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Robust Positively Invariant Polyhedral Sets and Constrained Control using Fuzzy T-S Models: a Bilinear Optimization Design Strategy ⋆

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Abstract: We propose a numerical method to compute stabilizing state feedback control laws and associated polyhedral invariant sets for nonlinear systems represented by Fuzzy Takagi-Sugeno (T-S) models, subject to state and control constraints, and persistent disturbances. Sufficient conditions are derived under which a given polyhedral set is positively invariant under a Parallel Distributed Compensation (PDC), in the form of bilinear algebraic inequalities. Then, a bilinear programming (BP) problem is proposed to compute the state feedback gains and an associated positively invariant polyhedron, with predefined complexity, which solve a constrained regulation problem for the Fuzzy T-S system. A numerical example illustrates the effectiveness of the method.

Keywords: Fuzzy T-S models, Parallel Distributed Compensation, Constrained Control, Invariant Sets, Bilinear Programming.

1. INTRODUCTION

Fuzzy Takagi-Sugeno (T-S) models provide a local representation of nonlinear plants as a convex combination of a number of linear models (Takagi and Sugeno, 1985). This feature has allowed the extension of linear systems analysis and design tools to handle nonlinear systems. In particular, the so-called Parallel-Distributed-Compensation (PDC) controllers can be designed in order to guarantee local asymptotical stability of a nonlinear system represented by a Fuzzy T-S model (Wang et al., 1996).

Most techniques described in the literature apply to continuous-time systems and consist in formulating analysis and synthesis conditions as convex optimization problems described in terms of Linear Matrix Inequalities (LMIs) (Tanaka and Wang, 2002; Feng, 2006). However, many of such techniques do not take into account the fact that Fuzzy T-S models are usually valid only locally. Then, the performance computed through LMIs can only be achieved if the state trajectory is included in the region of validity of the Fuzzy T-S model.

Enforcing state trajectories into a given region can usually be achieved by computing an invariant set associated to a given stabilizing control law. A number of set-invariance techniques have been developed to solve control problems for linear systems subject to state and control constraints, and to persistent disturbances (Blanchini and Miani, 2015). In particular, polyhedral invariant sets have proven to result in larger regions of attraction than the ellipsoidal sets obtained from quadratic Lyapunov functions delivered by LMI-based techniques.

Recent work has been reported on the construction of polyhedral invariant sets for discrete-time Fuzzy T-S systems. In Arino et al. (2014), an analysis technique is proposed to compute polyhedral approximations of the maximal positively invariant polyhedron contained in the set of state constraints, for a given stabilizable PDC, computed via LMI techniques which do not account for the size of the resulting polyedral set. In Arino et al. (2013), this technique is extended to systems with bounded disturbances and an approximation of the minimal reachable set is also computed. Polya expanded T-S models are used to reduce conservatism of such sets. Their computation is based on the iterative algorithms developed for linear systems. In Arino et al. (2017), a controlled invariant λ-contractive polyhedron is computed and an explicit piecewise-affine control law is computed from the solution of a multiparametric linear programming. The complexity of such a law can become very large, though.

In this paper, for discrete-time T-S systems subject to state and control constraints, and to bounded disturbances, we establish sufficient conditions for a polyhedral set to be positively invariant under a static state feedback PDC law. Then, we propose a bilinear programming design approach to simultaneously compute stabilizing feedback

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gains and two associated positively invariant polyhedral sets: a larger one which stands for the region of attraction and a smaller one which stands for the set where the state trajectories converge to, in spite of the disturbances. Compared to the aforementioned approaches, the main advantage of our optimization technique is the possibility of delivering feedback gains likely to result in large polyhedral sets accounting for the regions of attraction, with fixed complexity in the sense that the maximal number of inequalities defining the invariant sets is *a priori* established.

2. PROBLEM STATEMENT

Consider a discrete-time nonlinear system given by:

$$x(k+1) = f(x(k), u(k), d(k)), \quad y(k) = Cx(k). \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $d \in \mathbb{R}^q$ is the disturbance and $y \in \mathbb{R}^p$ is the output.

This system can be expressed locally in a compact region of the state-space, denoted here as region of validity Ω, as a Fuzzy Takagi-Sugeno system (Fuzzy T-S system) with r rules (or local models) in the form:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k) + E_i d(k)) \\ y(k) &= Cx(k), \end{aligned} \tag{2}$$

where $\alpha_i(x(k))$ represent membership functions such that the vector of membership functions $\alpha(x(k))$ belongs to the $(r-1)$-dimensional standard simplex $\Delta \in \mathbb{R}^r$, defined as:

$$\Delta = \{\alpha \in \mathbb{R}^r \ : \ \sum_{i=1}^{r} \alpha_i = 1, \ \alpha_i \geq 0\}. \tag{3}$$

For the sake of simplifying the notation, from this point on, we can drop the explicit dependency of the membership functions α on the state $x(k)$.

Considering now a fuzzy PDC (Parallel Distributed Compensation) state-feedback controller, given by:

$$u(k) = \sum_{i=1}^{r} \alpha_i(x(k)) F_i x(k), \tag{4}$$

the closed-loop system takes the form:

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j((A_i + B_iF_j)x(k) + E_i d(k)). \tag{5}$$

The region of validity Ω is considered to be polyhedral. The disturbance $d(k)$ is unknown but bounded to a compact polyhedron containing the origin:

$$d(k) \in \mathbb{D} = \{d \ : \ Wd \leq \mathbf{1}.\} \tag{6}$$

System (2) is subject to state and control constraints:

$$x(k) \in \mathbb{X} = \{x \ : \ Gx(k) \leq \mathbf{1}\}, \tag{7}$$

$$u(k) \in \mathbb{U} = \{u \ : \ Vu(k) \leq \mathbf{1}\}, \tag{8}$$

The sets $\mathbb{X}$ and $\mathbb{U}$ are compact polytopes containing the origin. Since we are interested in trajectories belonging to the region of validity Ω, we can consider that $\mathbb{X}$ is the result of the intersection between Ω and the additional state constraints. The possibility of enforcing state trajectories into a given set is characterized by the following definition:

Definition 1. A set $\mathbb{X} \subset \Omega$ is said to be *Robust Positively Invariant* (RPI) with respect to system (5) if $\forall x(0) \in \mathbb{X}$ and $\forall d(k) \in \mathbb{D}$, $x(k) \in \mathbb{X}$, $\forall k \geq 0$.

We are interested in finding state-feedback matrices F_i, $i = 1, \cdots, r$ and a polyhedral set

$$\mathbb{X}_{\text{inv}} = \{x \ : \ Lx \leq \mathbf{1}\}$$

which is Robust Positively Invariant under the control law (4) and such that any trajectory starting from $\mathbb{X}_{\text{inv}}$ converges, in finite-time, to a homothetic set $\mathbb{X}_{\text{ub}} \subseteq \mathbb{X}_{\text{inv}}$, given by $\mathbb{X}_{\text{ub}} = \{x \ : \ Lx \leq \rho\mathbf{1}\}$, $0 < \rho \leq 1$, $\rho \in \Re$, and remains *ultimately bounded* (UB) in $\mathbb{X}_{\text{ub}}$.

3. MAIN RESULTS

Proposition 1. The polyhedral set $\mathbb{X}_{\text{inv}}$ is Robust Positively Invariant with respect to the closed-loop system (5), with contraction rate λ, $0 < \lambda < 1$, and associated UB-set $\mathbb{X}_{\text{ub}}$, if there exist matrices H_{ii}, Z_{ii}, $i = 1, \cdots, r$ and H_{ij}, Z_{ij}, $i = 1, \cdots, r$, $j = i+1, \cdots, r$ such that:

$$\begin{aligned} &H_{ii}L = L(A_i + B_iF_i), \ H_{ii} \geq 0, \\ &Z_{ii}W = LE_i, \ Z_{ii} \geq 0, \\ &H_{ii}\mathbf{1} + Z_{ii}\mathbf{1} \leq \lambda\mathbf{1}, \\ &H_{ij}L = L\frac{(A_i + B_iF_j + A_j + B_jF_i)}{2}, \ H_{ij} \geq 0, \\ &Z_{ij}W = L\frac{(E_i + E_j)}{2}, \ Z_{ij} \geq 0, \\ &H_{ij}\mathbf{1} + Z_{ij}\mathbf{1} \leq \lambda\mathbf{1}, \\ &H_{ii}\rho\mathbf{1} + Z_{ii}\mathbf{1} \leq (1-\epsilon)\rho\mathbf{1}, \ H_{ij}\rho\mathbf{1} + Z_{ij}\mathbf{1} \leq (1-\epsilon)\rho\mathbf{1}, \end{aligned} \tag{9}$$

where $0 < \epsilon \in \Re$ is given and sufficiently small.

Proof: Consider that $Lx(k) \leq \mathbf{1}$. Then:

$$\begin{aligned} Lx(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j L((A_i + B_iF_j)x(k) + E_i d(k)) \\ &= \sum_{i=1}^{r} \alpha_i(x(k))^2 L((A_i + B_iF_i)x(k) + E_i d(k)) + \\ &+ \sum_{i=1}^{r}\sum_{j=i+1}^{r} \alpha_i\alpha_j L((A_i + B_iF_j + A_j + B_jF_i)x(k) + \\ &+ (E_i + E_j)d(k)) \\ &= \sum_{i=1}^{r} \alpha_i^2 (H_{ii}Lx(k) + Z_{ii}Wd(k)) + \\ &+ \sum_{i=1}^{r}\sum_{j=i+1}^{r} \alpha_i\alpha_j (H_{ij}Lx(k) + Z_{ij}Wd(k) \\ &\leq \sum_{i=1}^{r} \alpha_i^2 (H_{ii}\mathbf{1} + Z_{ii}\mathbf{1}) + \sum_{i=1}^{r}\sum_{j=i+1}^{r} \alpha_i\alpha_j (H_{ij}\mathbf{1} + Z_{ij}\mathbf{1}) \\ &\leq \sum_{i=1}^{r} \alpha_i^2 \lambda\mathbf{1} + \sum_{i=1}^{r}\sum_{j=i+1}^{r} 2\alpha_i\alpha_j\lambda\mathbf{1} \\ &= \lambda\left(\sum_{i=1}^{r}\alpha_i\right)^2 \mathbf{1} = \lambda\mathbf{1} < \mathbf{1}. \end{aligned} \tag{10}$$

This proves that the external polyhedron $\mathbb{X}_{\text{inv}}$ is RPI and contractive. Thus, by considering now the two last inequalities in (9), similar arguments can show that the internal polyhedron $\mathbb{X}_{\text{ub}}$ is also RPI. Hence, any trajectory that reaches $\mathbb{X}_{\text{ub}}$ or that emanates from it, remains inside. □

One should notice that this Proposition does not consider a particular membership function, providing membership-shape independent, possibly conservative, conditions.

Proposition 1 assures that if $x(0) \in \mathbb{X}_{\text{inv}}$, then $x(k) \in \mathbb{X}_{\text{inv}}$ for $k = 1, 2, \cdots, \forall d(k) \in \mathbb{D}$. Moreover, with $0 < \lambda < 1$ and without disturbances, it guarantees the contraction of state trajectories inside $\mathbb{X}_{\text{inv}}$, i.e., if $x(k) \in \mathbb{X}_{\text{inv}}$, then $x(k+1) \in \lambda\mathbb{X}_{\text{inv}}$. Therefore, if $x(0) \in \mathbb{X}_{\text{inv}}$, then $x(k) \in \lambda^k\mathbb{X}_{\text{inv}}$. A direct consequence is that if the conditions of Proposition 1 are satisfied, then $x(k) \to 0$ as $k \to \infty$.

When persistent disturbances enter the picture, convergence to the origin is not guaranteed anymore. In this case, the state can be proven to converge to a smaller RPI set around the origin. The two following results show that under a controller which satisfies the conditions of Proposition 1, the trajectory of $x(k)$ is guaranteed to converge to $\mathbb{X}_{\text{ub}}$

Proposition 2. Let $\mathbb{X}_{\text{inv}} = \{x : Lx \leq \mathbf{1}\}$ be a RPI set with respect to (5), with contraction rate λ, satisfying conditions (9). Then, any set $\mathbb{X}_\beta = \beta^{-1}\mathbb{X}_{\text{inv}} = \{x : \beta Lx \leq \mathbf{1}\} \supset \mathbb{X}_{\text{inv}}$, $0 < \beta < 1$, is a RPI set w.r.t. (5) with contraction rate $\lambda_\beta < \lambda$.

Proof: If $\mathbb{X}_{\text{inv}}$ is RPI w.r.t. (5) satisfying (9), then it is clear that the conditions in (9) (excepting the two last equations, related to $\mathbb{X}_{\text{ub}}$) are satisfied if we replace L by βL and Z_{ij} by $\tilde{Z}_{ij} = \beta Z_{ij}$. In this case:

$$H_{ij}\mathbf{1} + \tilde{Z}_{ij}\mathbf{1} = H_{ij}\mathbf{1} + \beta Z_{ij}\mathbf{1} < H_{ij}\mathbf{1} + Z_{ij}\mathbf{1} \leq \lambda\mathbf{1}.$$

Then, it is clear that there exists $\lambda_\beta < \lambda$ such that $H_{ij}\mathbf{1} + \tilde{Z}_{ij}\mathbf{1} \leq \lambda_\beta\mathbf{1}$, which proves, according to Proposition 1, that $\mathbb{X}_\beta$ is RPI with contraction rate $\lambda_\beta < \lambda$. □

Proposition 3. Let $\mathbb{X}_{\text{inv}} = \{x : Lx \leq \mathbf{1}\}$ and $\mathbb{X}_{\text{ub}} = \rho\mathbb{X}_{\text{inv}}$ be RPI sets with respect to (5) satisfying conditions (9). Then, any state trajectory starting from $x(0) \in \mathbb{X}_{\text{inv}}$ converges to $\mathbb{X}_{\text{ub}}$ in a finite number of steps.

Proof: By definition, $\mathbb{X}_{\text{ub}} = \rho\mathbb{X}_{\text{inv}}$, with $0 < \rho \leq 1$. From Proposition 2, all the sets $\beta^{-1}\mathbb{X}_{\text{ub}}$, with $\rho < \beta < 1$ are RPI with contraction rate $\lambda < \lambda_\beta \leq (1-\epsilon) \leq 1$. Then, it is clear that, starting from $x(0) \in \mathbb{X}_{\text{inv}}$, $x(k) \in \beta^{-1}\mathbb{X}_{\text{ub}}$ with monotonically increasing values of β, until $x(k)$ eventually reaches $\mathbb{X}_{\text{ub}}$. □

4. A CONSTRAINED REGULATION PROBLEM

The characterization of Robust Positive Invariance of $\mathbb{X}_{\text{inv}}$ under the PDC state-feedback law (4) allows to propose a solution for the following constrained regulation problem:

Problem 1. Compute a state-feedback law (4) such that the state and control constraints (7), (8) are satisfied $\forall d(k) \in \mathbb{D}$ and $x(k) \in \mathbb{X}_{\text{ub}}$ for $k \geq \bar{k}$, for a finite $\bar{k}$.

Proposition 4. Problem 1 has a solution if for given (sufficiently small) scalar $\epsilon > 0$, there exist matrices F_i, N_i, H_{ij}, Z_{ij}, $i = 1, \cdots, r$, $j = i, \cdots, r$, L and M, and a scalar λ such that:

$$\begin{aligned}
H_{ij}L &= L\frac{(A_i + B_iF_j + A_j + B_jF_i)}{2}.\mu_{ij},\ H_{ij} \geq 0, \\
Z_{ij}W &= L\frac{(E_i + E_j)}{2}.\mu_{ij},\ Z_{ij} \geq 0, \\
H_{ij}\mathbf{1} &+ Z_{ij}\mathbf{1} \leq (\lambda\mathbf{1}).\mu_{ij} \\
H_{ij}\rho\mathbf{1} &+ Z_{ij}\mathbf{1} \leq ((1-\epsilon)\rho\mathbf{1}).\mu_{ij},
\end{aligned} \tag{11}$$

$$N_iL = VF_i,\ N_i \geq 0,\ N_i\mathbf{1} \leq \mathbf{1},\ i = 1, \cdots, r \tag{12}$$

$$ML = G,\ M \geq 0,\ M\mathbf{1} \leq \mathbf{1}, \tag{13}$$

where $0 < \epsilon \in \Re$ is given and sufficiently small and $\mu_{ij} = \begin{cases} 1, & \text{if } i = j\ , \\ 2, & \text{otherwise.} \end{cases}$

Proof: According to Proposition 1, conditions (11) imply positive invariance of $\mathbb{X}_{\text{inv}}$ with contraction rate $0 < \lambda < 1$. That implies $x(k) \in \mathbb{X}_{\text{inv}}$, $k = 1, 2, \cdots, \forall d(k) \in \mathbb{D}$ if $x(0) \in \mathbb{X}_{\text{inv}}$ and, according to Proposition 3, $x(k)$ reaches $\mathbb{X}_{\text{ub}}$ in finite time.

According to the so-called Extended Farkas Lemma (see, e.g. Castelan and Hennet (1992)), conditions (12) imply $VF_ix \leq \mathbf{1}$, $\forall x \in \mathbb{X}_{\text{inv}}$. Since $u(k)$ is given by (4), then, $\forall x \in \mathbb{X}_{\text{inv}}$ one has:

$$\begin{aligned}
Vu(k) &= \sum_{i=1}^{r}\alpha_iVF_ix(k) = \sum_{i=1}^{r}\alpha_iN_iLx(k) \\
&\leq \sum_{i=1}^{r}\alpha_iN_i\mathbf{1} \leq \sum_{i=1}^{r}\alpha_i\mathbf{1} = \mathbf{1},
\end{aligned}$$

which guarantees satisfaction of control constraints (8).

Again using the Extended Farkas Lemma, conditions (13) imply $\mathbb{X}_{\text{inv}} \subset \mathbb{X}$, guaranteeing thereby satisfaction of state constraints (7). □

In the particular case where only the output $y(k)$ is measured and the membership functions depend only on $y(k)$, the approach proposed here can be easily extended to treat both static and dynamic output-feedback control design:

Static output-feedback control: $u(k) = \sum_{i=1}^{r}\alpha_iK_iy(k)$.
It amounts to replace F_i by K_iC in the expressions derived in Propositions 1 and 4.

Dynamic output-feedback control: Consider the following compensator:

$$\begin{aligned}
x_c(k+1) &= \sum_{i=1}^{r}\alpha_i\left(A_{ci}x_c(k) + B_{ci}y(k)\right), \\
u(k) &= \sum_{i=1}^{r}\alpha_i\left(C_{ci}x_c(k) + D_{ci}y(k)\right).
\end{aligned} \tag{14}$$

The closed-loop system can be written in the following augmented form:

$$\begin{aligned}
\begin{bmatrix} x(k+1) \\ x_c(k+1) \end{bmatrix} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\left(\begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} + \right. \\
&\left.\begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} D_{cj} & C_{cj} \\ B_{cj} & A_{cj} \end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\right)\begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}
\end{aligned} \tag{15}$$

which is clearly equivalent to the static output-feedback case with matrices A_i, B_i and C replaced by the corresponding augmented matrices.

5. BILINEAR OPTIMIZATION DESIGN STRATEGY

The proposed solution to Problem 1, given by Proposition 4, carry some products among pairs of matrix variables, including the control gains F_{ij} to be synthesized, as well as other bilinear terms arising from the products between matrices and scalars or vectors. These bilinear products can be appropriately considered as design constraints, and adapted nonlinear optimization techniques can be used to find solutions to Problem 1, as discussed in this section.

5.1 Design strategy

The proposed design strategy considers a shape-set approach to enlarge the size of the set of admissible initial conditions $\mathbb{X}_{\text{inv}}$, and a weighted objective function is set such that the relative size of the UB-set $\mathbb{X}_{\text{ub}}$ can also be diminished. Thus, consider a polyhedral shape-set $\mathcal{S} = \{x_k : Sx_k \leq \beta\mathbf{1}\}$, with $0 < \beta \in \mathbb{R}$ and $S \in \mathbb{R}^{l_s \times n}$, that must verify $\mathcal{S} \subseteq \mathbb{X}_{\text{inv}}$. According to the Extended Farkas' Lemma, this inclusion is equivalent to the existence of a matrix $0 \leq J \in \mathbb{R}^{r \times l_s}$ and a scalar $0 \leq \gamma \in \Re$ such that:

$$JS = L, \;\; J\mathbf{1} \leq \gamma\mathbf{1}. \tag{16}$$

Notice that matrix S, and hence the shape of $\mathcal{S}$, is a designer choice, whereas the coefficient $\gamma = \beta^{-1}$ is an optimization variable that allows to enlarge the set $\mathcal{S} \subseteq \mathbb{X}_{\text{inv}}$.

Next, consider the scalar variable $\rho \in (0, 1]$ that defines the limits of the UB-set $\mathbb{X}_{\text{ub}} \subset \mathbb{X}_{\text{inv}}$. Thus, the following basic bilinear optimization problem is proposed to find solutions to the design Problem 1, by considering that $r > n$, S and a scalar $\bar{\lambda} < 1$ are given:

$$\begin{array}{rl} \min\limits_{\Gamma} & \Phi(\gamma,\rho) = \gamma + \theta\rho \\ \text{subject to} & (11) - (13), (16), \\ & 0 \leq \lambda \leq \bar{\lambda}, \\ & f_\ell(\cdot) \leq \varphi_\ell, \quad \ell = 1, ..., \bar{\ell}, \end{array} \tag{17}$$

with $\Gamma = (L, F_i, H_{ij}, Z_{ij}, N_i, M, J, \lambda, \gamma, \rho)$, and where:

i) the proposed weighted objective function $\Phi(\gamma, \rho)$, with chosen weighting design parameter $0 \leq \theta \in \Re$, allows to trade-off the maximization of the size of $\mathbb{X}_{\text{inv}}$ and the minimization of the relative size of UB-set $\mathbb{X}_{\text{ub}}$;

ii) the additional constraints, represented by $f_\ell(\cdot) \leq \varphi_\ell$, may be imposed for different purposes, including numerical ones as briefly discussed in this section.

Notice that for different chosen weighting parameter θ, the optimization problem (17) can provide different solutions to Problem 1, or even fail, depending also on the choices of S and r.

Remark 1. A possible choice for the shape-set $\mathcal{S}$ is to consider that it has the same shape and complexity of the constraint set $\mathbb{X}$, setting $S = G$ and $r = l_g$.

5.2 Implementation Issues

Different non-linear optimization techniques could be considered to solve (17) as, for instance, the ones described in Conn et al. (2009); Kennedy and Eberhart (1995). In this work, the KNITRO solver (Byrd et al., 2006) was used to generate the results. It is based on interior point and active set methods that can efficiently deal with bilinear optimization problems such as (17). We emphasize that KNITRO is not guaranteed to find global optimal solutions. However, local minima are found upon convergence. Furthermore, the non-linear optimization problem (17), whose constraints we formulated under a matrix form, can be rewritten in the element-wise form that is more suitable for the language AMPL employed by KNITRO. Therefore, this reformulation fits well into the approach of bilinear optimization and for control design purposes (see, for instance, Brião et al. (2018)).

Besides, the constraints $f_\ell(\cdot) \leq \varphi_\ell$ are established to determine lower and upper bounds to variables, such as for the elements of matrices L and M:

$$\underline{l} \leq l_{\nu\iota} \leq \bar{l} \;\; \text{and} \; \underline{m} \leq m_{s\iota} \leq \bar{m},$$

with $\nu \in \{1, ..., r\}$, $s \in \{1, ..., r\}$ and $\iota \in \{1, ..., n\}$. Likewise, upper and lower bounds are also imposed on the elements of the other involved vector and matrix decision variables. Such bounds are usually imposed in nonlinear programming problems in order to restrain the search space. In the present case, upper bounds on the elements of matrix L do not jeopardize the possibility of finding a solution, because the optimization problem is tailored to deliver a small L, that would result in a large invariant set.

6. NUMERICAL EXAMPLES

KNITRO solver from the NEOS Server (Gropp and Moré, 1997), a free internet-based service for solving numerical optimization problems, was used generate the results we report now. The default solver configurations were used, together with the multi-algorithm option. Also, we assigned the following lower and upper bounds pair $(-10^3, 10^3)$ to each element of matrices L and F_i. Furthermore, from the inequalities that involve the non-negative matrices H_{ij}, Z_{ij}, M, and N_i, one can see that their elements should be upper limited by 1. Moreover, the non-negative scalars ρ and γ had lower and upper bounds pairs set as $(0, 1)$ and $(1, 10^3)$ respectively, and the variables ϵ and λ were set as $\lambda = 0.9999$ and $\epsilon = 10^{-6}$. Additionally, the auxiliary matrices U and J were bounded by the lower and upper bound pairs $(-10^8, 10^8)$ and $(0, 10^3)$, respectively.

Consider the Fuzzy T-S system (1), borrowed from Arino et al. (2013), with

$$A_1 = \begin{bmatrix} -8.3 & -4.3 \\ -4.3 & -1.3 \end{bmatrix}, \; B_1 = \begin{bmatrix} -45.5 \\ -20.6 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.01 \\ 0.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -10.8 & -6.6 \\ -6.6 & -4.4 \end{bmatrix}, B_2 = \begin{bmatrix} -34.3 \\ -22.3 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.05 \\ 0.6 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -17.1 & -5.5 \\ -5.5 & -1.2 \end{bmatrix}, B_3 = \begin{bmatrix} -42.7 \\ -12.2 \end{bmatrix}, E_3 = \begin{bmatrix} 0.0033 \\ 0.9 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} -3.5 & -2.5 \\ -2.5 & -1.9 \end{bmatrix}, \; B_4 = \begin{bmatrix} -37 \\ -19.3 \end{bmatrix}, \; E_4 = \begin{bmatrix} 0.025 \\ 1.2 \end{bmatrix},$$

subject to bounded unknown disturbance $d(k)$ as in (6), and to state and control constraints as in (7) and (8), with

$$W = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \;,\; G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ -0.1 & 0 \\ 0 & -0.1 \end{bmatrix} \;,\; V = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}.$$

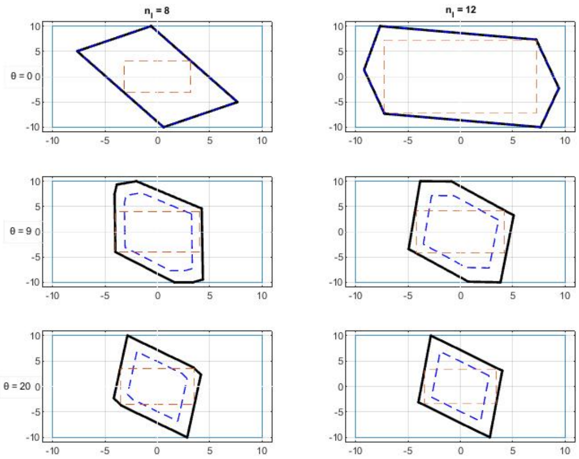

Fig. 1. Symmetric Polyhedrons

θ	n_l	γ	$\beta = \gamma^{-1}$	ρ	$\Phi(\gamma, \rho)$
0	8	3.16308	0.31615	0.999984	3.16308
	12	1.37705	0.72619	0.999823	1.37705
9	8	2.49875	0.40020	0.770132	9.42994
	12	2.36603	0.42265	0.715764	8.80791
20	8	2.82347	0.35417	0.684156	16.5066
	12	2.93000	0.34130	0.685668	16.6434

Table 1. Symmetric Polyhedrons

By applying the bilinear design strategy (17), with a *shape set* homothetic to the set $\mathbb{X}$ of state constraints (see Remark 1), different solutions to Problem 1 were encountered. In the sequel, we report the results obtained for three different values of the weighting factor θ, present in the objective function, and by considering that the maximum complexity of the invariant set, given by n_l = number of rows of L, can take two different values.

The results depicted by Figure 1 and Table 1 were obtained by imposing the invariant polyhedron $\mathbb{X}_{\rm inv}$ to be symmetric, which requires $L = \begin{bmatrix} L_s^T & -L_s^T \end{bmatrix}^T$, $L_s \in \Re^{\frac{n_l}{2} \times n}$. Figure 2 and table 2 depict the results obtained by considering that $\mathbb{X}_{\rm inv}$ may be asymmetric, in which case no special structure is imposed on L. In the two figures, the outermost set represents the state constraints $\mathbb{X}$, and the innermost one represents the shape set $\beta\mathbb{X}$. The two other sets correspond to $\mathbb{X}_{\rm inv}$ and $\mathbb{X}_{\rm ub}$ that are depicted in full black and dashed blue lines, respectively.

In both symmetric and asymmetric cases, the maximum allowed complexity were $n_l = 8$ or 12. For both cases, we can observe from Tables 1,2 and Figures 1, 2 the following:

- for each given value of n_l, the bigger is the weight θ, bigger is the value of the objective function $\Phi(\gamma, \rho)$ and, furthermore, smaller is the coefficient ρ that relates the size of $\mathbb{X}_{\rm ub}$ to the size of $\mathbb{X}_{inv}$. Also, by visual inspection, it can be observed that the size of the external set $\mathbb{X}_{inv}$ decreases as far as θ increases, as well as the size of the shape-set S.
- by considering the values $\theta = 0$ and 9, that the value of the objective function decreases by increasing the complexity n_l from 8 to 12, this behavior being more critical in the symmetric case possibly because all the imposed constraints are also symmetric.

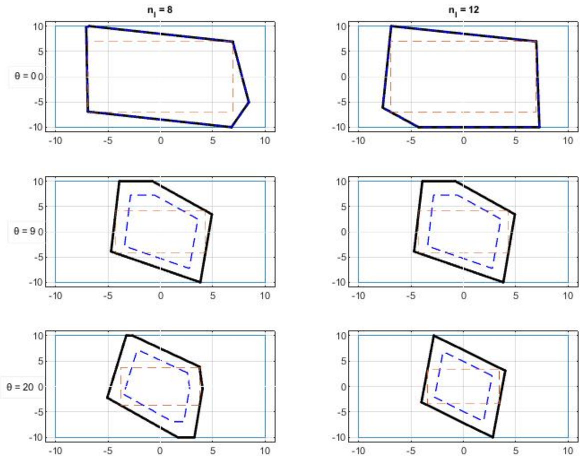

Fig. 2. Free-form Polyhedrons

θ	n_l	γ	$\beta = \gamma^{-1}$	ρ	$\Phi(\gamma, \rho)$
0	8	1.44482	0.69213	0.999826	1.44482
	12	1.43585	0.69645	0.999953	1.43585
9	8	2.35677	0.42431	0.726073	8.89143
	12	2.35818	0.42406	0.725913	8.89140
20	8	2.66983	0.37456	0.691345	16.4967
	12	2.92861	0.34146	0.685662	16.6419

Table 2. Free-form Polyhedrons

It is also noteworthy from figures 1, 2, that the obtained invariant polyhedrons have different shapes and actual complexities less than n_l. Which solution to choose should be decided from a comparative analysis of the solutions that could consider, apart from time-domain simulations, estimates of the maximum and minimum robust invariant sets contained in $\mathbb{X}$.

In Table 3, we show the gain matrices F_i, $i = 1, \ldots, 4$, obtained with $\theta = 9$, which are displayed, together with the corresponding matrix L_s in the symmetric cases, or L for the asymmetric polyhedron.

We performed simulations of state trajectories by considering a particular vector of membership functions, and a particular disturbance that randomly takes its value at the maximum or the minimum values of $w \in W$. In Figure 3 simulated state trajectories are depicted, related to the symmetric case with $n_l = 12$ (plots a) and c)), and to the asymmetric case with $n_l = 8$ (plots b) and d). In a) and b), the trajectories that emanate from different initial conditions in the $\mathbb{X}_{\rm inv}$ sets show the convergence from $\mathbb{X}_{\rm inv}$ to $\mathbb{X}_{\rm ub}$, while the plots in c) and d) aim to show that the trajectories generated from the origin should remain in $\mathbb{X}_{\rm ub}$. It is interesting to recall that other trajectory patterns could be obtained either by using other allowed disturbance sequences or other choices of membership functions.

7. CONCLUSIONS

In this paper we proposed sufficient conditions for a polyhedral set to be Robust Positively Invariant with respect

Case	n_l	F_i	L_s / L
Sym	8	$\begin{bmatrix}-0.17871 - 0.08775\end{bmatrix}_1$ $\begin{bmatrix}-0.30399 - 0.18560\end{bmatrix}_2$ $\begin{bmatrix}-0.40794 - 0.12486\end{bmatrix}_3$ $\begin{bmatrix}-0.10894 - 0.07597\end{bmatrix}_4$	$\begin{bmatrix}-0.00000 & -0.10000 \\ 0.04966 & -0.08308 \\ 0.10462 & 0.12093 \\ 0.23400 & 0.00188\end{bmatrix}$
Sym	12	$\begin{bmatrix}-0.17673 & -0.08717\end{bmatrix}_1$ $\begin{bmatrix}-0.30536 & -0.18944\end{bmatrix}_2$ $\begin{bmatrix}-0.40562 & -0.12175\end{bmatrix}_3$ $\begin{bmatrix}-0.10810 & -0.07950\end{bmatrix}_4$	$\begin{bmatrix}-0.00861 & -0.00450 \\ 0.00111 & 0.10043 \\ -0.00795 & -0.00424 \\ -0.12560 & -0.11100 \\ -0.21465 & 0.01760 \\ -0.00721 & -0.00421\end{bmatrix}$
Asym	8	$\begin{bmatrix}-0.17852 & -0.08650\end{bmatrix}_1$ $\begin{bmatrix}-0.30518 & -0.19065\end{bmatrix}_2$ $\begin{bmatrix}-0.40328 & -0.12106\end{bmatrix}_3$ $\begin{bmatrix}-0.10814 & -0.07929\end{bmatrix}_4$	$\begin{bmatrix}-0.22308 & 0.01259 \\ 0.00000 & 0.10000 \\ 0.12624 & 0.10943 \\ -0.09815 & -0.13753 \\ 0.00874 & 0.00220 \\ 0.21542 & -0.01762 \\ 0.00545 & 0.00112 \\ 0.00510 & 0.00088\end{bmatrix}$

Table 3. Numerical Results - $\theta = 9$

to a nonlinear system represented by a Fuzzy T-S model, subject to state and control constraints, and to persistent bounded disturbances. The conditions were then translated into constraints of a bilinear programming problem, whose solution delivers a stabilizing PDC controller and two invariant polyhedra: a larger one, which is a guaranteed region of attraction and a smaller one, into which the state trajectory is ultimately bounded.

The advantages of our approach compared to other approaches which deal with polyhedral invariant sets for Fuzzy T-S systems are the simultaneous computation of the sets and the stabilizing controllers, and the possibility of limiting the complexity of the invariant sets, in terms of the number of inequalities defining them.

Reliable solutions were obtained for the constrained control problem dealt with in the numerical experiments, even though the global optimum of the associated bilinear programming problem is not reached for sure. Future work should focus on a numerical method tailored for our problem, with the goal of further enlarging the size of the invariant polyhedron which serves as approximation of the region of attraction. The extension of the proposed approach to Polya expanded T-S models, which is expected to achieve less conservative results, as in Arino et al. (2014, 2017), is quite straightforward, and will be object of future work as well.

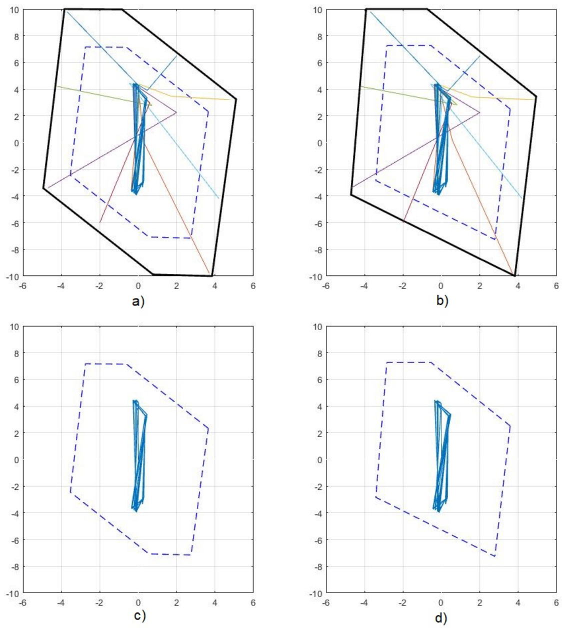

Fig. 3. Simulated Trajectories

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