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Normal Forms for Flat Two-input Control Systems Linearizable via a Two-fold Prolongation ⋆

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Abstract: We present normal forms for nonlinear two-input control systems that become static feedback linearizable after a two-fold prolongation of a suitably chosen control, which is one of the simplest dynamic feedback. They form a particular class of flat systems, namely those of differential weight $n + 4$, where n is the number of states. We also show that the dynamic feedback creates singularities in the control space depending on the state and we discuss them.

Keywords: Normal forms, nonlinear control systems, flatness, dynamic linearization.

1. INTRODUCTION

In this paper, we give normal forms for flat control-affine systems of the form

$$\Sigma \ : \ \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \tag{1}$$

where x is the state defined on a open subset X of $\mathbb{R}^n$ and $u = (u_1, u_2)$ is the control taking values in an open subset U of $\mathbb{R}^2$ (more generally, an n-dimensional manifold X and a *two*-dimensional manifold U, resp.), and where f, g_1 and g_2 are smooth. The word smooth will always mean $\mathcal{C}^\infty$-smooth. The notion of flatness was introduced in control theory in the 1990s, by Fliess, Lévine, Martin and Rouchon (Fliess et al. [1995], see also Isidori et al. [1986], Jakubczyk [1993], Aranda-Bricaire et al. [1995], Pomet [1995]) and has attracted a considerable interest Fliess et al. [1999], Pomet [1997], Van Nieuwstadt et al. [1998], Pereira da Silva and Corrêa Filho [2001] because of its important applications in the problem of motion planning and constructive controllability (see, e.g., Martin et al. [2003], Lévine [2009], Tang et al. [2011], Kolar et al. [2017]). The system $\Xi : \dot{x} = F(x, u)$, where $x \in X \subset \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$, is *flat* if we can find locally m functions $\varphi_i(x, u, \dots, u^{(r)})$, for some $r \geq 0$, such that

$$x = \gamma(\varphi, \dots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \dots, \varphi^{(s)}),$$

for a certain integer s and suitable smooth maps γ and δ, where $\varphi = (\varphi_1, \dots, \varphi_m)$ is called a *flat output*. Therefore, the evolution in time of all state and control variables can be recovered from that of flat outputs without integration and all trajectories of the system can be completely parameterized.

Systems linearizable via invertible static feedback are flat and their normal forms are well known: they are static feedback equivalent to the Brunovský canonical form. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see Fliess et al. [1995], Pomet [1995, 1997]. In Nicolau and Respondek [2019], the authors presented normal forms for the class of flat systems that are the closest to static feedback linearizable ones, namely those that are feedback linearizable via the simplest dynamic feedback, which is a one-fold prolongation of a suitably chosen control. The goal of this paper is to generalize those results to the case of a two-fold prolongation. We will consider the case of two-input control systems only. Solving that problem in the simplest case of two controls is interesting for few reasons; first, it yields a complete analysis for a well defined class of flat systems, and second, it shows what kind of difficulties one may face when trying to give normal forms or to characterize flatness in the general case. Our aim is to give normal forms for nonlinear flat control systems of differential weight $n+m+2 = n+4$ (see Respondek [2003], and Section 2 for the notion of differential weight) and to discuss how the geometry of that class of systems is reflected by the normal forms (necessary and sufficient geometric conditions describing flatness of control-affine differential weight $n+m+2 = n+4$ were presented in Nicolau and Respondek [2016a]).

It is well known (see, e.g., Jakubczyk and Respondek [1980], Hunt and Su [1981]) that any static feedback linearizable and controllable system is feedback equivalent to the Brunovský canonical form that consists of m independent chains of integrators. In Nicolau and Respondek [2019], we proposed for multi-input systems dynamically linearizable via a one-fold prolongation (or, equivalently, flat systems of differential weight $n+m+1$) a modification of the Brunovský canonical form that contains at most $m - 1$ nonlinearities (at most only one nonlinearity per each chain). For the particular case of two-input control systems, one (and only one) nonlinearity is present. In this paper, we show that two-input systems dynamically linearizable via a two-fold prolongation can be brought into a normal form generalizing that of Brunovský as well as that characterizing flatness of differential weight $n+3$. Namely, at most two nonlinearities (at most one more than for flatness of differential weight $n+3$) are present. Interest in those normal forms is three-fold. First, to understand that

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systems linearizable dynamically via a two-fold prolongation differ from static feedback linearizable ones (resp., from dynamically linearizable via a one-fold prolongation) by at most two (resp., one) non-removable nonlinearities and to identify where those nonlinearities may appear and on which variables they may depend. Second, for a flat system we can express all state and control variables with the help of flat outputs and their derivatives and the proposed normal forms allow to express all but at most two special variables of the transformed system by pure derivations as well as to identify those special variables and to compute them via the implicit function theorem. Third, like for flatness of differential weight $n+3$, the proposed normal forms allow to describe the singularities that the two-fold dynamic prolongation may create in the input space and thus to identify the control values at which the system ceases to be flat. Fourth, like the Brunovský canonical form, the presented normal forms are compatible with flat outputs: if (φ_1, φ_2) is a flat output, then there exists an invertible static feedback transformation bringing the system into that normal form with (φ_1, φ_2) playing the role of the top variables.

In Nicolau and Respondek [2016a], we gave a geometric characterization of control-affine systems that become static feedback linearizable after a two-fold prolongation. The proposed normal forms apply to all systems described there, but we do not use those results to construct our normal forms. The paper is self-contained and all presented results can be proved independently of those of Nicolau and Respondek [2016a] although can be seen as their illustration and their continuation. The paper is organized as follows. In Section 2, we recall the definitions of flatness and of differential weight. In Section 3, we give our main results and illustrate them by two examples in Section 4.

2. FLATNESS

For $l \geq -1$, denote $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$, with $\bar{u}^{-1}$ empty.

Definition 1. The system $\Xi : \dot{x} = F(x, u)$, $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, is *flat* at $(x_0, \bar{u}_0^l) \in X \times U \times \mathbb{R}^{ml}$, for $l \geq -1$, if there exist a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer s and smooth functions γ_i, $1 \leq i \leq n$, and δ_j, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

for any C^{l+s}-control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

If $\varphi_i = \varphi_i(x)$, for all $1 \leq i \leq m$, we say that the system is x-flat. The minimal number of derivatives of components of a flat output, needed to express x and u, is called differential weight Respondek [2003] of that flat output and is formalized as follows. By definition, for any flat output φ of Ξ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{aligned} \tag{2}$$

Moreover, we can choose $(s_1, \ldots, s_m)$, γ and δ such that (see Respondek [2003]) if for any other m-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ and functions $\tilde{\gamma}$ and $\tilde{\delta}$, we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$ the differential weight of φ. A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ. The *differential weight* of a flat system equals the differential weight of a minimal flat output. The differential weight is $n + m + p$, where $p \geq 0$ can be interpreted as the minimal dimension of a precompensator that dynamically linearizes the system. Indeed, $p = 0$ corresponds to static feedback linearizable systems, while the case $p = 1$ corresponds to systems linearizable via a one-fold prolongation of a suitably chosen control Nicolau and Respondek [2016b, 2017]. Presenting normal forms for the case $p = 1$ was the subject of Nicolau and Respondek [2019]. The goal of this paper is to give normal forms for two-input flat control systems of differential weight $n + m + 2 = n + 4$.

We say that control-affine systems Σ and $\tilde{\Sigma}$ given, resp., by $\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, $x \in X$, $u \in \mathbb{R}^m$ and $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \sum_{i=1}^m \tilde{u}_i \tilde{g}_i(\tilde{x})$, $\tilde{x} \in \tilde{X}$, $\tilde{u} \in \mathbb{R}^m$, with X and $\tilde{X}$ open subsets of $\mathbb{R}^n$, are (locally) static feedback equivalent if there exist a (local) diffeomorphism $\tilde{x} = \phi(x)$ and an invertible static feedback transformation of the form $u = \alpha(x) + \beta(x)\tilde{u}$ which transform Σ into $\tilde{\Sigma}$, i.e., $\tilde{f}(\phi(x)) = \frac{\partial \phi(x)}{\partial x}(f(x) + \alpha(x)g(x))$ and $\tilde{g}(\phi(x)) = \frac{\partial \phi(x)}{\partial x} g(x)\beta(x)$, where $g = (g_1, \ldots, g_m)$ and $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m)$.

The control-affine system Σ is static feedback linearizable if it is static feedback equivalent to a linear controllable system $\Lambda : \dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$. The problem of static feedback linearization was solved by Brockett [1979] (for a smaller class of transformations) and then by Jakubczyk and Respondek [1980] and, independently, by Hunt and Su [1981], who gave the following geometric necessary and sufficient conditions. Define the distributions $\mathcal{D}^{j+1} = \mathcal{D}^j + [f, \mathcal{D}^j]$, where $\mathcal{D}^0 = \text{span}\{g_1, \ldots, g_m\}$ and $[f, \mathcal{D}^j] = \{[f, \xi] : \xi \in \mathcal{D}^j\}$. The system Σ is locally static feedback linearizable if and only if for any $j \geq 0$, the distributions $\mathcal{D}^j$ are of constant rank, *involutive* and $\mathcal{D}^{n-1} = TX$. Therefore, the following nested sequence of involutive distributions summarizes the geometry of static feedback linearizable systems: $\mathcal{D}^0 \subset \mathcal{D}^1 \subset \ldots \subset \mathcal{D}^{n-1} = TX$. It is well known that systems linearizable via invertible static feedback are flat. Their description (2) uses the minimal possible, which is $n + m$, number of time-derivatives of the components φ_i. Indeed, a feedback linearizable system is static feedback equivalent to the Brunovský canonical form (see Brunovský [1970], and also Aranda-Bricaire et al. [1995], Pomet et al. [1992] for related works)

$$(Br) : \begin{cases} \dot{z}_i^j &= z_i^{j+1} \\ \dot{z}_i^{\rho_i} &= v_i, \end{cases}$$

where $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, and $\sum_{i=1}^m \rho_i = n$, and is clearly flat with $\varphi = (\varphi_1, \ldots, \varphi_m) = (z_1^1, \ldots, z_m^1)$ being a minimal flat output (of differential weight $n + m$). In fact, an equivalent way of describing static feedback linearizable systems is that they are flat systems of differential weight $n + m$, see Theorem 2.2 in Nicolau and Respondek [2017].

Assumption (A1). *We will work under constant ranks assumption implying that all results are valid on an open and dense subset of X and hold locally, around any given point x_0 of that set, where all involved ranks are constant.*

We study flat systems of differential weight $n+m+2$, thus Σ is not static feedback linearizable (not flat of differential weight $n+m$). It follows that there exists the smallest integer $0 \leq k \leq n-1$ such that the linearizability conditions (either involutivity or constant rank) are not satisfied for $\mathcal{D}^k$ (flat systems are always accessible so $\mathcal{D}^{n-1} = TX$ holds). Under Assumption (A1), only the case $\mathcal{D}^k$ noninvolutive can occur and we denote by $\overline{\mathcal{D}}^k$ its involutive closure. Any two-input flat system of differential weight $n+m+2 = n+4$ becomes static feedback linearizable after a two-fold prolongation of a suitably chosen control, as asserted by the following result.

Proposition 1. The following are equivalent:

(i) Σ *is flat at* $(x_0, u_0, \dot{u}_0, \ldots, u_0^{(l)})$, *of differential weight* $n+4$, *for a certain* $l \geq -1$;

(ii) Σ *is x-flat at either* x_0 *or* (x_0, u_0), *of differential weight* $n+4$;

(iii) There exists, around x_0, *an invertible static feedback transformation* $u = \alpha(x) + \beta(x)\tilde{u}$, *bringing* Σ *into the form* $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{g}_2(x)$, *such that the prolongation*

$$\tilde{\Sigma}^{(2,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{g}_2(x) \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= v_1 \end{cases}$$

is locally static feedback linearizable around (x_0, y_0), *with* $y_1 = \tilde{u}_1$, $v_2 = \tilde{u}_2$, $\tilde{f} = f + \alpha g$ *and* $\tilde{g} = g\beta$, *where* $g = (g_1, g_2)$ *and* $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$.

A system Σ satisfying (iii) will be called dynamically linearizable via an invertible two-fold prolongation. $\tilde{\Sigma}^{(2,0)}$ is, indeed, obtained by applying an invertible static feedback $u = \alpha + \beta\tilde{u}$ and then prolonging the first control $\tilde{u}_1$ twice as $v_1 = \ddot{\tilde{u}}_1$ and not prolonging $\tilde{u}_2$ (which explains the notation $\tilde{\Sigma}^{(2,0)}$). Before giving our main results, we introduce the notion of corank, and state Proposition 2 needed in the proofs, but also having an independent interest.

Notation 1. (Corank). Let $\mathcal{A}$ and $\mathcal{B}$ be two distributions of constant rank. Denote $[\mathcal{A}, \mathcal{B}] = \{[a, b] : a \in \mathcal{A},\ b \in \mathcal{B}\}$. If $\mathcal{A} \subset \mathcal{B}$, the corank of the inclusion $\mathcal{A} \subset \mathcal{B}$ equals the rank of the quotient $\mathcal{B}/\mathcal{A}$, i.e., $\operatorname{cork}(\mathcal{A} \subset \mathcal{B}) = \operatorname{rk}(\mathcal{B}/\mathcal{A})$.

Proposition 2. Suppose that Σ *is dynamically linearizable via invertible two-fold prolongation and let* $\mathcal{D}^k$ *be its first noninvolutive distribution. Then the distribution* $\mathcal{D}^k$ *is feedback invariant and satisfies* $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) \leq 2$. *Moreover, if* $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 2$, *then* $\operatorname{rk}\mathcal{D}^k = 2k+2$.

According to Proposition 2, at most two independent directions of $\overline{\mathcal{D}}^k$ stick out of $\mathcal{D}^k$. In this paper, we study only the case when the noninvolutivity of $\mathcal{D}^k$ is maximal, i.e., $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 2$. The normal forms for the particular case $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 1$ remind those for flatness of differential weight $n+3$ (for which the first noninvolutive distribution necessarily satisfies $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 1$), but are slightly different and will be treated elsewhere. To sum up, we make the following assumption:

Assumption (A2). *The integer* k *is the smallest such that* $\mathcal{D}^k$ *is not involutive and, moreover, we suppose* $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 2$.

3. MAIN RESULTS: NORMAL FORMS

The main results are given by Theorems 1 and 2 that present four normal forms for the class of flat two-input control-affine systems of differential weight $n+m+2 = n+4$.

3.1 Normal forms. Given a system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ that is flat at x_0 (at (x_0, u_0), if $k = 0$ or $k = 1$), the normal forms are obtained under local static feedback transformations

$$z = \phi(x), \qquad u = \alpha(x) + \beta(x)\tilde{u}, \tag{3}$$

and are flat at z_0 (at $(z_0, \tilde{u}_0)$, if $k = 0$ or $k = 1$), where

$$z_0 = \phi(x_0), \qquad u_0 = \alpha(x_0) + \beta(x_0)\tilde{u}_0. \tag{4}$$

For $k \geq 2$, we will give two normal forms NF1 and NF2, that are static feedback equivalent, each of them having its advantage: for NF1 we see immediately the control to be prolonged, whereas for NF2 the role of k is explicit. The integers ρ_i and μ_i that show up in the normal forms are such that $\rho_1 + \rho_2 + 2 = n$ and $\mu_1 + \mu_2 + 2k = n$. For $i = 1, 2$, denote $\bar{z}_i^j = (z_i^1, \cdots, z_i^j)$ and $\bar{w}_i^j = (w_i^1, \cdots, w_i^j)$.

Theorem 1. Suppose $k \geq 2$. *The following are equivalent:*

(i) Σ *is flat at* x_0 *of differential weight* $n+4$;

(ii) Σ *is locally, around* x_0, *static feedback equivalent in a neighborhood of* $z_0 \in \mathbb{R}^n$ *to:*

$$NF1 : \begin{cases} \dot{z}_1^j = z_1^{j+1} & \dot{z}_2^j = z_2^{j+1} \\ \dot{z}_1^{\rho_1} = \tilde{u}_1 & \dot{z}_2^{\rho_2} = z_2^{\rho_2+1} + b(z)\tilde{u}_1 \\ & \dot{z}_2^{\rho_2+1} = z_2^{\rho_2+2} + d(z)\tilde{u}_1 \\ & \dot{z}_2^{\rho_2+2} = \tilde{u}_2 \end{cases}$$

where $1 \leq j \leq \rho_i - 1$, $\rho_i \geq k+1$, *and the functions* $b = b(\bar{z}_1^{\rho_1-k+2}, \bar{z}_2^{\rho_2-k+2})$ *and* $d = d(\bar{z}_1^{\rho_1-k+3}, \bar{z}_2^{\rho_2-k+3})$ *and are such that* k *is as in Assumption (A2);*

(iii) Σ *is locally, around* x_0, *static feedback equivalent in a neighborhood of* $w_0 \in \mathbb{R}^n$ *to:*

$$NF2 : \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_2^j = w_2^{j+1} \\ \dot{w}_1^{\mu_1-1} = w_1^{\mu_1} & \dot{w}_2^{\mu_2-1} = p(w) + q(w)w_1^{\mu_1+2} \\ \dot{w}_1^l = w_1^{l+1} & \dot{w}_2^l = w_2^{l+1} \\ \dot{w}_1^{\mu_1+k} = \tilde{u}_1 & \dot{w}_2^{\mu_2+k} = \tilde{u}_2 \end{cases}$$

where $1 \leq j \leq \mu_i - 2$, $\mu_i \leq l \leq \mu_i + k - 1$, $\mu_1 \geq 1$, $\mu_2 \geq 3$, *the functions* $p = p(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2})$ *and* $q = q(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2})$ *are such that* $\left(\frac{\partial p}{\partial w_2^{\mu_2}} + \frac{\partial q}{\partial w_2^{\mu_2}} w_1^{\mu_1+2}\right)(w_0) \neq 0$ *and verify additional regularity conditions such that* k *is as in Assumption (A2).*

Moreover, the functions $(\varphi_1, \varphi_2) = (z_1^1, z_2^1)$ *for NF1, and* $(\varphi_1, \varphi_2) = (w_1^1, w_2^1)$ *for NF2, are flat outputs of differential weight* $n+2+2 = n+4$.

The functions b and d of NF1 (resp., p and q of NF2) are briefly discussed in Section 3.1.1 below.

Flatness described by Theorem 1 (treating the case $k \geq 2$) is local around x_0 but, like for flat systems of differential weight $n+m$ or $n+m+1$ (with $k \geq 1$), is global with respect to the control u. This changes if $k = 1$ or $k = 0$, in which cases we have to consider flatness at (x_0, u_0) as described by Theorem 2. Observe that we face a similar situation for flatness of differential weight $n+m+1$ when $k = 0$, Nicolau and Respondek [2017, 2019]. Below, $\tilde{u}_{10}$ stands for the first component of $\tilde{u}_0$ given by (4).

Theorem 2. Assume $k = 0$ *or* $k = 1$. *The following are equivalent:*

(i) Σ *is flat at* (x_0, u_0) *of differential weight* $n+4$;

(ii) Σ *is locally, around* x_0*, static feedback equivalent in a neighborhood of* $z_0 \in \mathbb{R}^n$ *to:*

$$NF3: \begin{cases} \dot{z}_1^j = z_1^{j+1} & \dot{z}_2^j = z_2^{j+1} \\ \dot{z}_1^{\rho_1} = \tilde{u}_1 & \dot{z}_2^{\rho_2} = z_2^{\rho_2+1} + (\tilde{u}_1 - \tilde{u}_{10})b(z) \\ & \dot{z}_2^{\rho_2+1} = z_2^{\rho_2+2} + (\tilde{u}_1 - \tilde{u}_{10})d(z) \\ & \dot{z}_2^{\rho_2+2} = \tilde{u}_2 \end{cases}$$

where $1 \le j \le \rho_i - 1$, $\rho_i \ge k+1$, *the functions* $b = b(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1})$ *and* $d = d(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+2})$ *are such that* k *is as in Assumption (A2), that is, satisfy additionally the conditions of Section 3.1.2 below;*

(iii) Σ *is locally, around* x_0*, static feedback equivalent in a neighborhood of* $z_0 \in \mathbb{R}^n$ *to:*

$$NF4: \begin{cases} \dot{z}_1^j = z_1^{j+1} & \dot{z}_2^j = z_2^{j+1} \\ \dot{z}_1^{\rho_1} = \tilde{u}_1 & \dot{z}_2^{\rho_2} = A(z, \tilde{u}_1) \\ & \dot{z}_2^{\rho_2+1} = B(z, \tilde{u}_1) \\ & \dot{z}_2^{\rho_2+2} = \tilde{u}_2 \end{cases}$$

where $1 \le j \le \rho_i - 1$, $\rho_i \ge k+1$, *and*

$$\frac{\partial A}{\partial z_2^{\rho_2+1}}(z_0, \tilde{u}_{10}) \neq 0, \quad \frac{\partial B}{\partial z_2^{\rho_2+2}}(z_0, \tilde{u}_{10}) \neq 0, \text{ and} \tag{5}$$

- *either* $k = 0$ *and then*

$$A(z, \tilde{u}_1) = a(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1}) + z_2^{\rho_2+1}\tilde{u}_1, \tag{6}$$

$$B(z, \tilde{u}_1) = c(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+2}) + z_2^{\rho_2+2}\tilde{u}_1, \tag{7}$$

- *or* $k = 1$ *and then*

$$\text{either } A(z, \tilde{u}_1) = a(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1}) + z_2^{\rho_2+1}\tilde{u}_1$$

$$\text{or } A(z, \tilde{u}_1) = z_2^{\rho_2+1} + b(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1})\tilde{u}_1, \tag{8}$$

$$\text{and } B(z, \tilde{u}_1) = z_2^{\rho_2+2} + d(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1})\tilde{u}_1, \tag{9}$$

where the function b *of the second case of (8) satisfies some regularity conditions assuring that* $k = 1$ *is as in Assumption (A2).*

Moreover, for NF3 and NF4, $(\varphi_1, \varphi_2) = (z_1^1, z_2^1)$ *is a flat output of differential weight* $n+4$.

3.1.1 Nonlinearities and invariants of NF1 and NF2. For NF1, the value of k is encoded when looking more precisely at the functions b and d, which depend on $\bar{z}_1^{\rho_1-k+2}$ and $\bar{z}_2^{\rho_2-k+2}$ for b and on $\bar{z}_1^{\rho_1-k+3}$ and $\bar{z}_2^{\rho_2-k+3}$ for d. The involutivity of the distributions $\mathcal{D}^j$, for $0 \le j \le k-1$, imposes on b and d three more conditions (which can be computed by a straightforward calculation) that we do not present here. Moreover, by Assumption (A2), the distribution $\mathcal{D}^k$ is noninvolutive and $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 2$ implying, see Proposition 2, $\operatorname{rk} \mathcal{D}^k = 2k+2$. Hence the integers ρ_i of NF1 are such that $\rho_i \ge k+1$, the integers μ_i of NF2 are such that $\mu_1 \ge 1$ and $\mu_2 \ge 3$, and the functions b and d of NF1, resp., p and q of NF2, should also satisfy some regularity conditions assuring $\operatorname{rk} \overline{\mathcal{D}}^k(x_0) = 2k+4$. One may also distinguish the subcase when the grow vector of $\mathcal{D}^k$ is $(2k+2, 2k+3, 2k+4)$ from that when the grow vector of $\mathcal{D}^k$ is $(2k+2, 2k+4)$.

3.1.2 Conditions on the functions b and d of NF3. If $k = 0$, the first distribution $\mathcal{D}^0$ is noninvolutive which, together with Assumption (A2), implies that $\operatorname{rk}(\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]) = 3$ and $\operatorname{rk} \overline{\mathcal{D}}^0 = 4$ (equivalently, the grow vector of $\mathcal{D}^0$ is (2, 3, 4)). The first condition implies $\frac{\partial d}{\partial z_2^{\rho_2+2}}(z_0) \neq 0$ and the second implies $\frac{\partial b}{\partial z_2^{\rho_2+1}}(z_0) \neq 0$. If $k = 1$, then $b = b(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1})$ and $d = d(\bar{z}_1^{\rho_1}, \bar{z}_2^{\rho_2+1})$ and have to satisfy additionally some conditions assuring that $\mathcal{D}^1$ is indeed noninvolutive and that $\operatorname{cork}(\mathcal{D}^1 \subset \overline{\mathcal{D}}^1) = 2$. As for NF1, one may also distinguish the cases $\overline{\mathcal{D}}^1 = \mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1]$ (corresponding to the grow vector of $\mathcal{D}^1$ being $(4, 6)$) and $\operatorname{cork}(\mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1] \subset \overline{\mathcal{D}}^1) = 1$ (corresponding to the grow vector of $\mathcal{D}^1$ being $(4, 5, 6)$).

3.2 Discussion of the normal forms. All normal forms are valid around $z_0 \in \mathbb{R}^n$, which may be zero or not. Thus all forms can be used around any point (equilibrium or not).

All forms and the minimal x-flat outputs are compatible, that is, for a given flat system Σ of differential weight $n+4$, we can always simultaneously normalize Σ and a priori given minimal flat output φ, as asserted by:

Proposition 3. Let Σ *be flat at* x_0 *(at* (x_0, u_0)*, if* $k = 0$ *or* $k = 1$*) and* φ *a minimal flat output of differential weight* $n+4$ *of* Σ*. Then* Σ *is locally around* x_0 *static feedback equivalent to NF1 or NF2, if* $k \ge 2$*, (resp. to NF3 or NF4, if* $k = 0$ *or* $k = 1$*), where* $\varphi = (z_1^1, z_2^1)$ *for NF1 (resp. for NF3 and NF4) and* $\varphi = (w_1^1, w_2^1)$ *for NF2.*

All normal forms become locally static feedback linearizable after a two-fold prolongation of $\tilde{u}_1$. In the cases $k = 1$ and $k = 0$ (and only in those two cases!), the precompensator creates singularities in the control space (depending on the state), see Section 3.3 below.

Normal forms NF1, NF3 and NF4 always contain a linear chain $\frac{\mathrm{d}^{\rho_1}}{\mathrm{d}t^{\rho_1}} z_1^1 = \tilde{u}_1$, called z_1-chain whose control has to be prolonged twice. For each form there are (at most) two nonlinearities of different possible forms (see Table 1) associated to the z_2-chain (which is called nonlinear). Observe that the normal forms for $k \ge 1$ may actually present only one nonlinear function (see Example 2): this happens only if the function d involved in the expression of $\dot{z}_2^{\rho_2+1}$ is identically zero. On the other hand, the normal forms for $k = 0$ always involve two nonlinearities. The number of possible nonlinearities of the forms presented in this paper, is due to the fact that the first noninvolutive distribution $\mathcal{D}^k$ is actually squeezed between two involutive ones, namely $\mathcal{D}^{k-1} \subset \mathcal{D}^k \subset \overline{\mathcal{D}}^k$ and both inclusions are of corank two, see Nicolau and Respondek [2016a].

NF2 also contains a linear w_1-subsystem that is a chain of pure ρ_1-fold integrator $\frac{\mathrm{d}^{\rho_1}}{\mathrm{d}t^{\rho_1}} w_1^1 = \tilde{u}_1$ (we actually have $\rho_1 = \mu_1 + k$). The w_2-chain has two nonlinearities $p(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2})$ and $q(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2})$ defining the only nonlinear component $\dot{w}_2^{\mu_2-1} = p(w) + q(w) w_1^{\mu_1+2}$. In NF2, the integer k appears explicitly, so the noninvolutive distribution $\mathcal{D}^k$ is easier to be analyzed with the help of NF2. From NF2, it is obvious that in the case $k \ge 2$, the flat outputs provide a parametrization of system's trajectories that is global with respect to controls. If $k = 1$, then the system is static feedback equivalent to a form that reminds NF2, namely to

$$NF2_{k=1}: \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_2^j = w_2^{j+1} \\ \dot{w}_1^{\mu_1-1} = w_1^{\mu_1} & \dot{w}_2^{\mu_2-1} = p(w) + q(w)\tilde{u}_1 \\ \dot{w}_1^{\mu_1} = w_1^{\mu_1+1} & \dot{w}_2^{\mu_2} = w_2^{\mu_2+1} \\ \dot{w}_1^{\mu_1+1} = \tilde{u}_1 & \dot{w}_2^{\mu_2+1} = \tilde{u}_2 \end{cases}$$

(the only difference with NF2 is that the control $\tilde{u}_1$ replaces the variable $w_1^{\mu_1+2}$ in the only nonlinear equation $\dot{w}_2^{\mu_2-1} = p(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2}) + q(\bar{w}_1^{\mu_1+1}, \bar{w}_2^{\mu_2}) w_1^{\mu_1+2}$).

3.2.1 Comparison with flat systems of differential weight $n+2$ and $n+3$. For two-input flat control systems of differential weight $n+m+1 = n+3$ (or equivalently, linearizable via a one-fold prolongation), we proposed in Nicolau and Respondek [2019], without constant rank assumption, four normal forms that are analogous to those presented here. Each of them contains only one nonlinear function and they differ notably by the role played by the first distribution destroying feedback linearizability (k being defined as the smallest integer such that either involutivity or constant rank is not satisfied for $\mathcal{D}^k$). We distinguished the cases $k \geq 1$ and $k = 0$. To summarize normal forms NF1, NF3 and NF4 as well as those for differential weight $n+2$ and $n+3$, we state:

Proposition 4. Σ is flat at x_0 or (x_0, u_0) of differential weight at most $n+4$ if and only if it is locally, around x_0, static feedback equivalent in a neighborhood of $z_0 \in \mathbb{R}^n$ to:

$$NF0: \begin{cases} \dot z_1^j = z_1^{j+1} & \dot z_2^j = z_2^{j+1} \\ \dot z_1^{\rho_1} = \tilde u_1 & \dot z_2^{\rho_2} = a(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1}) + b(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})\tilde u_1 \\ & \dot z_2^{\rho_2+1} = c(z) + d(z)\tilde u_1 \\ & \dot z_2^{\rho_2+2} = \tilde u_2 \end{cases}$$

where $1 \leq j \leq \rho_i - 1$, a, b, c, d are smooth functions verifying

$$\frac{\partial(a + b\tilde u_{10})}{\partial z_2^{\rho_2+1}}(z_0) \neq 0, \quad \frac{\partial(c + d\tilde u_{10})}{\partial z_2^{\rho_2+2}}(z_0) \neq 0.$$

Moreover, $(\varphi_1, \varphi_2) = (z_1^1, z_2^1)$ is a minimal flat output of differential weight at most $n+4$.

Normal form NF0 presents four nonlinearities, but we can always normalize at least two of them. Table 1 (where d.w. stands for differential weight) presents all possible cases.

Table 1. Comparison

d.w.	Form	Nonlinearities	
$n+2$	(Br)	$a = z_2^{\rho_2+1}$	$b \equiv 0$
		$c = z_2^{\rho_2+2}$	$d \equiv 0$
$n+3$	All forms	$a = z_2^{\rho_2+1}$	$b \equiv 0$
	NF1, $k \geq 1$	$c = z_2^{\rho_2+2}$	$d(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any
	NF3, $k = 0$	$c + \tilde u_{10} d = z_2^{\rho_2+2}$	$d(z)$ any
	NF4, $k = 0$	$c(z)$ any	$d = z_2^{\rho_2+2}$
		or*	
		$c = z_2^{\rho_2+2}$	$d(z)$ any
$n+4$	NF1, $k \geq 2$	$a = z_2^{\rho_2+1}$	$b(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any
		$c = z_2^{\rho_2+2}$	$d(z)$ any
	NF3, $k = 0$ or 1	$a + \tilde u_{10} b = z_2^{\rho_2+1}$	$b(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any
		$c + \tilde u_{10} d = z_2^{\rho_2+2}$	$d(z)$ any
	NF4, $k = 1$	$a(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any	$b = z_2^{\rho_2+1}$
		or	
		$a = z_2^{\rho_2+1}$	$b(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any
		and	
		$c = z_2^{\rho_2+2}$	$d(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any
	NF4, $k = 0$	$a(\bar z_1^{\rho_1}, \bar z_2^{\rho_2+1})$ any	$b = z_2^{\rho_2+1}$
		$c(z)$ any	$d = z_2^{\rho_2+2}$

*Under the constant rank assumption, only the first form shows up.

3.3 Identifying singularities in the control space. When $k = 0$ or $k = 1$, the system exhibits flatness singularities in the control space and we will explain that, according to Theorem 2, there are two ways to deal with them (reminding very much the case of flatness of differential weight $n+m+1$ with $k = 0$, Nicolau and Respondek [2019]). Normal forms NF3 and NF4 are local, around z_0, but global with respect to $\tilde u = (\tilde u_1, \tilde u_2) \in \mathbb{R}^2$ and thus allow to identify all points $(z, \tilde u)$ at which the system is x-flat and distinguish them from $(z^s, \tilde u^s)$ at which it is not. Let u_0 be a nominal control. Its value is involved in NF3 in such a way that NF3 is flat around (x_0, u_0). On the other hand, NF4 does not use the knowledge of u_0 and in order to verify that NF4 is flat around (x_0, u_0) one needs to check conditions (5). More precisely, the value of $\tilde u_{10}$ appears explicitly in NF3 and it yields, at the nominal point $(z_0, \tilde u_0)$, a zero multiplying the functions b and d and allowing us to normalize $a(z) + b(z)\tilde u_{10}$ as $z_2^{\rho_2+1}$ and $c(z) + d(z)\tilde u_{10}$ as $z_2^{\rho_2+2}$. The value of $\tilde u_{10}$ does not appear in NF4 and the normalization of the nonlinearities is different. In the case $k = 0$, due to noninvolutivity of $\mathcal{D}^0$ whose grow vector is (2, 3, 4), we can always normalize b and d to $b = z_2^{\rho_2+1}$ and $d = z_2^{\rho_2+2}$, resp. If $k = 1$, the function d does not depend on $z_2^{\rho_2+2}$ (so we can always normalize c to $c = z_2^{\rho_2+2}$) and from $\frac{\partial(a + \tilde u_{10} b)}{\partial z_2^{\rho_2+1}}(z_0) \neq 0$, it follows that we can normalize either a or b. Forms NF3 and NF4 hold on $\mathcal{O} \times \mathbb{R}^2$, where $\mathcal{O}$ is a neighborhood of z_0. The identification of all points at which the system is not flat can be performed as follows. Define by $a(z) + b(z)\tilde u_1$ and $c(z) + d(z)\tilde u_1$ the expressions for $\dot z_2^{\rho_2+1}$ and $\dot z_2^{\rho_2+2}$, resp., for both NF3 and NF4, independently of the normalization. Set $S_1(z, \tilde u_1) = \frac{\partial(a(z) + \tilde u_1 b(z))}{\partial z_2^{\rho_2+1}}$ and $S_2(z, \tilde u_1) = \frac{\partial(c(z) + \tilde u_1 d(z))}{\partial z_2^{\rho_2+2}}$, which depend (in an affine way) on $\tilde u_1$, and fix $z \in \mathcal{O}$. For $k = 1$, flatness singularities are $(z, \tilde u^s(z)) \in \mathcal{O} \times \mathbb{R}^2$, where $\tilde u^s(z) = (\tilde u_1^s(z), \tilde u_2^s(z))$, with $\tilde u_1^s(z)$ being the unique root of $S_1(z, \tilde u_1) = 0$ and any $\tilde u_2^s(z) \in \mathbb{R}$. Those singularities always exist and form, for a fixed z, one line in the control space $\mathbb{R}^2$. Similarly, if $k = 0$, flatness singularities are $(z, \tilde u^s(z)) \in \mathcal{O} \times \mathbb{R}^2$, where $\tilde u_1^s(z)$ is a root of the product $S_1(z, \tilde u_1) \cdot S_2(z, \tilde u_1) = 0$. Notice that, for each fixed $z \in \mathcal{O}$, the above product admits either two or one distinct real roots. Therefore for a given $z \in \mathcal{O}$, the values of the singular controls form, resp., two lines or one line in $\mathbb{R}^2$.

4. EXAMPLES

Example 1. Flatness of differential weight $n+4$ for four-dimensional control systems. The simplest two-input control system that may satisfy the assumptions under which we work (that is $\operatorname{cork}(\mathcal{D}^k \subset \overline{\mathcal{D}}^k) = 2$) are those in dimension four. The problem of flatness for four-dimensional control systems with two inputs has been solved by Pomet [1997] whose results can be interpreted in terms of dynamic linearizability via a p-fold prolongation of a suitably chosen control with $p \leq 3$ (or equivalently in terms of flatness of differential weight $n+m+p = 6+p$). While the cases $p \leq 2$ correspond to x-flatness (which is consistent with Proposition 1 and Nicolau and Respondek [2017]), the last case $p = 3$ describes (x, u)-flatness (i.e., all possible flat outputs depend explicitly on u). We will focus on the case $p = 2$ (which is the subject of this paper). For four-dimensional two-input control systems that satisfy Assumption (A2), the first noninvolutive distribution is necessarily $\mathcal{D}^0$, i.e., $k = 0$, and as we have already seen, under the hypotheses $\operatorname{cork}(\mathcal{D}^0 \subset \overline{\mathcal{D}}^0) = 2$, only the grow vector (2, 3, 4) for $\mathcal{D}^0$ is possible (and in particular, $\overline{\mathcal{D}}^0 = TX$). In fact, if a system with four states and two controls satisfying $\operatorname{cork}(\mathcal{D}^0 \subset \overline{\mathcal{D}}^0) = 1$ is flat, then it is necessarily dynamically linearizable via a one-fold prolongation (and thus of of differential weight $n+3 = 7$). Therefore the condition cork $(\mathcal{D}^0 \subset \overline{\mathcal{D}}^0) = 2$ is actually necessary for the differential weight $n+4 = 8$ and, according to Theorem 2, all systems of differential weight 8 admit, around (x_0, u_0), one normal form that can be taken as either

$$NF3_{n=4}: \begin{cases} z_1^1 = \tilde{u}_1 & \dot{z}_2^1 = z_2^2 + (\tilde{u}_1 - \tilde{u}_{10})b(z_1^1, \bar{z}_2^2) \\ & \dot{z}_2^2 = z_2^3 + (\tilde{u}_1 - \tilde{u}_{10})d(z_1^1, \bar{z}_2^3) \\ & \dot{z}_2^3 = \tilde{u}_2, \end{cases}$$

where $\frac{\partial b}{\partial z_2^2}(z_0) \neq 0$ and $\frac{\partial d}{\partial z_2^3}(z_0) \neq 0$, or

$$NF4_{n=4}: \begin{cases} z_1^1 = \tilde{u}_1 & \dot{z}_2^1 = a(z_1^1, \bar{z}_2^2) + z_2^2\tilde{u}_1 \\ & \dot{z}_2^2 = c(z_1^1, \bar{z}_2^3) + z_2^3\tilde{u}_1 \\ & \dot{z}_2^3 = \tilde{u}_2, \end{cases}$$

where $\frac{\partial a}{\partial z_2^2}(z_0) + \tilde{u}_{10} \neq 0$ and $\frac{\partial b}{\partial z_2^3}(z_0) + \tilde{u}_{10} \neq 0$. Form $NF4_{n=4}$ agrees with that of Pomet [1997] corresponding to linearizability via a two-fold prolongation and our regularity conditions coincide with one case of Pomet [1997] while the other case of Pomet [1997] is excluded by the constant rank Assumption (A1).

Example 2. The PVTOL aircraft. The following model of a planar vertical take off and landing (PVTOL) aircraft was introduced in Hauser et al. [1992] and has attracted a lot of attention in the last years (see, e.g., Martin et al. [1996], Lozano et al. [2004]). The configuration of the system is (θ, x_1, y_1), with θ the angle the aircraft makes with the horizontal axis and (x_1, y_1) the position of its center of mass. After normalisation of m and J, the dynamics of the PVTOL aircraft is given by:

$$\begin{array}{lll} \dot{\theta} = \omega & \dot{x}_1 = x_2 & \dot{y}_1 = y_2 \\ \dot{\omega} = u_2 & \dot{x}_2 = -u_1 \sin\theta & \dot{y}_2 = -a_g + u_1 \cos\theta \\ & \quad +\epsilon u_2 \cos\theta & \quad +\epsilon u_2 \sin\theta, \end{array}$$

where u_1 and u_2 correspond, resp., to the body vertical force (minus the gravity) and to forces on the tips of the wings, a_g is the gravity acceleration and $\epsilon \neq 0$ is a fixed constant related to the geometry of the aircraft. The PVTOL aircraft has been shown to be locally flat with $\varphi = (x_1 - \epsilon \sin\theta, y_1 + \epsilon \cos\theta)$ a flat output of differential weight $n+4$, see Martin et al. [1996]. By a direct calculation we get $k = 1$ and $\overline{\mathcal{D}}^1 = \mathcal{D}^1 + [\mathcal{D}^1, \mathcal{D}^1] = TX$. We show that the PVTOL model can be brought into NF4. Suppose that we work around a nominal point such that $\sin\theta_0 \neq 0$. By introducing the local coordinates $\tilde{x}_1 = x_1 - \epsilon \sin\theta$, $\tilde{x}_2 = L_f\tilde{x}_1$, $\tilde{y}_1 = y_1 + \epsilon\cos\theta$, $\tilde{y}_2 = L_f\tilde{y}_1$, $\tilde{\theta} = -\cot\theta$, $\tilde{\omega} = L_f\tilde{\theta}$, followed by a suitable invertible feedback transformation (where $\tilde{u}_1 = \epsilon\omega^2 \sin\theta - u_1 \sin\theta$), we get

$$\begin{array}{ll} \dot{\tilde{x}}_1 = \tilde{x}_2 & \dot{\tilde{y}}_1 = \tilde{y}_2 \\ \dot{\tilde{x}}_2 = \tilde{u}_1 & \dot{\tilde{y}}_2 = -a_g + \tilde{\theta}\tilde{u}_1 \\ & \dot{\tilde{\theta}} = \tilde{\omega} \\ & \dot{\tilde{\omega}} = \tilde{u}_2. \end{array}$$

This is normal form NF4 for $k = 1$ for which, with respect to the Brunovský canonical form, only one component (the third from the bottom) of the second chain is modified. In these coordinates, we have $\varphi = (\tilde{x}_1, \tilde{y}_1)$ and in order to express all states and controls, we need to differentiate $\tilde{u}_1$ twice (thus obtaining the differential weight $n+4$). The input $\tilde{u}_1$ is also the control that has to be prolonged twice in order to obtain a static feedback linearizable prolonged system. Finally, notice that $\tilde{u}_{10} = 0$ is a singular control for flatness of differential weight $n+4$.

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