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Analysis of Singular Perturbations for a Class of Interconnected Homogeneous Systems: Input-to-State Stability Approach ⋆

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Abstract: In this work an interconnection of two singularly perturbed homogeneous systems of different degrees is considered. Under relaxed restrictions on the smoothness of the right-hand sides of the system, and some standard assumptions, the conditions of local or practical asymptotic stability of the interconnection are established by means of ISS properties and the Small-Gain Theorem. Moreover, the domains of stability and attractions are estimated. Finally, the results are illustrated through an example with a homogeneous system of negative degree.

Keywords: Stability of Nonlinear Systems; Singular Perturbations; Input-to-State Stability; Homogeneity; Lyapunov Methods.

1. INTRODUCTION

Homogeneous systems (Bacciotti and Rosier, 2005; Zubov, 1964) constitute a subclass of nonlinear dynamics and admit interesting properties, e.g., scalability of solutions, robustness, different rates of convergence, etc. A very important feature of homogeneous controllers with a negative degree is that they are not Lipschitz, possessing an infinite gain near to the origin and providing finite-time convergence (Bhat and Bernstein, 1997; Levant, 2005; Cruz-Zavala and Moreno, 2017)

Commonly, control systems are affected by parasitic dynamics. Such interactions can be seen as the interconnection of systems with different scales of time, which are represented by singularly perturbed models. For smooth singularly perturbed systems, methods of stability analysis are based on Klimushchev-Krasvoskii Theorem (Klimushchev and Krasovskii, 1961) where asymptotic stability of the interconnection is concluded from uniform exponential stability of the slow and fast dynamics. Relaxing the last assumption, (Saberi and Khalil, 1984) addresses asymptotic and exponential stability by means of quadratic-type Lyapunov functions (LF) and estimates the domain of attraction with the upper bound of the perturbation parameter. Moreover, the problems of controller design for singularly perturbed systems have been subject of a wide interest (see the list of references in the book by (Kokotovic et al., 1999)). It is clear that these methods cannot be used for analysis of homogeneous systems in general case.

On the other hand, for systems affected by external inputs (e.g., exogenous disturbances, measurement noise, etc.), the concept of *Input-to-State Stability* (ISS) was introduced by (Sontag, 1989). The connexion between Lyapunov stability and ISS has permitted the development of many stability concepts very useful for the analysis and design of nonlinear control systems (see the list of references in (Dashkovskiy et al., 2011)). For instance, the so-called *Small-Gain Theorem* provides a sufficient condition to warranty the stability of interconnected systems (Jiang et al., 1994, 1996; Dashkovskiy and Kosmykov, 2013). In this context, (Christofides and Teel, 1996) studies ISS properties of smooth singularly perturbed systems providing a kind of "total stability" under standard assumptions.

To the best of our knowledge, existent works about stability analysis of singularly perturbed systems assume enough smoothness of the vector fields, which is a strong hypothesis for homogeneous systems of negative degree. The aim of this paper is to analyze the effect of singular perturbations on the stability of the interconnection of homogeneous systems by means of ISS properties and the Small-Gain Theorem. Unlike previous works, our analysis permits to relax smoothness requirements till assuming only continuity and at most continuous differentiability. Moreover, estimations of the region of attraction and ultimate bounds for the systems trajectories are provided.

The rest of the paper has the following structure. In Section 2 some useful definitions and preparatory results are presented.

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The problem statements and the main results of the paper with the stability proof are given in Section 3 and 4, respectively. In Section 5, the obtained results are illustrated by an example, where a homogeneous system with a negative degree is considered. Finally, the conclusions are presented in Section 6.

Notation

- $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.
- Expressions of the form $|\bullet|^\gamma sign(\bullet), \gamma \in \mathbb{R}$ are written as $\lceil\bullet\rfloor^\gamma$.
- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.
- The space $\mathcal{L}_\infty^m$ is defined as the set of piecewise continuous, bounded functions $u : [0, \infty) \to \mathbb{R}^m$ such that
$$\|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} |u(t)| < \infty.$$

2. PRELIMINARIES

Consider the system

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is an input. In addition, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence and uniqueness of the system solutions at least locally in time and $f(0, 0) = 0$.

2.1 Weighted homogeneity

The presentation of this subsection follows by (Zubov, 1964; Bacciotti and Rosier, 2005; Bernuau et al., 2013). For real numbers $r_i > 0 \quad (i = 1, \dots, n)$ called weights and $\lambda > 0$, one can define

- the vector of weights $r = (r_1, \dots, r_n)^T$, $r_{\max} = \max_{1\leq j\leq n} r_j$ and $r_{\min} = \min_{1\leq j\leq n} r_j$;
- the dilation matrix function $\Lambda_r(\lambda) = \text{diag}(\lambda^{r_i})_{i=1}^n$, such that, for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$, $\Lambda_r(\lambda)x = (\lambda^{r_1}x_1, \dots, \lambda^{r_i}x_i, \dots, \lambda^{r_n}x_n)^T$ (along the paper the dilation matrix is represented by Λ_r wherever λ can be omitted);
- the r-homogeneous norm of $x \in \mathbb{R}^n$ is given by $||x||_r = \left(\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}}\right)^{\frac{1}{\rho}}$ for $\rho \geq r_{\max}$ (it is not a norm in the usual sense, since it does not satisfied the triangle inequality);
- for $s > 0$, the sphere and the ball in the homogeneous norm are defined as $S_r(s) = \{x \in \mathbb{R}^n : ||x||_r = s\}$ and $B_r(s) = \{x \in \mathbb{R}^n : ||x||_r \leq s\}$, respectively.

Definition 1. A function $g : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous with a degree $\mu \in \mathbb{R}$ if for all $\lambda > 0$ and all $x \in \mathbb{R}^n$:
$$\lambda^{-\mu} g(\Lambda_r(\lambda)x) = g(x).$$
A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous with a degree $\nu \geq -r_{\min}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$,
$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x).$$
The system (1) (with $u = 0$) is r-homogeneous of degree ν if the vector filed f is r-homogeneous of degree ν.

Following (Efimov et al., 2018), by its definition, $||\cdot||_r$ is a r-homogeneous function of degree 1, and there exists $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}_\infty$ such that
$$\underline{\sigma}(||x||_r) \leq ||x|| \leq \bar{\sigma}(||x||_r) \qquad \forall x \in \mathbb{R}^n, \tag{2}$$
i.e., there is an equivalence between the norms $||\cdot||$ and $||\cdot||_r$. Moreover, for $r_{max} = 1$ and $\rho \geq r_{max}$, $||\cdot||_r$ is locally Lipschitz continuous.

2.2 Input-to-state stability

The next definitions and theorems were introduced by (Bernuau et al., 2013; Dashkovskiy et al., 2011; Jiang et al., 1996).

Definition 2. The system (1) is said to be input-to-state practically stable (ISpS), if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and a constant $c \geq 0$, such that, for any $u \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$, the solution $x(t)$ with initial condition $x(0) = x_0$ satisfies
$$\|x(t)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|u(t)\|), c\} \tag{3}$$
for all $t \geq 0$. The function γ is called nonlinear asymptotic gain. The system (1) is called ISS if $c = 0$.

Definition 3. A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called the ISpS-LF for system (1) if there exist some $c \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$, such that, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$
$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{4}$$
and
$$\|x\| \geq \chi(\max\{\|u\|, c\}) \implies \tfrac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x\|). \tag{5}$$
hold. Moreover, if $c = 0$ then V is called a ISS-LF for system (1).

Remark 4. The function $\gamma(\cdot)$ in (3) can be computed from the functions $\alpha_1(\cdot), \alpha_2(\cdot)$ and $\chi(\cdot)$. It is given by
$$\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \chi(r). \tag{6}$$
Theorem 5. The system (1) is ISS (ISpS) iff it admits an ISS (ISpS) LF.

2.3 Input-to-state stability of interconnected systems

Consider the system

$$\dot{x} = f(x, y), \tag{7}$$
$$\dot{y} = g(x, y, u), \tag{8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$ and the origin $x = y = u = 0$ is an equilibrium point. The system (7)-(8) can be seen as two interconnected subsystems where y is an input to the system (7) and x, u is an input to the system (8). Assume that both systems are ISpS w.r.t. their corresponding inputs. Therefore, from condition (3) we have

$$\|x(t)\| \leq \max\{\beta_1(\|x_0\|, t), \gamma_1(\|y(t)\|), c_1\}$$
$$\|y(t)\| \leq \max\{\beta_2(\|y_0\|, t), \gamma_2(\|x(t)\|), \gamma_3(\|u(t)\|), c_2\}$$

where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ are the initial conditions for each variable, $u \in \mathcal{L}_\infty$, $\beta_1, \beta_2 \in \mathcal{KL}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}$ are some functions from the indicated classes.

Sufficient conditions for ISpS stability of the interconnected system (7)-(8) can be found in (Dashkovskiy et al., 2011) as follows.

Theorem 6. Let each subsystem (7) and (8) be ISpS. If there exists some $c_0 \geq 0$ such that

$$\gamma_1(\gamma_2(r)) < r, \quad \text{for all} \quad r > c_0. \tag{9}$$

then the interconnected system (7)-(8) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$ then the system is ISS.

The inequality (9) is commonly refered as the *small-gain condition*. In particular, if the nominal system (7)-(8) is ISS then its solutions with $u = 0$ are globally asymptotically stable.

Roughly speaking, the Small-Gain Theorem establishes that the interconnected system (7)-(8) is ISS, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction.

3. PROBLEM STATEMENT

Consider the interconnected system

$$\dot{x} = f(x, y), \tag{10}$$

$$\epsilon\dot{y} = g(x, y) = A(x)y + R(x), \tag{11}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the state variables, and $\epsilon > 0$ is a small parameter, $f \in C^0$, $A \in C^1$ with $\det(A(x)) \neq 0$ for all $x \in \mathbb{R}^n$ and $R \in C^1$ with $R(0) = 0$ and $R(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, the system (10) is r-homogeneous with a degree ν, while the system (11) is $\tilde{r}$-homogeneous with a degree μ for the corresponding vector of weights r and $\tilde{r}$. Under the introduced restrictions on A and R, the equation $g(x, h(x)) = 0$ admits a solution $h(x) = -A^{-1}(x)R(x)$, which is locally Lipschitz continuous.

If we consider a small parameter $\epsilon \approx 0$ then the trajectories y of the system (11) remain in a neighborhood of the solution $\bar{y} = h(x)$ as it is predicted by the Tikhonov's theorem (Vasil'eva et al., 1995). However, in most of the existence results smoothness (at least continuous differentiability) of the vector fields of the system (10)-(11) is required.

In following section, two different problems will be addressed:

- Under what conditions the stability of the interconnected system (10)-(11) can be ensured.
- Moreover, is it possible to estimate the region of convergence and domain of attraction for the system solutions using their homogeneity?

4. MAIN RESULTS

The next theorem presents the main result of this paper and it establishes the conditions ensuring stability of the interconnected system (10)-(11). First, let us introduce the hypotheses under which we propose a solution.

Assumption 7. For the system (10)-(11):

A. The system
$$\dot{x} = f(x, h(x)), \tag{12}$$
is GAS at the origin.

B. The system
$$\dot{y} = A(x)y \tag{13}$$
is GAS at the origin, uniformly w.r.t. x, and there exists $P = P^\top > 0$ and $Q = Q^\top > 0$ such that
$$A^\top(x)P + PA(x) \leq -Q \tag{14}$$
for all $x \in \mathbb{R}^n$.

Assumption 7 is standard for singular perturbation analysis (Vasil'eva et al., 1995; Kokotovic et al., 1999). Nevertheless, in the most of existing results, a sufficient smoothness of the vector fields involved in the analysis is also required. However, as it will be shown below in our main result, such a restrictive requirement can be relaxed for the class of r-homogeneous.

Theorem 8. Let the subsystems (10) and (11) be r-homogeneous with a degree ν and $\tilde{r}$-homogeneous with a degree μ, respectively. Moreover, consider functions $f \in C^0$ and $A, R \in C^1$. If Assumption 7 is satisfied then there is $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ the interconnected system (10)-(11) is

- globally asymptotically stable for $\mu = \nu$,
- locally asymptotically stable, for $\mu < \nu$,
- globally asymptotically practically stable, for $\mu > \nu$.

Proof. Following the ideas of the singular perturbation theory (see Vasil'eva et al. (1995); Kokotovic et al. (1999)), let us define an "error" variable

$$z = y - h(x),$$

such that, the system (10)-(11) in the new coordinates is given by

$$\dot{x} = f(x, z + h(x)), \tag{15}$$

$$\dot{z} = \tfrac{1}{\epsilon}A(x)z - \tfrac{\partial h(x)}{\partial x}f(x, z + h(x)). \tag{16}$$

The main advantage of this representation is that the system (15)-(16) has an equilibrium point at the origin instead of the system (10)-(11) where the trajectories y tend to $h(x)$. Hence, by the homogeneity of the system (10)-(11), the following expressions hold

$$f(\Lambda_r x, \Lambda_{\tilde{r}} y) = \lambda^\nu \Lambda_r f(x, y),$$
$$A(\Lambda_r x)\Lambda_{\tilde{r}} y + R(\Lambda_r x) = \lambda^\mu \Lambda_{\tilde{r}} A(x)y + \lambda^\mu \Lambda_{\tilde{r}} R(x),$$
$$h(\Lambda_r x) = \Lambda_{\tilde{r}} h(x).$$

First, let's analyze the stability of the subsystem (15). Since the system (12) is GAS and r- homogeneous of degree ν for $z = 0$, there exists a LF $V(x)$ satisfying

$$V(\Lambda_r x) = \lambda^\kappa V(x), \tag{17}$$

$$\underline{a}_x \|x\|_r^\kappa \leq V(x) \leq \bar{a}_x \|x\|_r^\kappa, \tag{18}$$

$$\tfrac{\partial V(x)}{\partial x} f(x, h(x)) \leq -b_x \|x\|_r^{\nu+\kappa}, \tag{19}$$

$$\sup_{\|x\|_r \leq 1} \|\tfrac{\partial V(x)}{\partial x}\| \leq c_x, \tag{20}$$

for all $x \in \mathbb{R}^n$ and for some $\underline{a}_x, \overline{a}_x, b_x, c_x > 0$. Using $V(x)$ as an ISS-LF candidate for the system (15), its derivative is given by

$$\begin{aligned}\dot{V}(x) &= \tfrac{\partial V(x)}{\partial x} f(x, z + h(x)) \\ &= \tfrac{\partial V(x)}{\partial x} f(x, h(x)) + \tfrac{\partial V(x)}{\partial x}\left(f(x, z + h(x)) - f(x, h(x))\right), \\ &\leq -b_x \|x\|_r^{\nu+\kappa} \\ &\quad + c_x \|x\|_r^{\nu+\kappa} \|f(\xi, h(\xi) + \Lambda_{\tilde{r}}^{-1}(\|x\|_r)z) - f(\xi, h(\xi))\|,\end{aligned}$$

where the dilations $\Lambda_r(\|x\|_r)$ and $\Lambda_{\tilde{r}}(\|x\|_r)$ have been applied and $x = \Lambda_r(\|x\|_r)\xi$ for some $\xi \in S_r(1)$. By the continuity on the unit sphere of $f(x, z + h(x))$, we have that for any b_x, c_x and $0 < \theta < 1$ there exist δ such that if

$$\|\Lambda_{\tilde{r}}^{-1}(\|x\|_r)z\|_{\tilde{r}} \leq \delta$$

then

$$\|f(\xi, h(\xi) + \Lambda_{\tilde{r}}^{-1}(\|x\|_r)z) - f(\xi, h(\xi))\| \leq \frac{\theta b_x}{c_x},$$

for all $\|\xi\|_r = 1$. Therefore,

$$\dot{V}(x) \leq -b_x\|x\|_r^{\nu+\kappa} + \theta b_x\|x\|_r^{\nu+\kappa}, \\ \leq -(1-\theta)b_x\|x\|_r^{\nu+\kappa}, \qquad \text{if} \qquad \|x\|_r \geq \delta^{-1}\|z\|_{\tilde{r}},$$

where the properties $0 < \theta < 1$ and $\|\Lambda_{\tilde{r}}^{-1}(\|x\|_r)z\|_{\tilde{r}} = \|x\|_r^{-1}\|z\|_{\tilde{r}}$ were used. Accordingly, these properties imply that the system (15) is ISS w.r.t. an input z. Moreover, from Definition 2, the solution $x(t)$ of the system (15) is bounded by

$$\|x(t)\|_r \leq \max\{\beta_1(\|x_0\|_r, t), \gamma_1(\sup_{\tau\in[0,t)} \|z(\tau)\|_{\tilde{r}})\} \quad (21)$$

for all $t \geq 0$, where β_1 is a $\mathcal{KL}$ function and, considering equations (6) and (18), γ_1 is a $\mathcal{K}$ function given by

$$\gamma_1(s) = \delta^{-1}\frac{\bar{a}_x}{\underline{a}_x}s. \quad (22)$$

Now, let's study the stability of the systems (16). Taking into account Assumption 7.B., there exists a LF $W(z) = z^\top Pz$, where P is a solution of equation (14), satisfying

$$W(\Lambda_{\tilde{r}}z) = \lambda^2 W(z) \quad (23)$$

$$\underline{a}_z\|z\|_{\tilde{r}}^2 \leq W(z) \leq \bar{a}_z\|z\|_{\tilde{r}}^2, \quad (24)$$

$$\tfrac{\partial W(z)}{\partial y}A(x)z \leq -b_z\|z\|_{\tilde{r}}^{\mu+2}, \quad (25)$$

$$\sup_{\|\zeta\|_{\tilde{r}}\leq 1} \|\tfrac{\partial W(\zeta)}{\partial \zeta}\| \leq c_z, \quad (26)$$

for all $z \in \mathbb{R}^m$ and for some $\underline{a}_z, \bar{a}_z, b_z, c_z > 0$. The function $W(z)$ can be used as an ISpS-LF candidate for the system (16), such that

$$\dot{W}(z) = \tfrac{1}{\epsilon}\tfrac{\partial W(z)}{\partial z}A(x)z - \tfrac{\partial W(z)}{\partial z}\tfrac{\partial h(x)}{\partial x}f(x, z+h(x)), \\ \leq \tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+2} + \lambda^{2+\nu}\|\tfrac{\partial W(\zeta)}{\partial \zeta}\|\|\tfrac{\partial h(\xi)}{\partial \xi}f(\xi, \zeta+h(\xi))\|.$$

where $\lambda = \max\{\|x\|_r, \|z\|_{\tilde{r}}\}$ and the dilations $\Lambda_r(\lambda^{-1})$ and $\Lambda_{\tilde{r}}(\lambda^{-1})$ were applied, such that, $\|\xi\|_r = \frac{\|x\|_r}{\lambda} \leq 1$ and $\|\zeta\|_{\tilde{r}} = \frac{\|z\|_{\tilde{r}}}{\lambda} \leq 1$. By arguments of continuity of functions $h(x)$ and $f(x, z+h(x))$ on the unit ball,

$$\sup_{\|\xi\|_r\leq 1, \|\zeta\|_{\tilde{r}}\leq 1} \|\tfrac{\partial h(\xi)}{\partial \xi}f(\xi, \zeta+h(\xi))\| \leq \frac{b_z\eta}{c_z}. \quad (27)$$

for some constant η. Thus,

$$\dot{W}(z) \leq -\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+2} + b_z\eta\lambda^{2+\nu} \\ \leq -\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+2} + b_z\eta\left(\|x\|_r^{2+\nu} + \|z\|_{\tilde{r}}^{2+\nu}\right) \\ \leq -(1-\tilde{\theta})\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^{\mu+2},$$

if

$$\|z\|_{\tilde{r}}^{2+\mu} \geq \max\{\tfrac{\eta\epsilon}{\tilde{\theta}_1}\|x\|_r^{2+\nu}, \tfrac{\eta\epsilon}{\tilde{\theta}_2}\|z\|_{\tilde{r}}^{2+\nu}\} \quad (28)$$

where $\tilde{\theta}_1 + \tilde{\theta}_2 = \tilde{\theta}$ and $0 < \tilde{\theta} < 1$. Thus, $\dot{W}(z) < 0$, if

$$\|z\|_{\tilde{r}} > \left(\tfrac{\eta\epsilon}{\tilde{\theta}_1}\right)^{\frac{1}{\mu+2}}\|x\|_r^{\frac{\nu+2}{\mu+2}}, \quad (29)$$

or, on the other hand,

$$\|z\|_{\tilde{r}}^{\mu-\nu} > \left(\tfrac{\epsilon\eta}{\tilde{\theta}_2}\right). \quad (30)$$

where three different behaviors of system (16) can be observed:

- If $\mu = \nu$, the system (16) is ISS w.r.t. x for a sufficiently small ϵ.
- If $\mu < \nu$, the system (16) is locally ISS w.r.t. x.
- If $\mu > \nu$, the system (16) is ISpS w.r.t. x.

Combining all these cases, from Definition 2 the trajectories of the system (15) are bounded by

$$\|z(t)\|_{\tilde{r}} \leq \max\{\beta_2(\|z_0\|_{\tilde{r}}, t), \gamma_2(\sup_{\tau\in[0,t)} \|x(\tau)\|_r), \rho\}$$

for all $t \geq 0$, where β_2 is a $\mathcal{KL}$ function, ρ is a constant which can be estimated from (30), i.e., $\rho = 0$ for $\mu \leq \nu$, and $\rho = \frac{\bar{a}_z}{\underline{a}_z}\left(\frac{\epsilon\eta}{\tilde{\theta}}\right)^{\frac{1}{\mu-\nu}}$ for $\mu > \nu$, and considering equations (6), (29) and (24), γ_2 is a class $\mathcal{K}$ function given by

$$\gamma_2(s) = \tfrac{\bar{a}_z}{\underline{a}_z}\left(\tfrac{\eta\epsilon}{\tilde{\theta}}\right)^{\frac{1}{\mu+2}} s^{\frac{\nu+2}{\mu+2}}. \quad (31)$$

Finally, the internal stability of the interconnection (15)-(16) is investigated by using the Small-Gain Theorem. Likely, from equations (9), (22) and (31) we have

$$\gamma_1(\gamma_2(s) = \delta^{-1}\tfrac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\left(\tfrac{\eta\epsilon}{\tilde{\theta}}\right)^{\frac{1}{\mu+2}} s^{\frac{\nu+2}{\mu+2}}.$$

According to the small-gain condition (9), the stability of the interconnected system (15)-(16) is insured for

$$\delta^{-1}\tfrac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\left(\tfrac{\eta\epsilon}{\tilde{\theta}}\right)^{\frac{1}{\mu+2}} < s^{\frac{\mu-\nu}{\mu+2}}.$$

Thus, depending on the homogeneity degrees of systems (15) and (16), we have three different cases:

- For $\mu = \nu$, the system (15)-(16) is globally asymptotically stable if
$$\epsilon < \tfrac{\tilde{\theta}}{\eta}\left(\delta\tfrac{\underline{a}_x\underline{a}_z}{\bar{a}_x\bar{a}_z}\right)^{\mu+2}. \quad (32)$$
.
- For $\mu < \nu$, the system (15)-(16) is locally asymptotically stable if
$$\|x\|_r < \left(\delta^{-1}\tfrac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\right)^{\frac{\mu+2}{\mu-\nu}}\left(\tfrac{\eta\epsilon}{\tilde{\theta}}\right)^{\frac{1}{\mu-\nu}}, \quad (33)$$
- For $\mu > \nu$, the system (15)-(16) is globally asymptotically practically stable if
$$\|x\|_r > \left(\delta^{-1}\tfrac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z}\right)^{\frac{\mu+2}{\mu-\nu}}\left(\tfrac{\eta\epsilon}{\tilde{\theta}}\right)^{\frac{1}{\mu-\nu}}, \quad (34)$$

Since the changes of variables $z = y - h(x)$ is a difeomorphism, the results (32),(33) and (34) are preserved in the coordinates x, y for the system (10)-(11) and this concludes the proof of Theorem 8.

In addition, the following corollary provides the estimations of the domains of attraction and the ultimate bounds for the trajectories of the system (10)-(11) with initial condition $x_0 = x(0)$ and $y_0 = y(0)$.

Corollary 9. Let the system (10)-(11)satisfies all the requirements of Theorem 8.

- For $\mu = \nu$,
$$\lim_{t\to\infty}\|x(t)\|_r = 0, \\ \lim_{t\to\infty}\|y(t)\|_r = 0,$$
for all $x_0 \in \mathbb{R}^n$ and all $y_0 \in \mathbb{R}^m$.
- For $\mu < \nu$,
$$\lim_{t\to\infty}\|x(t)\|_r = 0, \\ \lim_{t\to\infty}\|y(t)\|_r = 0,$$
for all
$$\|x_0\|_r < \left(\tfrac{\tilde{\theta}}{\eta\epsilon}\left(\tfrac{\underline{a}_x\underline{a}_z\delta}{\bar{a}_x\bar{a}_z}\right)^{\mu+2}\right)^{\frac{1}{\nu-\mu}},$$

and

$$\|y_0 - h(x_0)\|_r < \left(\frac{\tilde{\theta}}{\eta\epsilon}\left(\frac{\underline{a}_z}{\bar{a}_z}\right)^{\mu+2}\left(\frac{\underline{a}_x\delta}{\bar{a}_x}\right)^{\nu+2}\right)^{\frac{1}{\nu-\mu}}.$$

- For $\mu > \nu$,

$$\lim_{t\to\infty}\|x(t)\|_r \le \left(\frac{\eta\epsilon}{\tilde{\theta}}\left(\frac{\bar{a}_x\bar{a}_z}{\underline{a}_x\underline{a}_z\delta}\right)^{\mu+2}\right)^{\frac{1}{\mu-\nu}},$$

$$\lim_{t\to\infty}\|y(t) - h(x(t))\|_r \le \left(\frac{\eta\epsilon}{\tilde{\theta}}\left(\frac{\bar{a}_z}{\underline{a}_z}\right)^{\mu+2}\left(\frac{\bar{a}_x}{\underline{a}_x\delta}\right)^{\nu+2}\right)^{\frac{1}{\mu-\nu}}$$

for all $x_0 \in \mathbb{R}^n$ and all $y_0 \in \mathbb{R}^m$.

Remember that by definition $z = y - h(x)$ and $h(0) = 0$ hence the estimation for variable $y(t)$ and its initial condition y_0 can be readily derived in the same way that (32), (33) and (34) in the proof of Theorem (8).

Note that for $\mu \neq \nu$ the system (15)-(16) always possesses some kind of stability (locally or practically), and by decreasing the value of ϵ it is possible to enlarge the domain of attraction for $\mu < \nu$ or to decrease the size of the neighborhood of the origin that attracts all the solution for $\mu > \nu$.

5. EXAMPLE: A HOMOGENEOUS SYSTEM OF NEGATIVE DEGREE

In (Cruz-Zavala et al., 2018), a second order systems

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -k_1\lceil r(x)\rfloor^{\frac{2r_2-r_1}{r_1}}. \tag{35}$$

where

$$r(x) = k_2^{-\frac{r_1}{r_2}}\lceil x_2\rfloor^{\frac{r_1}{r_2}} + x_1,$$

and k_1, k_2 are gains, was introduced. The parameters $r_1, r_2 \in \mathbb{R}_+$ satisfy $2r_2 > r_1 > r_2$, such that, the systems (35) is (r_1, r_2)-homogeneous with a degree $\nu = r_2 - r_1 < 0$. Moreover, a strict LF

$$V(x) = \ell_1 d\left(\lceil x_1\rfloor^{\frac{\varrho-r_2}{r_1}}x_2 + \frac{r_1}{\varrho k_2}|x_1|^{\frac{\varrho}{r_1}}\right) + (\ell_2+\ell_1)\left(\frac{1}{2}|x_2|^2 + \frac{r_1}{2r_2}k_1|x_1|^{\frac{2r_2}{r_1}}\right)^{\frac{\varrho}{2r_2}}, \tag{36}$$

where $\ell_1, \ell_2 \in \mathbb{R}_+$, and

$$d^2 = \frac{\varrho}{2\ell^2 r_2}\left[\frac{r_1k_1\varrho}{2r_2(\varrho-r_2)}\right]^{\frac{\varrho-r_2}{r_2}}$$

with $\ell = \frac{\ell_1}{\ell_1+\ell_2}$, was provided.

Proposition 10. (Cruz-Zavala et al. (2018)). Under $2r_2 > r_1 > r_2 > 0$, the origin of (35) is globally finite-time stable for all $k_1, k_2 \in \mathbb{R}_+$. Moreover, for each $\varrho \ge r_1 + r_2$, there exists a small enough ℓ such that (36) is a strict LF for (35).

In order to illustrate the results presented in Section 4, let put a parasitic dynamic in the system (35), such that,

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -k_1\lceil y\rfloor^{\frac{2r_2-r_1}{r_1}}, \tag{37}$$

$$\epsilon\dot{y} = -a(x)y + r(x), \tag{38}$$

where

$$a(x) = \sin\left(\frac{\lceil x_1\rfloor^{\frac{\rho_0}{r_1}} + \lceil x_2\rfloor^{\frac{\rho_0}{r_2}}}{\|x\|_r}\right) + 2 > 0, \quad \text{for all} \quad x_1, x_2 \in \mathbb{R},$$

where $\rho_0 > r_1$. Furthermore, the system (38) is r_1-homogeneous with a degree $\mu = 0$. Note that the function $\lceil\cdot\rfloor^q$ describing system (37) is just continuous for any $0 < q < 1$, i.e., it belongs to the class C^0 and $a(x), r(x) \in C^1$. Moreover, $1 < a(x) < 3$.

For $\epsilon = 0$, the system (35) is recovered with

$$\bar{k}_1 = \frac{k_1}{a(x)^{\frac{2r_2-r_1}{r_1}}}.$$

It can be readily seen that the stability of the system (35) is kept despite of the term $a(x)$ since it is bounded and positive for any $x \in \mathbb{R}^2$. On the other hand, by construction the function (36) is positive definite (see (Cruz-Zavala et al., 2018, Lemma 7)) and (r_1, r_2)-homogeneous with a degree ϱ hence it is also radially unbounded (Bhat and Bernstein, 2005, Lemma 4.1). Therefore, we can assume that (36) satisfies conditions (18)-(20) for certain $\underline{a}_x, \bar{a}_x, b_x, c_x \in \mathbb{R}_+$.

Assuming a small parameter $\epsilon > 0$, we can define the variable $z = y - \frac{r(x)}{a(x)}$, such that, the system (37)-(38) in the new coordinates is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\bar{k}_1\lceil r(x) + a(x)z\rfloor^{\frac{2r_2-r_1}{r_1}}, \tag{39}$$

$$\dot{z} = -\frac{a(x)}{\epsilon}z + \frac{\partial}{\partial x}\frac{r(x)}{a(x)}\begin{bmatrix} x_2 \\ -\bar{k}_1\lceil r(x) + a(x)z\rfloor^{\frac{2r_2-r_1}{r_1}}\end{bmatrix} \tag{40}$$

To show that the system (39) is ISS w.r.t. z, we can use the LF (36) such that its derivative along the trajectories of the system (39) satisfies

$$\dot{V}(x) \le -b_x\|x\|_r^{\nu+\varrho} + k_1\left\|\frac{\partial V(x)}{\partial x}\right\| \times \left\|\lceil r(x)\rfloor^{\frac{2r_2-r_1}{r_1}} - \lceil r(x) + a(x)z\rfloor^{\frac{2r_2-r_1}{r_1}}\right\|,$$

where the property $\bar{k}_1 \le k_1$ was used. Taking advantage of the homogeneity of the system (39) and the LF $V(x)$, and using the dilations $\Lambda_r(\|x\|_r^{-1})$ and $\Lambda_{\tilde{r}}(\|x\|_r^{-1})$, it is obtained

$$\dot{V}(x) \le -b_x\|x\|_r^{\nu+\varrho} + k_1c_x\|x\|_r^{\nu+\varrho} \times \left\|\lceil r(\xi)\rfloor^{\frac{2r_2-r_1}{r_1}} - \lceil r(\xi) + a(\xi)\Lambda_{\tilde{r}}(\|x\|_r^{-1})z\rfloor^{\frac{2r_2-r_1}{r_1}}\right\|,$$

where $\xi \in S_r(1)$. By continuity arguments and taking into account that $\|\xi\|_r = 1$, we can suppose that for any $k_1, b_x, c_x \in \mathbb{R}_+$ and $0 < \theta_1 < 1$ there exists $\delta_1 \in \mathbb{R}_+$ such that

$$\left\|\lceil r(\xi)\rfloor^{\frac{2r_2-r_1}{r_1}} - \lceil r(\xi) + a(\xi)\Lambda_{\tilde{r}}(\|x\|_r^{-1})z\rfloor^{\frac{2r_2-r_1}{r_1}}\right\| \le \frac{b_x\theta_1}{c_xk_1},$$

for all $\frac{\|z\|_{\tilde{r}}}{\|x\|_r} \le \delta_1$. Therefore,

$$\begin{aligned}\dot{V}(x) &\le -b_x\|x\|_r^{\nu+\varrho} + \theta_1b_x\|x\|_r^{\nu+\varrho},\\ &\le -(1-\theta_1)b_x\|x\|_r^{\nu+\varrho}, \quad \forall \quad \|x\|_r \ge \delta_1\|z\|_{\tilde{r}},\end{aligned}$$

where $0 < \theta_1 < 1$, hence we can say that the system (39) is ISS w.r.t z. By Definition 2, we can say that

$$\|x(t)\|_r \le \max\{\beta(\|x_0\|_r, t), \gamma_1(\|z(t)\|_{\tilde{r}})\} \tag{41}$$

for all $t \ge 0$, where β is a $\mathcal{KL}$ function and, considering equation (6) and since LF (36) satisfies (18), we have

$$\gamma_1(\|z(t)\|_{\tilde{r}}) = \frac{\bar{a}_x}{\underline{a}_x}\delta_1\|z(t)\|_{\tilde{r}}. \tag{42}$$

On the other hand, the input-to state stability of the system (40) can be analyzed by the Lyapunov function $W(z) = \frac{1}{2}z^2$, such that

$$\dot{W}(z) \le -\frac{a(x)b_z}{\epsilon}\|z\|_{\tilde{r}}^2 + |z|\left\|\frac{\partial}{\partial x}\frac{r(x)}{a(x)}\begin{bmatrix} x_2 \\ -\frac{k_1}{a(x)}\lceil r(x) + a(x)z\rfloor^{\frac{2r_2-r_1}{r_1}}\end{bmatrix}\right\|.$$

For $\lambda = \max\{\|x\|_r, \|z\|_{\tilde{r}}\}$, we define the dilations $\Lambda_r(\lambda^{-1})$ and $\Lambda_{\tilde{r}}(\lambda^{-1})$ such that

$$\dot{W}(z) \leq -\tfrac{a(x)b_z}{\epsilon}\|z\|_{\tilde{r}}^2 \\ +\lambda^{\nu+2}\left\|\frac{\partial}{\partial\varsigma}\frac{r(\varsigma)}{a(\varsigma)}\begin{bmatrix} \varsigma_2 \\ -\frac{k_1}{a(\varsigma)}\lceil r(\varsigma)+a(\varsigma)\zeta\rfloor^{\frac{2r_2-r_1}{r_1}}\end{bmatrix}\right\|.$$

where $\|\varsigma\|_r \leq 1$ and $\|\zeta\|_{\tilde{r}} \leq 1$. By homogeneity and continuity arguments, we can assume that there is a constant η such that

$$\sup_{\|\varsigma\|_r\leq 1,\|\zeta\|_{\tilde{r}}\leq 1}\left\|\frac{\partial}{\partial\varsigma}\frac{r(\varsigma)}{a(\varsigma)}\begin{bmatrix} \varsigma_2 \\ -\frac{k_1}{a(\varsigma)}\lceil r(\varsigma)+a(\varsigma)\zeta\rfloor^{\frac{2r_2-r_1}{r_1}}\end{bmatrix}\right\| \leq b_z\eta.$$

Thus, we have

$$\dot{W}(z) \leq -\tfrac{a(x)b_z}{\epsilon}\|z\|_{\tilde{r}}^2 + b_z\eta\lambda^{\nu+2}, \\ \leq -(1-\theta_2)\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^2 - \tfrac{b_z\theta_2}{\epsilon}\|z\|_{\tilde{r}}^2 \\ +b_z\eta\left(\|x\|_r^{\nu+2} + \|z\|_{\tilde{r}}^{\nu+2}\right),$$

where $0 < \theta_2 < 1$. Accordingly, it can be concluded that

$$\dot{W}(z) \leq -(1-\theta_2)\tfrac{b_z}{\epsilon}\|z\|_{\tilde{r}}^2,$$

if

$$\|z\|_{\tilde{r}} \geq \max\{\left(\tfrac{\epsilon\eta}{\theta_3}\right)^{-\frac{1}{\nu}}, \left(\tfrac{\epsilon\eta}{\theta_4}\right)^{\frac{1}{2}}\|x\|_r^{\frac{\nu+2}{2}}\}$$

where $\theta_3 + \theta_4 = \theta_2$ and by its definition $\nu = r_2 - r_1 < 0$ and $1 < a(x) < 3$ for all $x \in \mathbb{R}^2$, thus system (40) is ISpS w.r.t x. Following Definition 2, considering equation (6) and the LF $W(z)$, it can be said that

$$\|z(t)\|_{\tilde{r}} \leq \max\left\{\beta(\|z_0\|_{\tilde{r}}, t), \left(\tfrac{\epsilon\eta}{\theta_2}\right)^{\frac{1}{2}}\|x\|_r^{\frac{\nu+2}{2}}, \left(\tfrac{\epsilon\eta}{\theta_2}\right)^{-\frac{1}{\nu}}\right\}, \tag{43}$$

for all $t \geq 0$, where β is a $\mathcal{KL}$ function. Finally, applying the *small-gain* theorem, the composition between functions (42) and (43) is given by

$$\gamma_1(\gamma_2(|x|)) = \tfrac{\bar{a}_x}{\underline{a}_x}\delta_1\left(\tfrac{\epsilon\eta}{\theta_2}\right)^{\frac{1}{2}}\|x\|_r^{\frac{\nu+2}{2}}, \tag{44}$$

such that the small-gain condition fulfills for

$$\|x\|_r > \left(\tfrac{\bar{a}_x}{\underline{a}_x}\delta_1\right)^{-\frac{2}{\nu}}\left(\tfrac{\epsilon\eta}{\theta_2}\right)^{-\frac{1}{\nu}},$$

with a small enough $\epsilon > 0$, hence we can conclude that the interconnected system (37)-(38) is ISpS.

6. CONCLUSION

This paper was devoted to study the stability of a class of r-homogeneous interconnected systems affected by a singular perturbation. According to the homogeneity degrees of the involved subsystems, sufficient conditions to warranty the stability (local or practical) of the inter-connexion were established. Moreover, estimations of the regions of attraction and ultimate bounds for the system trajectories were provided. Finally, the mentioned properties were illustrated by an example with an r-homogeneous system with a negative degree.

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