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Computing impulsive equilibrium sets with respect to target zone for linear impulsive systems

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Abstract: Linear impulsive control systems are convenient to formulate a venue of real-life problems, from diseases treatment to aerospace guidance. To take into account that the origin is the only formal equilibrium of such systems while usual control objective consist in stabilizing a desired target region that exclude it, a different concept of equilibrium – which includes periodic/orbital trajectories – has been defined. This work presents a new characterization of the equilibrium sets respecting this target region for linear impulsive systems. Based on the Lucasz-Markov theorem a tractable and non conservative description is obtained. Furthermore, to assess this description, the constrained equilibrium set is explicitly used in the formulation of a model-based controller.

Keywords: Impulsive control systems, sets characterization, target zone MPC

1. INTRODUCTION

In the last two decades, impulsive control systems have gained lot of interest from the control community. In fact, linear impulsive models arise in different application fields such as biomedical research (see González et al. [2017] and references therein), spacecraft rendezvous guidance and control (see Arantes Gilz et al. [2019] and references therein), for instance. Impulsive systems have been studied in the literature from different perspectives in terms of solutions existence and uniqueness and stability notions have been formally described (see Bainov and Simeonov [1995], Haddad et al. [2006]).

More recently, the Model Predictive Control community got involved in the control of such impulsive systems. In Pereira et al. [2015], a general framework for optimization-based predictive control is stated. The notions of invariance and stability are introduced, and some stability conditions are provided.

The case of linear impulsive systems has also drawn attention not only because of its importance in several application fields, but also from a theoretical perspective. In fact, linear impulsively controlled systems have some particularities with respect to classical linear control paradigm. Those particularities imply to adapt cardinal notions such as equilibrium and stability. For instance, the regulation of non-zero set-points is challenging in general: it is even claimed in Sopasakis et al. [2015] that it is not possible to stabilize any desired states of an impulsive system. In Sopasakis et al. [2015], invariance and stability have been redefined in order to design predictive controller. The proposed controller exploits polytopic invariant set that approximate the maximal invariant set. However, the computation of this polytopic invariant set still is an open problem in the general case. An intermediate approach that does not require invariant sets is proposed in Rivadeneira et al. [2015b] and uses the concept of zone control defined in González et al. [2009], Gonzalez and Odloak [2009]. In this framework, the state is steered to an equilibrium set instead of equilibrium point (that does not exist anywhere except at the zero point). The equilibrium point concept has been extended to the equilibrium set that consists of periodic controlled orbits included in a target zone Rivadeneira et al. [2018]). In Rivadeneira et al. [2018], the equilibrium set is clearly defined and its attractivity is established.
However, no formal description is provided and only approximating techniques are proposed in Sopasakis et al. [2015], Rivadeneira et al. [2018]. A first contribution of this paper is to answer the particular question of characterizing the constrained equilibrium set and to propose a tractable methodology to compute it. The free propagation of the states is first expressed as univariate polynomial thanks to the relevant change of variable. Then, the set of trajectories included in the target is characterized through the Linear Matrix Inequalities (LMI) conditions on the initial states. These conditions are based on the Lucasz-Markov theorem (see Nesterov [2000]) and they have been

inspired by the work Henrion et al. [2005]. To highlight the tractability of this description, a novel zone MPC algorithm is proposed, which follows the procedure exposed in Ferramosca et al. [2010], Gonzalez and Odloak [2009] but make an explicit use of the new set characterizations.

2. LINEAR IMPULSIVE SYSTEMS

Consider the impulsive control system (ICS)

$$\begin{cases} \dot{x}(t) & = Ax(t), \qquad\qquad t \neq \tau_k, \\ x(\tau_k^+) & = x(\tau_k) + Bu(\tau_k), \quad k \in \mathbb{N}, \end{cases}, \quad x(0) = x_0 \tag{ICS}$$

where $\tau_k = kT$, for a period time $T > 0$ and τ_k^+ denotes the time just after τ_k,

$$\tau_k^+ = \lim_{\substack{t \to \tau_k, \\ t > \tau_k}} t.$$

The output equation is also given by

$$y(t) = Cx(t). \tag{1}$$

Variables $x \in \mathcal{X} \subseteq \mathbb{R}^n$ represent the system states, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, the control inputs, and $y \in \mathbb{R}^p$, the system outputs.

Assumption 1. Set $\mathcal{X}$ is assumed to be closed and set $\mathcal{U}$, compact. Both, $\mathcal{X}$ and $\mathcal{U}$, are assumed to be convex and to contain the origin in their nonempty interior.

Roughly speaking, an system (ICS) evolves freely on any time interval $[\tau_k, \tau_{k+1}]$ and its history is described by

$$x(t) = \Phi(t)x(\tau_k^+), \text{ where } \Phi(t) = e^{At} \tag{2}$$

is the transition matrix. In addition, the system states trajectory shows discontinuities of the first order (so-called "jumps") enforced by the signal input at times τ_k. Clearly, the input signal is an impulse train:

$$u(t) = \sum_{k=0}^{\infty} u(\tau_k)\delta(t - \tau_k) \tag{3}$$

The control objective is twofold: (i) reach a given nonempty polytopic set $\mathcal{Z} \subseteq \mathcal{X}$, denoted as the target set, and (ii) maintain the system states inside $\mathcal{Z}$, indefinitely. In the context of linear impulsive system, periodic orbits to ensure that the system remains in a given set have been exploited in several work in the literature. In Arantes Gilz et al. [2019], the natural periodic orbits included in the target zone are stabilized by means of a predictive controller. In Rivadeneira et al. [2016], González et al. [2017], the equilibrium orbit notion is extended since the periodicity is forced by means of an admissible control input. The equilibrium set can be defined as follows (Rivadeneira et al. [2018]):

Definition 1. (Impulsive equilibrium set (IES)).
Consider the ICS (ICS). A nonempty convex set $\mathcal{X}_s \subset \mathcal{X}$ is an impulsive equilibrium set with respect to $\mathcal{Z}$ if for every $x \in \mathcal{X}$ exists $u \in \mathcal{U}$ such that (i) $\Phi(T)x + Bu = x$, and (ii) $\{\Phi(t)x,\ t \in [0, T]\} \in \mathcal{Z}$. Every single state x in $\mathcal{X}_s$ will be denoted as impulsive equilibrium state w.r.t. $\mathcal{Z}$.

From the latter definition, it is clear that $\mathcal{X}_s \subseteq \mathcal{Z}$.

3. IMPULSIVE EQUILIBRIUM SET WITH RESPECT TO A TARGET ZONE

The aim of this section is to characterize the IES, $\mathcal{X}_s$, in a tractable manner. Let x^+ denotes the state of system ICS (ICS) at times τ_k^+ (i.e., the states derived form the sampling of the system just after the discontinuity). Then, the following discrete-time system can be obtained

$$x^+(\tau_{k+1}) = \Phi(T)x^+(\tau_k) + Bu(\tau_{k+1}), \tag{4}$$

for $k \in \mathbb{N}$.

Assumption 2. The pair $[\Phi(T),\ B]$ is controllable.

Next two sets will be defined, the intersection of which allows us a proper computation of the IES, $\mathcal{X}_s$.
On one hand, the equilibrium set of the latter discrete-time system is, by definition, the set of states fulfilling condition (i) from in Definition 1 (Rivadeneira et al. [2015a], González et al. [2017]). That is,

$$\mathcal{X}_s^\circ = \{x \in \mathcal{X}\ :\ \exists u \in \mathcal{U} \text{ such that } (I_n - e^{AT})x = Bu\}, \tag{5}$$

is the set of states for which exists an admissible input u, such that the controlled trajectory describes a periodic orbit of period T.
On the other hand, the admissible initial set, with respect to $\mathcal{Z}$, $\mathcal{T}_\mathcal{Z}$ is defined to account for condition (ii) in the IES Definition 1. Particularly, the set $\mathcal{T}_\mathcal{Z}$ is the set of states from where the free propagated trajectories remain in $\mathcal{Z}$, over a complete T-period.

Definition 2. (Admissible initial set). Consider the system (ICS). The admissible initial set with respect to target set $\mathcal{Z}$ is given by:

$$\mathcal{T}_\mathcal{Z} = \{x \in \mathcal{X} : \Phi(t)x \in \mathcal{Z}, \quad t \in [0,\ T]\},$$

In Sopasakis et al. [2015], Rivadeneira et al. [2018], some techniques to approximate the set $\mathcal{T}_\mathcal{Z}$ are proposed. A contribution of this paper is to provide a formal description of this set, which is tractable enough to be used in the MPC algorithm.

Finally, $\mathcal{X}_s$, the IES w.r.t. $\mathcal{Z}$, can be defined as:

$$\mathcal{X}_s = \mathcal{X}_s^\circ \cap \mathcal{T}_\mathcal{Z}. \tag{6}$$

If the latter intersection is empty, then, it means that set $\mathcal{Z}$ is not well defined for the ICS (ICS). In the remainder of the paper, it is assumed that $\mathcal{X}_s$ is nonempty.

Admissible initial set characterization

Let $\mathcal{Z}$ be a polytopic subset described by its Cartesian coordinates $H \in \mathbb{R}^{l \times n}$ and $V \in \mathbb{R}^{l \times 1}$ (where l is a minimal number of hyperplanes necessary to describe $\mathcal{Z}$) such that

$$\mathcal{Z} = \{x \in \mathcal{X}\ :\ Hx \leq V\}.$$

In the sequel $\mathcal{T}_\mathcal{Z}$ is characterized in terms of linear matrix inequalities (LMI). First some assumptions are made on the flow dynamics of system (ICS).

Assumption 3. (i) Eigenvalues of the matrix A are rational numbers, $\lambda_s \in \mathbb{Q}$, with no imaginary part so that $\lambda_l = \frac{\eta_l}{\rho}$ where $\eta_l \in \mathbb{Z}$, $\rho \in \mathbb{N}$ and $l = 1, \ldots, n$. Eigenvalues are ranked in a increasing order such that η_1 is the largest negative integer and η_n is the largest positive integer; (ii) Eigenvalues of A are all distinct.

Remark 1. In the conclusion of the paper Henrion et al. [2005], arguments are provided to justify such assumptions. In particular, the cases where non-zero eigenvalue multiplicity is greater than 1 are isolated cases that are non robust to small perturbations. Consequently, this works will not account for this case.

Remark 2. The stability of the ICS flow is not required for the following development.

The admissible initial set is given by

$$\mathcal{T}_{\mathcal{Z}} = \{x(t) \in \mathcal{X} \ : \ H\Phi(t)x(t) \leq V, \ \ \forall t \in [0, \ T]\}. \quad (7)$$

Addressing the inequality from (7) row by row, it comes

$$h_i \sum_{j=1}^{n} \Phi_j x_j(t) \leq v_i, \quad i = 1, \dots, l \quad (8)$$

where h_i and v_i are respectively the i^{th} row and i^{th} singleton of H and V respectively. Φ_j is the j^{th} column of the transition matrix Φ. In the context described by the assumptions 3, each singleton of the transition matrix $\Phi(t)$ is given by

$$\Phi_{ij} = \sum_{r=1}^{n} \phi_r^{ij} e^{\lambda_r t}, \qquad \phi_r^{ij} \in \mathbb{R}$$

where λ_s are the eigenvalues of the dynamic matrix A. Then, inequalities (8) can be rewritten

$$\sum_{j=1}^{n} \sum_{r=1}^{n} \beta_r^{ij} e^{\lambda_r t} x_{0,j} \leq v_i, \quad t \in [0, \ T], \quad i = 1, \dots, l \quad (9)$$

with $\beta_r^{ij} \in \mathbb{R}$. Let us propose the change of independent variable:

$$w = e^{\frac{-1}{\rho} t}, \ \text{ so that } \ t \in [0, \ T] \Leftrightarrow w \in [W, \ 1] \quad (10)$$

where $W = e^{\frac{-T}{\rho}}$.

Exploiting the previous change of variable, inequalities (9) are equivalent to the following inequalities:

$$\sum_{r=1}^{n} \gamma_r^j(x_0, \phi, h_i) w^{-\eta_r} - v_i \leq 0, \quad w \in [W, \ 1], \quad i = 1, \dots, l \quad (11)$$

where $\gamma_r^j(\cdot)$ are linear function of the initial state x_0, the coefficients ϕ_s^{ij} of the transition matrix Φ collected in the vector ϕ and the vector h_i.

Since $\eta_s \in \mathbb{Z}$, (11) are rational inequalities, it comes that

$$\frac{\sum_{l=1}^{n} \gamma_s^j(x, \phi) w^{-\eta_s + \eta_n} - v_i w^{\eta_n}}{w^{\eta_n}} \leq 0, \quad w \in [W, \ 1]. \quad (12)$$

The monomial term w^{η_n} being positive on $[W, \ 1]$, inequality (12) is equivalent to a polynomial positivity problem with a given structure:

$$P(w) = \sum_{d=0}^{\bar{\eta}} \pi_d w^d \geq 0, \quad w \in [W, \ 1]. \quad (13)$$

where $\bar{\eta} = \eta_n - \eta_1$. If 0 does not belong to the spectrum of A $(0 \notin \{\eta_s\})$,

$$\text{where } \pi_d = \begin{cases} -\gamma_d(x, \phi) & \text{if } d \in \{-\eta_1 + \eta_n, \dots, \eta_n\} \\ v_i & \text{if } d = \eta_n \\ 0 & \text{else} \end{cases}.$$

If 0 belongs to the spectrum of A $(0 \in \{\eta_s\})$,

$$\text{where } \pi_d = \begin{cases} -\gamma_d(x, \phi) & \text{if } d \in \{-\eta_1 + \eta_n, \dots, 0\} \\ v_i - \gamma_j(x, \phi) & \text{if } d = \eta_n \text{ and } \eta_j = 0 \\ 0 & \text{else} \end{cases}.$$

Applying the Lucasz-Markov theorem (see Nesterov [2000]), inequalities (13) are satisfied if and only if $P(w)$ can be written as weighted sum of squares:

$$P(w) = \begin{cases} \langle q_1, u(t) \rangle^2 + (w - W)(1 - w) \langle q_2, v(t) \rangle^2, & \text{if } \bar{\eta} \text{ is even} \\ (w - W) \langle q_1, u(t) \rangle^2 + (1 - w) \langle q_2, u(t) \rangle^2, & \text{if } \bar{\eta} \text{ is odd} \end{cases}$$

where $u(t) = \{1, t, \dots, t^{\bar{\eta}/2}\}$, $v(t) = \{1, t, \dots, t^{\bar{\eta}/2-1}\}$, $q_1 \in \mathbb{R}^{\bar{\eta}/2}$ and $q_2 \in \mathbb{R}^{\bar{\eta}/2-1}$ or $q_2 \in \mathbb{R}^{\bar{\eta}/2}$ depending on $\bar{\eta}$ being even or odd. Equivalently, $P(w)$ will be non negative if its coefficient vector π is the image of two positive semi-definite matrices Y_1 and Y_2 through linear operator Λ_1^* and Λ_2^*:

$$\pi(x) = \Lambda_1^*(Y_1) + \Lambda_2^*(Y_2), \quad Y_1, Y_2 \succeq 0. \quad (14)$$

The operators Λ_1^* and Λ_2^* are defined in Nesterov [2000] and conveniently described in terms of Hankel matrices (see [Arantes Gilz 2018, Appendix B] for a detailed example). Thus, $\mathcal{T}_{\mathcal{Z}}$ is described as a semi-algebraic set as the initial condition x needs to satisfy the LMI condition (14):

$$\mathcal{T}_{\mathcal{Z}} = \{x \in \mathcal{X} \ : \ \pi(x) = \Lambda_1^*(Y_1) + \Lambda_2^*(Y_2), \ Y_1, Y_2 \succeq 0\} \quad (15)$$

As $\mathcal{T}_{\mathcal{Z}}$ is independent from the input variable u, this set can be extended such that

$$\mathcal{T}_{\mathcal{Z}}^u = \{(x, u) \in \mathcal{X} \times \mathbb{R}^m \ : \ x \in \mathcal{T}_{\mathcal{Z}} \text{ and } u \in \mathcal{U}\} \quad (16)$$

In this section, the set $\mathcal{X}_s$ has been described as a slice of an spectrahedron as the intersection of a set given by linear equalities and a set defined by LMI conditions. If it exists, this set has the property to be non empty and convex. It is also tractable enough to be included as constraint in a convex program that can be solved efficiently with interior-point method solver. The next section takes advantage of it to design a model predictive controller.

4. ZONE MPC CONTROL

This section is devoted to the description of a zone model predictive controller. Its main goal is to steer the ICS to the target zone $\mathcal{Z}$, that may exclude the origin, and to make it hover this specific zone. In other words, the MPC must be able to stabilize the IES with respect to $\mathcal{Z}$. The idea is to use the discrete-time model (4) for predictions, and to use as a target set the set $\mathcal{T}_{\mathcal{Z}}$ (instead of the discrete-time equilibrium $\mathcal{X}_s^\circ$, since it does not fulfill the conditions in Definition 1).

The cost function to be minimized on-line by the MPC is given by

$$J_N(x, \mathbf{u}, x_s, u_s) = \sum_{j=0}^{N-1} \alpha \| x^+(\tau_j) - x_s \|_2 + \beta \| u(\tau_j) - u_s \|_2 + \gamma \text{dist}_{\mathcal{T}_{\mathcal{Z}}}(x_s) \quad (17)$$

where $x = x(\tau_0)$ represents the current state, $\mathbf{u} = \{u(\tau_0), u(\tau_1), \dots, u(\tau_{N-1})\}$ is the predicted sequence of inputs, x_s and u_s are additional optimization equilibrium variables fulfilling $\phi(T)x_s + Bu_s = x_s$, and α, β and γ are penalization scalars.

Remark 3. The distance term $\text{dist}_{\mathcal{T}_\mathcal{Z}}(x_s)$ is understood as the minimal 2-norm distance between any state point x_s and a state point x^* that belong to $\mathcal{T}_\mathcal{Z}$. $\text{dist}_{\mathcal{T}_\mathcal{Z}}(x_s) = \min_{x^* \in \mathcal{T}_\mathcal{Z}} \|x_s - x^*\|$ is evaluated by using an additional optimization variable x^*, which is forced to be in $\mathcal{T}_\mathcal{Z}$. Addition, the term $\|x_s - x^*\|_2$ replaces $\text{dist}_{\mathcal{T}_\mathcal{Z}}(x_s)$ in the cost function such that

$$J_N(x, \mathbf{u}, x_s, u_s) = \sum_{j=0}^{N-1} \alpha \|x^+(\tau_j) - x_s\|_2 + \beta \|u(\tau_j) - u_s\|_2 + \gamma \|x_s - x^*\|_2, \quad (18)$$

and constraint $x^* \in \mathcal{T}_\mathcal{Z}$ is included as an additional constraint in the optimization (19), introduced next.

The optimization problem to be solved at each time τ_k is as follows:

$$\begin{aligned} &\min_{\mathbf{u}, x_s, u_s} J_N(x, \mathbf{u}, x_s, u_s) \\ &\text{s.t.} \begin{cases} x^+(\tau_0) = x + Bu(\tau_0) \\ x^+(\tau_{j+1}) = \phi(T)x^+(\tau_j) + Bu(\tau_{j+1}), & j \in \mathbb{N}_{N-1} \\ x^+(\tau_j) \in \mathcal{X}, \ u(\tau_j) \in \mathcal{U}, & j \in \mathbb{N}_{N-1} \\ x^+(\tau_N) = x_s, \\ x^* \in \mathcal{T}_\mathcal{Z}, \\ \phi(T)x_s + Bu_s = x_s, & (x_s \in \mathcal{X}_s^\circ). \end{cases} \end{aligned} \quad (19)$$

In the latter optimization problem, x is the optimization parameter while $\mathbf{u}, x_s, u_s$ are the optimization variables. The terminal constraint $x(\tau_N) = x_s$ forces the last state on the control horizon to reach the artificial equilibrium state x_s, to ensure stability. The constraint $x^* \in \mathcal{T}_\mathcal{Z}$ is included as constraint accordingly with Remark 3. The last constraint forces the additional variables (x_s, u_s) to be an equilibrium pair of the sampled system (4). This two constraints determine the region of feasibility of the optimization problem, which is given by the controllable set, in N-step, to $\mathcal{X}_s^\circ$:

$$\mathcal{C}_N(\mathcal{X}_s^\circ) = \{x^+ \in \mathcal{X} : x^+(\tau_j) \in \mathcal{X}, u(\tau_j) \in \mathcal{U}, \ j \in \mathbb{N}_{N-1}, \ x^\circ(\tau_N) \in \mathcal{X}_s^\circ\}, \quad (20)$$

where $x^+(\tau_{j+1}) = \Phi(T)x^+(\tau_j) + Bu(\tau_j)$, $x = x(\tau_0)$.

Remark 4. Note that the cost function (18) is a quadratic function, the equality constraints in (19) are linear as well as the membership conditions to $\mathcal{X}$ and $\mathcal{U}$. In addition, the $\mathcal{T}_\mathcal{Z}$ membership condition involves a LMI constraints as $\mathcal{T}_\mathcal{Z}$ is described by (15). Consequently, Problem (19) will be practically solved by means of SDP solver.

Once the MPC problem is solved at time τ_k, the optimal solution is given by the optimal input sequence

$$\mathbf{u^0}(x) = \{u^0(x, \tau_0), u^0(x, \tau_1), \ldots, u^0(x, \tau_{N-1})\}, \quad (21)$$

and the optimal additional variables $(x_s^0(x), u_s^0(x))$, while the optimal cost is denoted as $J_N^0(x) := J_N(x, \mathbf{u}^0, x_s^0, u_s^0)$. The control law, derived from the application of a receding horizon control policy (RHC), is given by $u(\tau_k) = \kappa_{MPC}(x(\tau_k)) = u^0(x, \tau_0)$, where $u^0(x, \tau_0)$ is the first control action in $\mathbf{u^0}(x)$.

The stabilizing properties of the controller are summarized in the following Property.

Property 1. The IES $\mathcal{X}_s$, w.r.t. $\mathcal{Z}$, is asymptotically stable for the closed-loop system ICS,

$$\begin{cases} \dot{x}(t) & = Ax(t), \quad t \neq \tau_k, \\ x(\tau_k^+) & = x(\tau_k) + B\kappa_{MPC}(x(\tau_k)), \qquad k \in \mathbb{N}, \end{cases} \quad (22)$$

with $x(0) = x_0$, and a domain of attraction given by $\mathcal{C}_N(\mathcal{X}_s^\circ)$.

Sketch of Proof: The proof follows the steps of the so called MPC for tracking, Ferramosca et al. [2010], Rivadeneira et al. [2018]. The time of the closed-loop is denoted by τ_k, as in (22), while the time for predictions, inside each optimization problem, is denoted by τ_j.

The recursive feasibility of the sequence of optimization problems follows from the fact that, for every $x \in \mathcal{C}_N(\mathcal{X}_s^\circ)$, the terminal constraint forces the system $x^\circ(\tau_{j+1}) = \phi(T)x^\circ(\tau_j) + Bu(\tau_j)$ to reach an invariant set (for instance the equilibrium set, $\mathcal{X}_s^0$), at the end of the control horizon. So, if the solution of the optimization problem for x, at time τ_k, is given by $\mathbf{u^0}(x)$, $x_s^0(x)$ and $u_s^0(x)$, then a feasible solution for the state x^+, at time τ_{k+1}, can be computed as $\tilde{\mathbf{u}}(x^+) = \{u^0(x;\tau_1), u^0(\tau_2), \ldots, u^0(\tau_{N-2}), \tilde{u}_s(x^+)\}$, $\tilde{x}_s(x^+) = x_s(x)$ and $\tilde{u}_s(x^+) = u_s(x)$. This feasible solution produces a feasible sequence of states, given by $\tilde{\mathbf{x}}(x^+) = \{x^0(x, \tau_1), x^0(x, \tau_2), \ldots, \tilde{x}_s(x^+), \tilde{x}_s(x^+)\}$.

The attractivity of $\mathcal{X}_s$ follows from the fact that, $\tilde{J}_N(x^+) \leq J_N^0(x) - \alpha\|x - x_s\|_2 - \beta\|u - u_s\|_2$, where $\tilde{J}_N(x^+) = J_N(\tilde{\mathbf{u}}(x^+), \tilde{x}_s(x^+), \tilde{u}_s(x^+))$ and u is the input injected to the system at time τ_k. Then, by optimality, it is $J_N^0(x^+) \leq \tilde{J}_N(x^+)$, which means that $J_N^0(\cdot)$ is a strictly decreasing positive function - i.e., $J_N^0(x^+) \leq J_N^0(x)$ - that only stops to decrease if $x = x(\tau_k) = x_s$ and $u(\tau_k) = u_s$. Furthermore, the fact that $x(\tau_k) \to x_s$ and $u(\tau_k) \to u_s$, as $k \to \infty$, implies that $x(\tau_k)$ tends also to $\mathcal{T}_\mathcal{Z}$ (by the effect of the cost term $\text{dist}_{\mathcal{T}_\mathcal{Z}}(x_s)$, as stated in Lemmas 1, 2 and 3, in Rivadeneira et al. [2018]). This way, $x(\tau_k)$ tends to the intersections of $\mathcal{X}_s^\circ$ and $\mathcal{T}_\mathcal{Z}$, which, by (6), represents the IES $\mathcal{X}_s$.

Remark 5. Note that it is not necessary to express explicitly the intersection (6) to formulate the MPC. In fact, such intersection is implicit in the controller formulation, by means of the additional variables (x_s, u_s) - that are forced to be in $\mathcal{X}_s^\circ$ -, and the cost term $\text{dist}_{\mathcal{T}_\mathcal{Z}}(x_s)$ which steers the states after the discontinuities, $x^+(\tau_k)$, to $\mathcal{T}_\mathcal{Z}$.

Remark 6. In Rivadeneira et al. [2018] a target set $\mathcal{X}_s^\circ$, already accounting for the properties of $\mathcal{X}_s$ w.r.t. $\mathcal{Z}$ (i.e., accounting for $\{\Phi(t)x_s, \ t \in [0, T]\} \in \mathcal{Z}$), needs to be outer-approximated by a polyhedron, and then explicitly used in the controller formulation. In contrast, the proposed MPC steers the system to the exact set $\mathcal{X}_s$, without the need of explicitly compute it.

Remark 7. Another benefits of the proposed MPC to be emphasized is that it steers the system to an equilibrium region that fulfill continuous-time constraints by only considering a sampled discrete-time system, as it is $x^\circ(\tau_{j+1}) = \phi(T)x^\circ(\tau_j) + Bu(\tau_j)$.

5. NUMERICAL STUDIES

Let us consider a linear time-invariant and impulsively controlled system described by the state space model ICS where

$$A = \begin{bmatrix} \frac{1}{2} & -2 \\ 0 & -\frac{3}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad C = [1\ 1]. \tag{23}$$

The eigenvalues of A are $\frac{1}{2}$ and $-\frac{3}{2}$ so that the flow dynamics of the impulsive control system is unstable and diverges naturally from the target set $\mathcal{Z}$. The control period T is set to 1 second. The admissible input control set, $\mathcal{U}$, is given by the interval $[-1,\ 1]$. The aim of the control is to stabilize and maintain the output y in the target interval $[0.5,\ 3]$. Consequently, the impulsive control system has to be controlled inside the target zone $\mathcal{Z}$, given by:

$$\mathcal{Z} = \{x \in \mathbb{R}^2 \mid Hx \leq V\}, \tag{24}$$

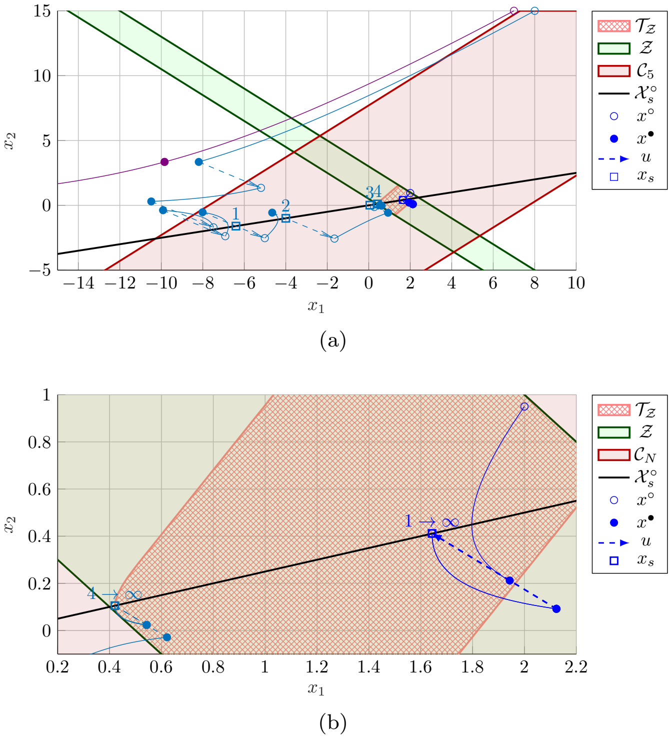
where

$$H = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ -0.5 \end{bmatrix}. \tag{25}$$

The MPC tuning parameters are given by $N = 5$, $\alpha = 1$, $\beta = 1$ and $\delta = 100$. Simulations have been conducted using MATLAB, Yalmip (Löfberg [2004]) and Mosek Solver (Andersen and Andersen [2000]).

Figures 1 expose the different sets described in the previous sections such as the target set $\mathcal{Z}$, the admissible initial set with respect to $\mathcal{Z}$, $\mathcal{T}_{\mathcal{Z}}$, and the controllable set to $\mathcal{X}_s^\circ$, $\mathcal{C}_N$, which represents the domain of attraction of the MPC. The set $\mathcal{X}_s^\circ$ is a straight line as the projection onto the (x_1, x_2) space of the intersection of two planes in the (x_1, x_2, u) space (see definition (5)). It should be noted that the initial admissible set can not be a polytope, by definition. Actually, $\mathcal{T}_{\mathcal{Z}}$ is spectrahedon, and it can be seen that its western corner is in fact rounded (see Figure 1b).

Three different initial conditions have been tested to assess the efficiency and the limits of the predictive controller. First, the initial state $x_0 = [7,\ 15]^T$ can not be steered to the target zone as it is just outside of the controller domain of attraction, $\mathcal{C}_N$. In this case, the optimization problem (19) is infeasible, given that the terminal constraint cannot be fulfilled. In general, an enlarged set $\mathcal{C}_N$ can be obtained for larger values of the control horizon N. However, it should be considered that a constrained, open-loop unstable system has a bounded maximal stabilizable set, outside of which no controller can stabilize any equilibrium point. The other trajectories start at the initial states that are in $\mathcal{C}_N$, respectively $x_0 = [8,\ 15]^T$ and $x_0 = [2,\ 0.9]^T$. As it can be seen on figures 1, both trajectories do converge to an impulsive equilibrium point in the IES $\mathcal{X}_s$.
For the initial state $x_0 = [8,\ 15]^T$, the controller steers the system to the boundary of the IES, from the left, given that according to the selected cost term $\text{dist}_{\mathcal{T}_{\mathcal{Z}}}(x_s)$, this is the cheapest way to approach and reach $\mathcal{X}_s$ (recall that $\mathcal{X}_s = \mathcal{T}_{\mathcal{Z}} \cap \mathcal{X}_s^\circ$). Figure 1a shows that the optimal additional variables x_s converges step-by-step to $\mathcal{T}_{\mathcal{Z}}$ along the sampled discrete-time system equilibrium set $\mathcal{X}_s^\circ$. Moreover, It reaches a particular state x_s characterized by a periodic orbit with the smallest value of u_s (see Figures 1a). The last initial point is located in $\mathcal{T}_{\mathcal{Z}}$. From this point the system converges to the closest point in the IES, with respect its impulsive dynamics and the norm used in $\text{dist}_{\mathcal{T}_{\mathcal{Z}}}(x_s)$).



(a)

(b)

Fig. 1. ICS controlled trajectories for different initial states

6. CONCLUSIONS

The formal description of the generalized equilibrium set with respect to a target set is proposed for impulsive control system. The main advantage of such description is that it enables a tractable computation that can be embedded in a convex program. Taking advantage of such properties a zone model predictive controller has been implemented to steer the impulsive control system to a given target zone. Once reached, the target zone is hovered with the certification that the equilibrium orbits remains inside it on the time continuum. This approach has been developed assuming that the flow dynamics of the impulsive control system is characterized by real eigenvalues with no multiplicity. If the need to account for multiplicities of the eigenvalue is arguable, complex eigenvalues can be easily met in the real applications. This fact clearly highlights the need for further investigation to enhance the proposition made in this work. Another axis of research is to evaluate the region of attraction of the impulsive equilibrium set taking advantage of its formal characterization.

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