of definition 11 is the generalized version of the explicit expression (5) with Hamiltonian which may be expressed as

$$H = \int_a^b \frac{1}{2} \left(\zeta^\top \mathcal{S}^* \mathcal{P} \zeta \right) dz$$

Let us now compute the time derivative of the Hamiltonian functional, using the boundary relations of a Lagrangian submanifold (10) and (12) then the properties of a Stokes-Dirac structure

$$\begin{aligned} \frac{dH}{dt} &= \frac{d}{dt} \int_a^b \frac{1}{2} \left(\zeta^\top \mathcal{S}^* \mathcal{P} \zeta \right) dz \\ &= \frac{1}{2} \int_a^b \left(\frac{\partial \zeta}{\partial t}^\top \mathcal{S}^* \mathcal{P} \zeta + \zeta^\top \mathcal{S}^* \mathcal{P} \frac{\partial \zeta}{\partial t} \right) dz \\ &= \int_a^b \left(\zeta^\top \mathcal{S}^* \mathcal{P} \frac{\partial \zeta}{\partial t} \right) dz + \left(\frac{d\phi_\partial}{dt}^\top, \frac{d\psi_\partial}{dt}^\top \right) J_{2m}^s \begin{pmatrix} \phi_\partial \\ \psi_\partial \end{pmatrix} \\ &= e_\partial^\top f_\partial + \frac{d\phi_\partial}{dt}^\top \psi_\partial - \frac{d\psi_\partial}{dt}^\top \phi_\partial \end{aligned}$$

If H is bounded from below, the system is dissipative with respect to the pair of port variables $(e_\partial,\, f_\partial)$ defined by the Stokes Dirac structure and, in addition, the pairs of variables $\left(\psi_\partial,\, \frac{d\phi_\partial}{dt}\right)$ and $\left(\phi_\partial,\, \frac{d\psi_\partial}{dt}\right)$ composed of the boundary energy variable (*and their time derivative*) associated with the Lagrangian subspace.

4. ILLUSTRATION ON A MODEL OF NANOROD

In this section, we shall illustrate the suggested definition with the example of the elasto-dynamical model developed for modelling carbon nanotubes [Cemal Eringen 1983] and treated in[Karličič et al. 2015] where the spectral analysis of a set of coupled beams is analyzed and in [Heidari and Zwart 2019] where a descriptor Port Hamiltonian formulation was used to prove the well-posedness of the model.

4.1 Nonlocal elasticity relation

Compared with the model of elastic rod presented in the section 4, the constitutive relation of elasticity is nonlocal and given by is written

$$\sigma = \left(1 - \mu \partial_z^2\right)^{-1} T\, \epsilon \tag{19}$$

where the coefficients μ and T are positive real numbers. This elasticity relation may be written in the image representation using the operators $\mathcal{P}_{el} = \left(1 - \mu \partial_z^2\right)^*$ and $\mathcal{S}_{el} = T$

$$\begin{pmatrix} \epsilon \\ \sigma \end{pmatrix} = \begin{pmatrix} \left(1 - \mu \partial_z^2\right)^* \\ T \end{pmatrix} \varepsilon \tag{20}$$

where the internal variable ε may be interpreted as a *virtual strain* and the elastic potential energy is

$$U\left(\varepsilon, \partial_z \varepsilon\right) = \int_Z \frac{1}{2} \sigma\, \varepsilon dz = \int_Z \frac{1}{2} T \left(\varepsilon^2 + \mu \left(\partial_z \varepsilon\right)^2 \right) dz$$

which variational derivative

$$\delta_\varepsilon U\left(\varepsilon, \partial_z \varepsilon\right) = T \left(1 - \mu \partial_z^2\right) \varepsilon$$

However the operator $\mathcal{P}_{el}$ is of order 2 which leads us to extend the space of internal variables with $\varsigma = \sqrt{\mu} \partial_z \varepsilon$ and write the elastic energy as

$$U_0\left(\varepsilon,\, \varsigma\right) = \int_Z \frac{1}{2} T \left(\varepsilon^2 + \varsigma^2 \right)$$

The total energy of the system $U_0\left(\varepsilon,\, \varsigma\right) + U_g\left(u\right) + K\left(p\right)$ with the definition of the variables, amounts to consider the (formal) Lagrangian subspace generated by the operators

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\sqrt{\mu}\partial_z & 0 \\ 0 & \sqrt{\mu}\partial_z & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{(\rho A)} \end{pmatrix} \tag{21}$$

which satisfy obviously the symmetry conditions of lemma 7. The boundary port variables are a linear combination of the local strain ζ and it derivative ς and the two associated stress variables at the boundary of the system

$$\begin{pmatrix} \phi_\partial \\ \psi_\partial \end{pmatrix} = \begin{pmatrix} J_b & -J_b \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} \zeta\left(a\right) \\ \varsigma\left(a\right) \\ \zeta\left(b\right) \\ \varsigma\left(b\right) \end{pmatrix} \tag{22}$$

with $J_b = -\sqrt{\mu}\, T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4.2 The port Hamiltonian dynamics

Inserting the image representation of the energy and co-energy variables in (8) and adding the definition of the internal variables ς(third line), the dynamics of the nanorod may then be expressed as the Differential-Algebraic Hamiltonian system

$$\partial_t \mathcal{P} \begin{pmatrix} u \\ \zeta \\ \varsigma \\ p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \partial_z \\ 0 & 0 & 0 & 0 \\ -1 & \partial_z & 0 & 0 \end{pmatrix} \mathcal{S} \begin{pmatrix} u \\ \zeta \\ \varsigma \\ p \end{pmatrix}$$

In order to include the power flows stemmming from the interconnection of the nanorod with its environment, this system extends to a Port Hamiltonian systems defined with respect to the Stokes-Dirac structure $\mathcal{D}_\mathcal{J}$ generated by the Hamiltonian operator

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \partial_z \\ 0 & 0 & 0 & 0 \\ -1 & \partial_z & 0 & 0 \end{pmatrix}$$

on the Stokes-Lagrangian subspace (15) associated with the operators (21).The boundary port variables associated with the Stokes-Dirac structure are (a linear combination of) the pairs of stress $T\zeta$ and velocities $\frac{p}{(\rho A)}$ at the boundary and the port variables associated with the boundary energy variables are (a linear combination of) the stresses and their time derivatives $\left(\frac{\partial \zeta}{\partial t},\, \sqrt{\mu}\, T\varsigma\right)$ and $\left(\frac{\partial \varsigma}{\partial t},\, \sqrt{\mu}\, T\zeta\right)$ at the boundary.

5. CONCLUSION

In this paper, we have suggested an extension of Boundary Port Hamiltonian systems by considering that the energy of the system is implicitely defined as a Lagrangian submanifold. We have considered the particular case when this Lagrangian submanifold is defined by an image representation based on two first-order linear differential operators. We have then derived port variables associated with this Lagrangian submanifold and stated the energy balance equation of the Boundary Port Hamiltonian systems. The right-hand side of this energy balance equation which consist in terms being the pairing of the power port variables of the Stokes-Dirac structure plus the pairing of the port variables of the Lagrangian submanifold and their time-derivative. This definition has been illustrated with the model of a voibrating nanorod with a non-local elasticity relation.

Of course this construction may be extended to higher-order linear differential operators and it would be intersting to see how the results on the well-posedness and the stabilizing control, based on the semi-group approach could be extended to this class of systems. A more abstract approach, valid on higher-dimensional domains may also be developed based on the suggested approach.

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