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Global control for a class of feedforward nonlinear systems with uncertain measurement functions ⋆

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Abstract: Based on the nested-saturation technique, this paper investigates the global stabilization problem for a class of upper-triangular nonlinear systems with uncertain measurement functions. By imposing certain assumptions on the uncertain powers, a state-feedback controller, only involving the known parameters, is designed to locally stabilize the nonlinear system. Then, a saturated controller is constructed by combining the nested function and the local stabilizer. With appropriate saturation level, it can be proved that the saturated controller is able to make the closed-loop system globally asymptotically stable. Finally, a simulation example is presented to demonstrate the effectiveness and robustness of the proposed control scheme.

Keywords: Nonlinear system; upper-triangular system; uncertain measurement function; nested saturation approach; robustness; global stabilization

1. INTRODUCTION

In the real world, the exact system model cannot be easily obtained due to the external disturbance, such as control coefficients, nonlinearities, state measurement functions or system powers are unknown. Adaptive control has been proved to be a powerful tool. With respect to the uncertain control coefficients, the work SUN and LIU (2008) has designed a global state feedback control law by combining the backstepping technique and adaptive control together, while the work Yu et al. (2011) has proposed a backstepping repetitive learning control method based on the Nussbaum-gain, but the nonlinear term needs to be known. For uncertain nonlinearities, two common assumptions for systems with uncertain nonlinear functions are nonlinearities satisfy the linear growth condition or local Lipschitz condition. When the linear growth rate is unknown, a universal output feedback has been constructed in Lei and Lin (2006) such that the system states can be regulated to the origin. It has been shown in Zha et al. (2016) that, due to the sensor's property, the output can be an uncertain nonlinear function, such as the infrared distance sensor. To address this issue, the work Zha et al. (2016) gave a new concept of the power drift upper-bound to allow the drifts to vary within limits and the designed controller was able to globally stabilize a family of nonlinear systems with different measurement drifts. Then the result has been further generalized to a class of high-order uncertain nonlinear systems in Zha et al. (2017). Moreover, it is shown in the work Su et al. (2017) that affected by the external environment, the powers p_i's in practical systems are not usually fixed but belong to suitable bounds. By adopting the new concept of interval homogeneity with monotone degrees to determine the allowable bounds for the unknown power drifts, global stabilizers have been constructed for the nonlinear systems. For the time-varying power, a smooth state feedback controller was designed in Chen et al. (2017).

In spite of these developments, the aforementioned results cannot be easily generalized to the feedforward nonlinear systems, since the backstepping method Gao et al. (2018) and the adding a power integrator technique cannot be used to design the controller. Therefore, the stabilization problem for the upper-triangular nonlinear systems has received much attention. Nested saturation Teel (1996) and the forwarding method Mazenc and Praly (1996) have been proved to be two useful tool for controller design. By imposing certain conditions on the nonlinear term, the paper Ding et al. (2009) has provided a solution to the global stabilization problem for a general class of feedforward systems by combining the adding a power integrator technique and the nested saturation method. Different from the nested saturation method, the forwarding approach relies on the bottom-to-top design procedure, including the smooth Qian and Lin (2012) and non-smooth versions Tian et al. (2014). However, the controller design in the aforementioned literature needs a priori knowledge of the measurement functions.

In this paper, we aim to address the global stabilization problem for a class of feedforward system with uncertain

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measurement functions. Motivated by Ding et al. (2009); Zha et al. (2018) and Zha et al. (2016), with the combination of the uncertain Lyapunov functions and the nested saturation functions, the designed controller can robustly stabilize the feedforward nonlinear system. The main contributions of this paper are as follows:
(i) Instead of designing adaptive laws or observers to estimate the system states, the controller is constructed using the uncertain information which reduces the complexity of the system model. only using the known bounds, is able to globally stabilize a family of upper-triangular nonlinear systems.
(ii) The proposed linear state feedback controller has simple structure and can be implemented easily.
(iii) The controller construction procedure only utilizes the known bounds, which leads to the robustness result for the system.

2. PROBLEM STATEMENT AND ASSUMPTIONS

This paper investigate the global stabilization problem for the following nonlinear systems

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + \phi_1(x_2(t), \cdots, x_n(t))\\ &\vdots\\ \dot{x}_{n-1}(t) &= x_n(t) + \phi_{n-1}(x_n(t))\\ \dot{x}_n(t) &= u(t),\\ y_i(t) &= x_i^{q_i}(t), \quad i = 1, \cdots, n,\end{aligned} \tag{1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) = (y_1(t), \cdots, y_n(t))^T$ are the system states, the control input and the measurements of system states, respectively. For $i = 1, \cdots, n$, the unknown powers $q_i \in \mathbb{R}_{odd}^{\geq 1} = \{q \in \mathbb{R} : q \geq 1$, q is a ratio of odd integers$\}$ satisfy $1 \leq a_i \leq q_i \leq b_i$ with known a_i, b_i, and the uncertain nonlinear functions $\phi_i(\cdot)$'s are $\mathcal{C}^1$ functions with $\phi_i(0) = 0$, $i = 1, \cdots, n-1$.

In order to design a linear state-feedback controller to globally stabilize the system (1), the following assumptions need to be imposed.

Assumption 2.1. For $i = 2, \cdots, n-1$, $b_i \leq \frac{2a_{i-1}a_{i+1}}{a_{i-1}+a_{i+1}}$ and $b_n \leq \frac{2a_{n-1}}{a_{n-1}+1}$.

Assumption 2.2. In a neighborhood of the origin, there exists a positive constant d, such that $\forall i = 1, \cdots, n-1$

$$|\phi_i(\cdot)| \leq d(|y_{i+1}|^{p_i} + \cdots + |y_n|^{p_i}) \tag{2}$$

with $q_i \geq \frac{1}{a_{i+1}}$.

3. LOCAL STABILIZATION FOR SYSTEM (1)

This section gives a local stabilizer design procedure for system (1) based on the notion of homogeneity with monotone degrees (HWMD). First, consider the corresponding nominal system

$$\dot{x}_i = x_{i+1}, \quad i = 1, \cdots, n-1, \ \dot{x}_n = u. \tag{3}$$

Step 1: Define $r_i = \frac{1}{q_i}$ and $\tau_i = r_{i+1} - r_i$. By choosing the Lyapunov function $V_1 = \frac{r_1}{2-\tau_1} x_1^{\frac{2-\tau_1}{r_1}} = \frac{1}{2}\xi_1^2$, the derivative of V_1 along the trajectory of system (3) is

$$\dot{V}_1 = x_1^{\frac{2-\tau_1-r_1}{r_1}}(x_2 - x_2^*) + x_1^{\frac{2-\tau_1-r_1}{r_1}} x_2^*,$$

where x_2^* is a virtual controller to be determined later.

With respect to the first Lyapunov function, the level set is constructed as

$$\Omega_1 = \{y \in \mathbb{R}^n | V_1(y_1) \leq N\},$$

for a positive constant N. With the linear virtual controller $x_2^* = -\beta_1^{r_2}\xi_1^{r_2}$, $\beta_1 > n^{b_2} > n^{\frac{1}{q_2}}$ and $\xi_1 = x_1^{r_1} = y_1$, the derivative of V_1 becomes

$$\begin{aligned}\dot{V}_1|_{\Omega_1} =& y_1^{2-r_2}(x_2 - x_2^*) + y_1^{2-r_2} x_2^*\\ \leq& -ny_1^2 + y_1^{2-r_2}x_2^*.\end{aligned} \tag{4}$$

Step k: Suppose that at step $k-1$, there exists a C^1 Lyapunov function V_{k-1}, and a set of virtual controllers defined for $i = 2, \cdots, k$,

$$\begin{aligned}x_1^* &= 0, & \xi_1 &= x_1^{r_1} - x_1^{*r_1},\\ x_i^* &= -\beta_{i-1}^{r_i}\xi_{i-1}^{r_i}, & \xi_i &= (x_i^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}})\end{aligned} \tag{5}$$

such that

$$\dot{V}_{k-1}|_{\Omega_{k-1}} \leq -(n-k+2)\sum_{i=1}^{k-1}\xi_i^2 + \xi_{k-1}^{2-r_k}(x_k - x_k^*), \tag{6}$$

where positive constants β_i's only involve the known upper and lower bounds, the level sets $\Omega_i \triangleq \{y \in \mathbb{R}^n | V_i(y_1, \cdots, y_i) \leq N\}$ satisfy $\Omega_{k-1} \subset \cdots \subset \Omega_1$.

In what follows, we claim that (6) also holds at step k. Construct the Lyapunov function $V_k = V_{k-1} + W_k$ with $W_k = \int_{x_k^*}^{x_k} (s^{\frac{1}{r_k}} - x_k^{*\frac{1}{r_k}})^{2-r_{k+1}} ds$ and the k-th level set $\Omega_k = \{y \in \mathbb{R}^n | V_k(y_1, \cdots, y_k) \leq N\}$. Obviously, $\Omega_k \subset \Omega_{k-1}$ holds. Then, the derivative of V_k can be calculated as

$$\begin{aligned}\dot{V}_k|_{\Omega_k} \leq& -(n-k+2)\sum_{i=1}^{k-1}\xi_i^2 + \xi_k^{2-r_{k+1}}x_{k+1}\\ &+ \xi_{k-1}^{2-r_k}(x_k - x_k^*) + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}x_{i+1}.\end{aligned} \tag{7}$$

Based on Lemmas A.1 and A.2, one has

$$\begin{aligned}\xi_k^{2-r_{k+1}}x_{k+1} \leq& 2^{1-r_k}|\xi_{k-1}|^{2-r_k}|\xi_k|^{r_k}\\ \leq& \frac{1}{2}\xi_1^2 + c_k\xi_k^2,\end{aligned} \tag{8}$$

where c_k is a positive constant dependent of the known bounds of q_k.

With the definition of ξ_i's, the estimate for the last term in the right-hand side of (7) is given as

$$\begin{aligned}&\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}x_{i+1}\\ =&(r_{k+1}-2)\sum_{i=1}^{k-1}\frac{\partial x_k^{*1/r_k}}{\partial x_i}x_{i+1}\int_{x_k^*}^{x_k}(s^{\frac{1}{r_k}} - x_k^{*\frac{1}{r_k}})^{1-r_{k+1}}ds\\ \leq& c|\xi_k|^{1-\tau_k}\sum_{i=1}^{k-1}(|\xi_i|^{1-r_i} + |\xi_{i-1}|^{1-r_i})\times(|\xi_i|^{r_{i+1}} + |\xi_{i+1}|^{r_{i+1}})\\ \leq& \frac{1}{2}\sum_{i=1}^{k-1}\xi_i^2 + h_k(y_1, \cdots, y_k)\xi_k^2,\end{aligned} \tag{9}$$

where the last relation holds owing to $\tau_1 \geq \cdots \geq \tau_k$, $c > 0$ is a constant and $h_k(\cdot)$ is a continuous function of $y_1, \cdots, y_k$.

Substituting (8)-(9) into (7), it can be concluded that

$$\begin{aligned}\dot{V}_k|_{\Omega_k} \leq &-(n-k+1)\sum_{i=1}^{k-1}\xi_i^2 + (c_k + h_k(\cdot))\xi_k^2 \\ &+ \xi_k^{2-r_{k+1}}(x_{k+1} - x_{k+1}^*) + \xi_k^{2-r_{k+1}}x_{k+1}^*. \quad (10)\end{aligned}$$

Since the continuous function $h_{k,1}(\cdot)$ is bounded on the set Ω_k, i.e. $h_k(\cdot) \leq \bar{h}_k$, the $k+1$th virtual controller designed as $x_{k+1}^* = -\beta_k^{r_{k+1}}\xi_k^{r_{k+1}}$, $\beta_k > (\bar{h}_k + c_k + n - k + 1)^{b_{k+1}}$, leads to that

$$\dot{V}_k|_{\Omega_k} \leq -(n-k+1)\sum_{i=1}^{k}\xi_i^2 + \xi_k^{2-r_{k+1}}(x_{k+1} - x_{k+1}^*). \quad (11)$$

This completes the inductive proof.

Last Step: Based on the mathematical induction, we can conclude that (11) holds for $k = n$. Therefore, there exists a linear controller

$$u = -\beta_n\big(y_n + \cdots + \beta_2(y_2 + \beta_1 y_1)\big) \quad (12)$$

such that $\dot{V}_n|_{\Omega_n} \leq -\sum_{i=1}^n \xi_i^2$, where $V_n = V_{n-1} + W_n = V_{n-1} + \int_{x_n^*}^{x_n}(s^{\frac{1}{r_n}} - x_n^{*\frac{1}{r_n}})^{2-r_{n+1}}ds$ and the level set $\Omega_n \triangleq \{y \in \mathbb{R}^n | V_n(y_1, \cdots, y_n) \leq N\}$ satisfying $\Omega_n \subset \cdots \subset \Omega_1$. Therefore, the closed-loop system (3)-(12) is globally asymptotically stable.

Then, the derivative of V_n along the trajectory of the system (1) is

$$\dot{V}_n|_{\Omega_n} \leq -\sum_{i=1}^{n}\xi_i^2 + \sum_{i=1}^{n-1}\frac{\partial V_n}{\partial x_i}\phi_i. \quad (13)$$

Based on Assumptions 2.1 and 2.2, one has

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{\partial V_n}{\partial x_i}\phi_i \leq &\sum_{i=1}^{n-1} g_i(\cdot)(|\xi_{i-1}|^{2-r_{i+1}} + \cdots + |\xi_n|^{2-r_{i+1}}) \\ &\times \big(\sum_{j=i+1}^{n}(|\xi_{j+1}|^{r_{i+1}q_i} + |\xi_j|^{r_{i+1}q_i}\big) \\ \leq &R(y)(\xi_i^2 + \cdots + \xi_n^2)\end{aligned}$$

with continuous functions $h(\cdot)$ and $R(\cdot)$. By appropriately selecting N, it can be concluded that $\forall y \in \Omega_n$, $R(y) < \frac{1}{2}$ holds. From (13), it is clear that

$$\dot{V}_n|_{\Omega_n} \leq -\frac{1}{2}\sum_{i=1}^{n}\xi_i^2, \quad (14)$$

which indicates that the closed-loop system (1)-(12) is locally asymptotically stable in the set Ω_n.

4. GLOBAL STABILIZATION FOR SYSTEM (1)

This section will further investigate the global stabilization problem for system (1). By combining the control law (12) and the saturation function $\sigma(s) = \{\begin{smallmatrix}\epsilon sign(s) & |s| > \epsilon \\ s & |s| \leq \epsilon\end{smallmatrix}$, a saturated controller is designed as

$$u = -\bar{\beta}_n\sigma\big(y_n + \cdots + \bar{\beta}_2\sigma(y_2 + \bar{\beta}_1\sigma(y_1))\big), \quad (15)$$

where $0 < \epsilon < 1$ is a constant to be determined later and the coefficients $\bar{\beta}_i > \beta_i$ are chosen to satisfy

$$\begin{aligned}&(i)(\frac{\bar{\beta}_1}{2})^{1/b_2} - 1 - 2^{1-1/b_2} > 0 \\ &(ii)(\frac{\bar{\beta}_{i+1}}{2})^{1/b_{i+2}} - 2^{1-1/a_{i+2}} - 1 - 4\alpha_i(\cdot)(1+\bar{\beta}_i)^{1/a_{i+1}} \\ &\quad > 0,\ \forall i = 2, \cdots, n-1 \\ &(iii)\bar{\beta}_n > 8\alpha_{n-1}(\cdot)(1+\bar{\beta}_{n-1})^{1/a_n} \quad (16)\end{aligned}$$

with $\alpha_1(\beta_1) = b_1\bar{\beta}_1\big((1+\bar{\beta}_1)^{1/a_2}+1\big)$ and $\alpha_i(\beta_1, \cdots, \beta_i) = b_i\bar{\beta}_i(1+\bar{\beta}_{i-1})^{1-1/b_i}\big((1+\bar{\beta}_i)^{1/a_{i+1}}+1\big) + \alpha_{i-1}(\cdot)$, $i = 2, \cdots, n-1$.

To begin with, the following lemma is introduced whose proof is similar to the one of Lemma 4.1 in **?** and therefore is omitted here.

Lemma 4.1. For the closed-loop system (1)-(15), there exists a constant $0 < \epsilon_1 < 1$ and $\alpha_i(\beta_1, \cdots, \beta_i)$, $i = 1, \cdots, n-1$ defined in (16), such that as long as $|y_j| \leq \epsilon(1+\bar{\beta}_{j-1})$, $j = i+1, \cdots, n$, $\forall 0 < \epsilon \leq \epsilon_1$, the following estimates hold

$$\begin{aligned}&(i)|\phi_i(\cdot)| \leq \epsilon^{1/q_{i+1}} = \epsilon^{r_{i+1}} \\ &(ii)|u_i(\bar{t}) - u_i(\underline{t})| \leq \epsilon^{1+\tau_{i+1}}\alpha_i(\cdot)(\bar{t} - \underline{t}),\quad \forall \bar{t} \geq \underline{t} \quad (17)\end{aligned}$$

with $u_i = -\bar{\beta}_i\sigma(y_i - u_{i-1})$.

Based on Lemma 4.1, we give the main result of this paper.

Theorem 4.1. Under Assumptions 2.1 and 2.2, the proposed controller (15) with an appropriately chosen $0 < \epsilon_1 \leq \epsilon$ is able to globally stabilize the upper-triangular nonlinear system (1).

Proof: In the previous section, the local stabilization result has been achieved for system (1) within Ω. If all the system states will enter and stay in the prescribed region Ω_n in a period of time, the global stabilization result can be achieved naturally.

In what follows, we first show that the saturated control (15) will make all the system states converge to a small region dependent of ϵ by using mathematical induction.

Initial Step: Firstly, it can be proved by contradiction that there exists $t_1 > 0$, such that

$$|y_n(t_1) - u_{n-1}(t_1)| \leq \frac{\epsilon}{2}. \quad (18)$$

Assume that $\forall t \geq 0$, $|y_n(t) - u_{n-1}(t)| > \frac{\epsilon}{2}$. With respect to the positive case, one has

$$\dot{x}_n(t) = -\bar{\beta}_n\sigma(y_n(t) - u_{n-1}(t)) < -\frac{\bar{\beta}_n}{2}\epsilon,$$

which implies

$$x_n(t) < x_n(0) - \frac{\bar{\beta}_n}{2}t, \quad \forall t > 0.$$

Due to the fact that $|u_{n-1}(t)| \leq \bar{\beta}_{n-1}\epsilon$, one can get $\forall t > 0$,

$$\frac{\epsilon}{2} - \bar{\beta}_{n-1}\epsilon < y_n(t) < (x_n(0) - \frac{\bar{\beta}_n}{2}\epsilon t)^{q_n}. \quad (19)$$

Since $q_n \in \mathbb{R}_{odd}^{\geq 1}$, as time goes to infinity, the term $(x_n(0) - \frac{\bar{\beta}_n}{2}\epsilon t)^{q_n}$ will go to negative infinity, which leads to a contradiction. In a similar way, the negative case can also be proved impossible. Therefore, it can be concluded that the inequality (18) holds.

Next, we will show that for all $t \geq t_1$, $|y_n(t) - u_{n-1}(t)| < \epsilon$. If this inequality does not hold, there are time instants $t_1' \in [t_1, +\infty)$ and $t_1'' \in [t_1, +\infty)$ such that

$$\begin{cases} |y_n(t_1') - u_{n-1}(t_1')| = \dfrac{\epsilon}{2}, \\ |y_n(t_1'') - u_{n-1}(t_1'')| = \epsilon, \\ \dfrac{\epsilon}{2} \leq |y_n(t) - u_{n-1}(t)| \leq \epsilon, \ \forall t \in [t_1', t_1''], \end{cases} \tag{20}$$

which includes both the positive and negative cases. With respective to the positive one, one has $\forall t \in [t_1', t_1'']$

$$\dot{x}_n(t) = -\bar{\beta}_n \sigma(y_n(t) - u_{n-1}(t)) < -\frac{\bar{\beta}_n}{2}\epsilon,$$

which leads to

$$x_n(t_1'') - x_n(t_1') \leq -\frac{\bar{\beta}_n}{2}(t_1'' - t_1'). \tag{21}$$

From (20), one can obtain

$$x_n(t_1') = (\frac{\epsilon}{2} + u_{n-1}(t_1'))^{1/q_n} \leq (1 + \bar{\beta}_{n-1})^{1/q_n}\epsilon^{1/q_n}, \tag{22}$$

$$x_n(t_1'') = (\epsilon + u_{n-1}(t_1''))^{1/q_n} \geq -(1 + \bar{\beta}_{n-1})^{1/q_n}\epsilon^{1/q_n}, \tag{23}$$

under which

$$\begin{aligned} t_1'' - t_1' <& \frac{2}{\bar{\beta}_n \epsilon}(x_n(t_1') - x_n(t_1'')) \\ <& \frac{4}{\bar{\beta}_n}(1 + \bar{\beta}_{n-1})^{1/q_n}\epsilon^{1/q_n - 1} \end{aligned} \tag{24}$$

Because $x_n(t_1'') \leq x_n(t_1')$, then $y_n(t_1'') \leq y_n(t_1')$ holds. Therefore, one has

$$\begin{aligned} y_n(t_1'') - u_{n-1}(t_1'') \leq& y_n(t_1') - u_{n-1}(t_1') \\ &+ u_{n-1}(t_1') - u_{n-1}(t_1''), \end{aligned} \tag{25}$$

which leads to

$$\frac{\epsilon}{2} \leq u_{n-1}(t_1') - u_{n-1}(t_1''). \tag{26}$$

On the other hand, due to the fact that $|u_{n-1}(t)| \leq \bar{\beta}_{n-1}\epsilon$, one has $|y_n(t)| \leq (1 + \bar{\beta}_{n-1})\epsilon$, $\forall t \in [t_1', t_1'']$, under which the following holds

$$|u_{n-1}(t_1'') - u_{n-1}(t_1')| \leq \epsilon^{2-r_n}\alpha_{n-1}(\cdot)(t_1'' - t_1') \tag{27}$$

according to Lemma 4.1. Combining (26) and (27) together, it yields

$$\frac{\epsilon}{2} \leq \frac{4}{\bar{\beta}_n}(1 + \bar{\beta}_{n-1})^{r_n}\alpha_{n-1}(\cdot)\epsilon. \tag{28}$$

With $\bar{\beta}_n$ chosen in (16), one has $\frac{\epsilon}{2} < \frac{\epsilon}{2}$. This contradiction shows that the positive case of (20) is impossible. In a similar way, it can be proved that the negative case will never happen neither. Hence, there is a time instant $t_1 > 0$ such that $\forall t \geq t_1$

$$Y_n(t) \in Q_n = \{Y_n(t) : |y_n(t) - u_{n-1}(t)| < \epsilon\}.$$

Inductive Step: Suppose that at step $i-1$, there exists $0 \leq t_1 \leq \cdots \leq t_{i-1}$, such that $\forall t \geq t_{i-1}$, $j = n-i+2, \cdots, n$, the following holds

$$Y_j(t) \in Q_j = \{Y_j(t) : |y_j(t) - u_{j-1}(t)| < \epsilon\}. \tag{29}$$

Then, we will prove that (29) also holds when $j = n-i+1$. Similar to the initial step, we first show that there is $t_i \geq t_{i-1}$, such that

$$|y_{n-i+1}(t_i) - u_{n-i}(t_i)| \leq \frac{\epsilon}{2}. \tag{30}$$

If this does not hold, it means that $|y_{n-i+1}(t) - u_{n-i}(t)| > \frac{\epsilon}{2}$, $\forall t \geq t_{i-1}$, which contains two cases, i.e., $y_{n-i+1}(t_i) - u_{n-i}(t_i) > \frac{\epsilon}{2}$ and $y_{n-i+1}(t_i) - u_{n-i}(t_i) < -\frac{\epsilon}{2}$. Consider the negative one, then $u_{n-i+1}(t) > \frac{\epsilon}{2}\bar{\beta}_{n-i+1}$. The derivative of x_{n-i+1} can be rewritten as

$$\begin{aligned} \dot{x}_{n-i+1} =& u_{n-i+1}^{r_{n-i+2}} + y_{n-i+2}^{r_{n-i+2}} - u_{n-i+1}^{r_{n-i+2}} + \phi_{n-i+1} \\ >& (\frac{\epsilon}{2}\bar{\beta}_{n-i+1})^{r_{n-i+2}} + y_{n-i+2}^{r_{n-i+2}} - u_{n-i+1}^{r_{n-i+2}} + \phi_{n-i+1}. \end{aligned} \tag{31}$$

By Lemma A.1, one has

$$\begin{aligned} &|y_{n-i+2}^{r_{n-i+2}} - u_{n-i+1}^{r_{n-i+2}}| \\ \leq& 2^{1-r_{n-i+2}}|y_{n-i+2} - u_{n-i+1}| \leq 2^{1-r_{n-i+2}}\epsilon^{2^{1-r_{n-i+2}}}. \end{aligned} \tag{32}$$

From $|y_j(t) - u_{j-1}(t)| < \epsilon$, $\forall j = n-i+2, \cdots, n$, we know that $|y_j| \leq (1 + \bar{\beta}_{j-1})\epsilon$ holds for any $Y_j \in Q_j$ and then by Lemma 4.1, one has $\phi_{n-i+1} \leq \epsilon^{r_{n-i+2}}$.

Thus, the equation (31) can be estimated as

$$\begin{aligned} \dot{x}_{n-i+1}(t) >& \Big((\frac{\bar{\beta}_{n-i+1}}{2})^{r_{n-i+2}} - 2^{1-r_{n-i+2}} - 1 \Big)\epsilon^{r_{n-i+2}} \\ \triangleq& \delta_{n-i+1}\epsilon^{r_{n-i+2}} > 0, \quad \forall t \geq t_{i-1} \end{aligned} \tag{33}$$

based on the selection of $\bar{\beta}_i$ in (16). It implies that $x_{n-i+1}(t) > x_{n-i+1}(t_{i-1}) + \delta_{n-i+1}\epsilon^{r_{n-i+2}}(t - t_{i-1})$, $\forall t \geq t_{i-1}$. With $u_{n-i} > -\bar{\beta}_{n-i}\epsilon$, one can reach that

$$\begin{aligned} &x_{n-i+1}(t_{i-1}) + \delta_{n-i+1}\epsilon^{r_{n-i+2}}(t - t_{i-1}) < x_{n-i+1}(t) \\ <& y_{n-i+1}(t) < -\frac{\epsilon}{2} + \bar{\beta}_{n-i}\epsilon, \quad \forall t \geq t_{i-1}. \end{aligned}$$

Analogous to (19), the above inequality will lead to a contradiction by letting t tend to infinity, which shows that the negative case will never happen. The same result for the positive case can be obtained easily referring to the one in the initial step. In conclusion, the inequality (30) holds.

In what follows, similar to the initial step, we will prove that

$$|y_{n-i+1}(t) - u_{n-i}(t)| < \epsilon, \quad \forall t \geq t_i$$

If it is not true, the following situation will occur:

$$\begin{cases} |y_{n-i+1}(t_i') - u_{n-i}(t_i')| = \dfrac{\epsilon}{2}, \\ |y_{n-i+1}(t_i'') - u_{n-i}(t_i'')| = \epsilon, \\ \dfrac{\epsilon}{2} \leq |y_{n-i+1}(t) - u_{n-i}(t)| \leq \epsilon, \ \forall t \in [t_i', t_i''] \end{cases} \tag{34}$$

with $t_i' \in [t_i, +\infty)$, $t_i' \in [t_i, +\infty)$. Then, we focus on the negative case. According to (33), one has $\dot{x}_{n-i+1}(t) > \delta_{n-i+1}\epsilon^{r_{n-i+2}}$, $\forall t \in [t_i', t_i'']$, which implies that

$$x_{n-i+1}(t_i'') - x_{n-i+1}(t_i') > \delta_{n-i+1}\epsilon^{r_{n-i+2}}(t_i'' - t_i'). \tag{35}$$

Moreover, with the definition of u_i's, we know that

$$x_{n-i+1}(t_1') \geq -(1 + \bar{\beta}_{n-i})^{r_{n-i+1}}\epsilon^{r_{n-i+1}}, \tag{36}$$

$$x_{n-i+1}(t_1'') \leq (1 + \bar{\beta}_{n-i})^{r_{n-i+1}}\epsilon^{r_{n-i+1}}, \tag{37}$$

under which

$$t_1'' - t_1' < \frac{2(1 + \bar{\beta}_{n-i})^{r_{n-i+2}}}{\mu_{n-i+1}}\epsilon^{-\tau_{n-i+1}}. \tag{38}$$

Since $x_{n-i+1}(t_i'') \geq x_{n-i+1}(t_i')$, then $y_{n-i+1}(t_i'') \geq y_{n-i+1}(t_i')$ holds. Therefore, one has

$$\begin{aligned} -\epsilon =& y_{n-i+1}(t_i'') - u_{n-i}(t_i'') \\ \geq& y_{n-i+1}(t_i') - u_{n-i}(t_i') + u_{n-i}(t_i') - u_{n-i}(t_i''), \end{aligned} \tag{39}$$

which leads to

$$-\frac{\epsilon}{2} \geq u_{n-i}(t_i') - u_{n-i}(t_i''). \tag{40}$$

Meanwhile, due to the fact that $\forall t \in [t_1', t_1'']$, $|y_j(t)| \leq (1+\bar{\beta}_{j-1})\epsilon$, $j = n-i+1, \cdots, n$, one has

$$|u_{n-i}(t_1'') - u_{n-i}(t_1')| \leq \epsilon^{1-r_{n-i+1}}\alpha_{n-i}(\cdot)(t_1'' - t_1') \tag{41}$$

according to Lemma 4.1. Substituting (41) into (40), it yields

$$-\frac{\epsilon}{2} \geq \frac{2(1+\bar{\beta}_{n-i})^{r_{n-i+1}}}{\mu_{n-i+1}}\alpha_{n-i}(\cdot)\epsilon. \tag{42}$$

Obviously, it is a contradiction with coefficients chosen in (16). The proof for the positive situation is similar to the negative one and therefore is omitted here.

The aforementioned arguments show that there exists a time instant $t_i \geq t_{i-1}$, such that $\forall t \geq t_i$

$$|y_j(t) - u_{j-1}(t)| \leq \epsilon, \quad j = n-i+1, \cdots, n.$$

Last Step: Based on the mathematical induction, we can conclude that there exists $t_{n-1} \geq t_{n-2} \geq \cdots \geq t_1 \geq 0$ such that $|y_j(t) - u_{j-1}(t)| < \epsilon$, $\forall t \geq t_{n-1}$, which leads to $|y_j(t)| < (1+\bar{\beta}_{j-1})\epsilon$, $j = 2, \cdots, n$. Therefore, $|\phi_1| \leq \epsilon^{r_2}$ holds and the derivative of x_1 arrives at

$$\dot{x}_1 = x_2 + \phi_1 = u_1^{r_2} + y_2^{r_2} - u_1^{r_2} + \phi_1.$$

Moreover, according to Lemmas A.1 and A.2, one can get

$$|y_2^{r_2} - u_1^{r_2}| \leq 2^{1-r_2}|y_2 - u_1|^{r_2} \leq 2^{1-r_2}\epsilon^{r_2}. \tag{43}$$

For the case that $y_1(t) > \frac{\epsilon}{2}$, $\forall t \geq t_{n-1}$, one has

$$\dot{x}_1(t) < -\Big((\frac{\bar{\beta}_1}{2})^{r_2} - 1 + 2^{1-r_2}\Big)\epsilon^{r_2} < 0,$$

while for $y_1(t) < -\frac{\epsilon}{2}$, $\forall t \geq t_{n-1}$, one gets

$$\dot{x}_1(t) > \Big((\frac{\bar{\beta}_1}{2})^{r_2} - 1 + 2^{1-r_2}\Big)\epsilon^{r_2} > 0$$

based on (16).

Thus, there is $t_n \geq t_{n-1}$, such that $|y_1(t)| \leq \frac{\epsilon}{2} < \epsilon$, $\forall t \geq t_n$. It can be obtained that after the time instant t_n, the system states will enter and stay in the region

$$\begin{aligned} Q = &\{Y_n : |y_1(t) < \epsilon|, \ |y_2(t) - u_1(t)| < \epsilon, \cdots, \\ &|y_n(t) - u_{n-1}(t)| < \epsilon\}. \end{aligned}$$

Note that the region Q is determined by the saturation level ϵ and the closed-loop system (1)-(12) is locally stable. Therefore, by appropriately tuning the parameter ϵ to guarantee that $Q \subset \Omega_n$, the saturated controller (15) will becomes the unsaturated one (12) after t_n and will stabilize system (1) in Q. Therefore, we can conclude the global stabilization result for system (1).

5. AN ILLUSTRATIVE EXAMPLE

In order to show the effectiveness of the control law proposed in Section 3, an numerical example is proposed. Consider the following upper-triangular nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.1x_3^4, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u \\ y_1 &= x_1^{q_1}, \quad y_2 = x_2, \quad y_3 = x_3 \end{aligned} \tag{44}$$

where q_1 is an unknown power satisfying $q_1 \in [1, 2]$. Clearly, with $a_1 = a_2 = a_3 = 1$ and $b_1 = 2, b_2 = b_3 = 1$, Assumptions 2.1 and 2.2 holds naturally. Therefore, according to Theorem 4.1, a saturated controller can be designed as

$$u = -8\sigma\big(y_3 + 5\sigma(y_2 + 2\sigma(y_1))\big) \tag{45}$$

with the saturation level $\epsilon = 0.2$.

In the simulation, we choose parameters $q_1 = \frac{5}{3} \in [1, 2]$ and the initial condition as $(x_1(0), x_2(0), x_3(0)) = (0.5, -1, -0.5)$. Fig 1 and Fig 2 give the simulation result. According to controller design procedure, we know that the designed controller is robust to the unknown powers. Therefore, another set of powers are selected as $q_1 = \frac{7}{5}$ in order to verify the conclusion. It is shown in Fig 3 that the system states still converge to the origin asymptotically by the controller (45), which is consistent with the theoretical analysis before.

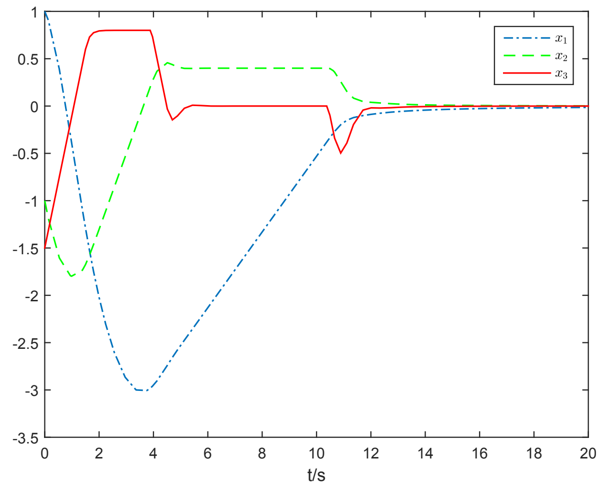

Fig. 1. The trajectories of $x(t)$ of the closed-loop system (44)-(45) with $q_1 = \frac{5}{3}$

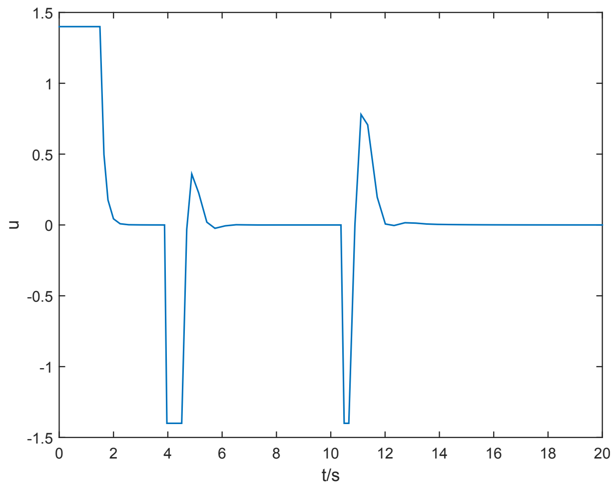

Fig. 2. The trajectory of $u(t)$ of the closed-loop system (44)-(45) with $q_1 = \frac{5}{3}$

6. CONCLUSION

This paper has proposed a saturated controller design procedure in the linear form based on the Lyapunov stability

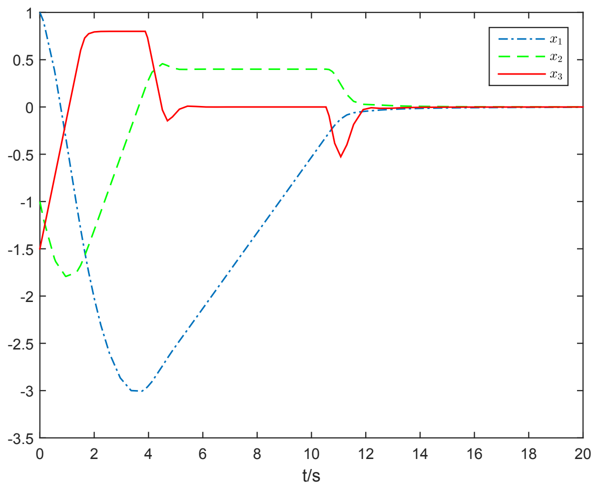

Fig. 3. The trajectories of $x(t)$ of the closed-loop system (44)-(45) with $q_1 = \frac{7}{5}$

theory. As long as the measurement function powers satisfy certain assumptions, the designed controller can globally stabilize the upper-triangular nonlinear systems. This is accomplished by generalizing the technique of nested saturation and the Lyapunov design method to the uncertain case. Moreover, the linear controller is easy implemented in real scenarios.

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REFERENCES

Chen, C., Qian, C., Lin, X., Sun, Z., and Liang, Y. (2017). Smooth output feedback stabilization for a class of nonlinear systems with time-varying powers. *International Journal of Robust & Nonlinear Control*.

Ding, S., Qian, C., and Li, S. (2009). Global stabilization of feedforward systems with lower-order vector field. In *48th IEEE Conference on Decision & Control*.

Gao, T., Liu, Y., Liu, L., and Li, D. (2018). Adaptive neural network-based control for a class of nonlinear purefeedback systes with time-varying full state constraints. *IEEE/CAA Journal of Automatica Sinica*, 5(05), 41–51.

Lei, H. and Lin, W. (2006). Universal adaptive control of nonlinear systems with unknown growth rate by output feedback. *Automatica*, 42(10), 1783–1789.

Mazenc, F. and Praly, L. (1996). Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE Transactions on Automatic Control*, 41(11), 1559–1578.

Qian, C. and Lin, W. (2012). Homogeneity with incremental degrees and global stabilisation of a class of high-order upper-triangular systems. *International Journal of Control*, 85(12), 1851–1864.

Su, Z., Qian, C., and Shen, J. (2017). Interval homogeneity-based control for a class of nonlinear systems with unknown power drifts. *IEEE Transactions on Automatic Control*, 62(3), 1445–1450.

SUN, Z. and LIU, Y. (2008). Adaptive stabilizing control for a class of high-order nonlinear uncertain systems. *Control Theory & Applications*, 25(1), 40–46.

Teel, A.R. (1996). A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Transactions on Automatic Control*, 41(9), 1256–1270.

Tian, W., Qian, C., and Du, H. (2014). A generalised homogeneous solution for global stabilisation of a class of non-smooth upper-triangular systems. *International Journal of Control*, 87(5), 951–963.

Yu, M., Ye, X., and Qi, D. (2011). Repetitive learning control for triangular systems with unknown control directions. *IET Control Theory & Applications*, 5(17), 2045–2051.

Zha, W., Qian, C., Zhai, J., and Fei, S. (2016). Robust control for a class of nonlinear systems with unknown measurement drifts. *Automatica*, 71, 33–37.

Zha, W., Qian, C., Zhai, J., and Fei, S. (2017). Robust control for a class of high-order uncertain nonlinear systems via measurement feedback. *International Journal of Control*, (4), 1–16.

Zha, W., Zhai, J., and Fei, S. (2018). Global control for a class of upper-triangular nonlinear systems with unknown powers. In *37th Chinese Control Conference*.

Appendix A. PRELIMINARY

In this section, we present some definitions of homogeneous system theory and some lemmas which play important roles in the controller design procedure.

Definition A.1. For a fixed choice of coordinates $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and positive real numbers $(r_1, \cdots, r_n) \triangleq r$, a one-parameter family of dilation is a map $\Delta_\epsilon^r : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Delta_\epsilon^r x = (\epsilon^{r_1} x_1, \cdots, \epsilon^{r_n} x_n)$, $\forall \epsilon > 0$ with r_i's being the weights of the coordinates.

Definition A.2. For a given dilation Δ_ϵ^r and a series of real monotone numbers $\tau_1 \geq \tau_2 \cdots \geq \tau_n$, a continuous vector field $f(x) = [f_1(x), \cdots, f_n(x)]^T$, $x \in \mathbb{R}^n$, is said to be homogeneous with monotone degrees (HWMD) $\tau_1, \cdots, \tau_n$, if $\forall x \in \mathbb{R}^n \backslash \{0\}$, $f_j(\Delta_\epsilon^r x) = \epsilon^{\tau_j + r_j} f_j(x)$, $j = 1, \cdots, n$.

When $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$, the definition of homogeneity with monotone degrees reduces to the traditional homogeneity with homogeneous degree τ.

Lemma A.1. For given $p \in \mathbb{R}_{odd}^{\geq 1}$ and any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, there holds

$$|x + y|^p \leq 2^{p-1} |x^p + y^p|,$$
$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} (|x| + |y|)^{\frac{1}{p}}.$$

If $p \geq 1$ and $p \in \mathbb{R}_{odd}^+$,

$$|x - y|^p \leq 2^{p-1} |x^p - y^p|,$$
$$|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{\frac{p-1}{p}} |x - y|^{\frac{1}{p}},$$
$$|x^p - y^p| \leq c|x - y| |(x - y)^{p-1} + y^{p-1}|$$

for a positive constant c.

Lemma A.2. For any positive real numbers c, d and any real-valued function $\gamma(x, y) > 0$, the following inequality holds:

$$|x|^c |y|^d \leq \frac{c}{c + d} \gamma(x, y) |x|^{c+d} + \frac{d}{c + d} \gamma^{-\frac{c}{d}}(x, y) |y|^{c+d}.$$