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Finite-time extremum seeking control for a class of unknown static maps

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Abstract: This paper proposes an extremum-seeking control design that achieves finite-time stability of the optimum an unknown measured cost function. The finite-time extremum-seeking control technique is shown to achieve finite-time practical stability of the optimum of the unknown cost function. The main characteristic of the proposed extremum seeking control approach is that the target averaged system considered achieves finite-time stability of the unknown optimum. A simulation study is presented to demonstrate the effectiveness of the approach.

Keywords: Extremum-seeking control, Finite-time stability, Nonlinear systems

1. INTRODUCTION

ESC is a feedback control mechanism designed to drive an unknown nonlinear dynamical system to the optimum of a measured variable of interest (Tan et al., 2010). The basic ESC properties were first outlined in the works of Krstic and Wang (2000) and Tan et al. (2006). Based on this initial theoretical work, a vast and growing literature has been evolving to complement, generalize and improve the basic schemes. Finite-time stability in control systems is a desirable property in many applications where the timing of control tasks is critical. Several studies have been recently conducted in the design and analysis of finite-time control systems. The general stability conditions were first developed in (Bhat and Bernstein, 2000). The finite-time stabilization of a class of controllable systems was considered in (Hong, 2002). The output feedback finite-time stabilization of nonlinear systems was considered in Hong et al. (2001) for the local case. A perspective of global finite-time stabilization was provided in (Hong et al., 2000). Robust finite-time stabilization was treated in (Hong and Jiang, 2006). The concept of finite-time input to state stability (FTISS) was presented in (Hong et al., 2010). This study provides a complete characterization of finite-time nonlinear systems subject to external inputs. The finite-stability property is usually associated with dynamical control systems that are either non-Lipschitz or discontinuous. In most existing work, finite-time stable systems are closely related to classes of nonlinear systems with continuous right hand side (Bhat and Bernstein, 2000). In a similar fashion, it can be shown that finite-time stabilization can be achieved using continuous feedback controllers (Hong, 2002). Recent developments in the area of gradient descents with finite-time convergence were recently proposed in (Garg and Panagou, 2018b) and (Garg and Panagou, 2018a) where a comprehensive stability analysis was provided to address the non-Lipschitz nature of finite-time systems.

In this manuscript, we propose an ESC design technique that can achieve finite-time stability in the practical sense. Given a measured cost function with an unknown mathematical formulation, the objective of this study is to design an extremum seeking control that brings the system to a neighbourhood of the unknown optimum value of the input in finite-time. The finite-time can be prescribed by choosing the tuning parameters of the control system.

The paper is structured as follows. The problem formulation is given in Section 2. In Section 3, a target averaged finite-time ESC system is proposed. The proposed ESC is presented in Section 3.3. A simulation study is given in Section 4 followed by brief conclusions in Section 5.

2. PROBLEM FORMULATION

In this study, we consider a class of unknown nonlinear systems described by the following dynamical system:

$$\dot{x} = u \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the input variable, and $y \in \mathbb{R}$ is the output variable. It is assumed that the function $h : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth.

The function h, is assumed to be unknown. The function $h(x)$ has an unknown minimizer x^* with an optimal value $y^* = h(x^*)$.

We make the following assumptions concerning the measured cost function, $h(x)$.

Assumption 1. The function $h(x)$ is such that its gradient vanishes only at the minimizer x^*, that is:

$$\left.\frac{\partial h}{\partial x}\right|_{x=x^*} = 0.$$

The Hessian at the minimizer is assumed to be positive and nonzero. In particular, there exists a positive constant α_h such that

$$\frac{\partial^2 h(x)}{\partial x \partial x^T} \geq \alpha_h I$$

for all $x \in \mathcal{X} \subset \mathbb{R}$.

The objective of this study is to develop an ESC design technique that guarantees finite-time convergence to the unknown minimizer, x^*, of the measured function $y = h(x)$.

3. FINITE TIME EXTREMUM SEEKING CONTROLLER DESIGN AND ANALYSIS

3.1 Finite-time Stability

In the following, we will establish that this equilibrium is finite-time stable. We first provide a formal definition of finite-time stability (as stated in Hong et al. (2010)) that will be used throughout the manuscript.

Consider the system:

$$\dot{X} = F(X) \tag{2}$$

where $X \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to X.

The continuity of the right hand side of (2) guarantees existence of at least one solution, which is possibly non-unique. We denote by $\mathcal{X}(t, t_0, X_0)$ the set of all solutions with initial conditions $X(t_0) = X_0$ for $t \geq t_0$. The set of all solutions of system (2) at time t is denoted by $X(t)$. It is assumed that the equilibrium $X_0 = 0$ is a unique solution of the system in forward time.

Definition 2. The equilibrium $X = 0$ of (9) is said to be finite-time locally stable if it is Lyapunov stable and such that there exists a settling-time function

$$T(X_0) = \inf \left\{ \bar{T} \geq t_0 \;\middle|\; \lim_{t \to \bar{T}} X(t) = 0\,;\, X(t) \equiv 0,\, \forall t \geq \bar{T} \right\}$$

in a neighbourhood U of $X = 0$. It is globally finite-time stable if $U = \mathbb{R}^n$.

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a called a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. It is a class $\mathcal{K}_\infty$ function if it is class $\mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

A continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized class $\mathcal{K}$ function if $\phi(0) = 0$ and

$$\begin{cases} \phi(s_1) > \phi(s_2) & \text{if}\, \phi(s_1) > 0,\, s_1 > s_2 \\ \phi(s_1) = \phi(s_2) & \text{if}\, \phi(s_1) = 0,\, s_1 > s_2. \end{cases} \tag{3}$$

A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(s, t)$ is a generalized $\mathcal{K}$ function and each fixed $s \geq 0$, the function $\beta(s, t)$ is such that $\lim_{t\to T} \beta(s, t) = 0$ for $T \leq \infty$.

Definition 3. System (2) is finite-time stable if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every solution $X(t)$ satisfies:

$$\|X(t)\| \leq \beta(\|X(0)\|, t) \tag{4}$$

with $\beta(r, t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

Definition 4. Let $V(X)$ be a continuous function. It is called a finite-time Lyapunov function if there exists class $\mathcal{K}_\infty$ functions ϕ_1 and ϕ_2 and a class $\mathcal{K}$ function ϕ_3 such that:

$$D^+V(X(t)) = \limsup_{s\to 0^+} \frac{V(X(t+s)) - V(X(t))}{s} \leq -\phi_3(\|X\|)$$

where, in addition, ϕ_3 satisfies:

$$c_1 V(X)^\alpha \leq \phi_3(\|X\|) \leq c_2 V(X)^a$$

for some positive constants c_1 and c_2.

If we now consider the system:

$$\dot{X} = F(X, v(t)) \tag{5}$$

where $X \in \mathbb{R}^n$. The function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is measurable and locally essentially bounded and the vector value function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous in X and $v(t)$.

Definition 5. System (5) is finite-time input to state stable ($FTISS$) if there exists a generalized $\mathcal{KL}$ function β : $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every solution $X(t)$ satisfies:

$$\|X(t)\| \leq \beta(\|X(0)\|, t) + \alpha(\|v(t)\|_\infty) \tag{6}$$

with $\beta(r, t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

Definition 6. Let $V(X)$ be a continuous function. It is called an $FTISS$ Lyapunov function if there exists class $\mathcal{K}_\infty$ functions ϕ_1 and ϕ_2 and class $\mathcal{K}$ functions ϕ_3 and ϕ_4 such that:

$$\|X(t)\| \geq \phi_4(\|v(t)\|) \Rightarrow D^+V(X(t)) \leq -\phi_3(\|X\|)$$

where, in addition, ϕ_3 satisfies:

$$c_1 V(X)^\alpha \leq \phi_3(\|X\|) \leq c_2 V(X)^\alpha$$

for $0 < \alpha < 1$ and some positive constants c_1 and c_2.

Theorem 1. (Hong et al., 2010) System (5) is $FTISS$ if admits an $FTISS$ Lyapunov function.

Finally, we consider the interconnection of two $FTISS$ systems:

$$\begin{aligned} \dot{X}_1 &= F_1(X_1, X_2) \\ \dot{X}_2 &= F_2(X_2, X_1) \end{aligned} \tag{7}$$

where $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$ with $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $F_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ are continuous in X_1 and X_2 with a unique equilibrium at $X_1 = 0$ and $X_2 = 0$. The equilibrium is the unique solution to (7) forward in time.

Theorem 2. Suppose (7) are FTISS, with X_2 as an input for the X_1 subsystem and with X_1, the input of the X_2 subsystem. Suppose that the solutions of each system satisfy:

$$\|X_1(t)\| \leq \beta_1(\|X_1(0)\|, t) + \alpha_1(\|X_2(t)\|_\infty)$$
$$\|X_2(t)\| \leq \beta_2(\|X_2(0)\|, t) + \alpha_2(\|X_1(t)\|_\infty)$$

where β_1 and β_2 are generalized $\mathcal{KL}$ functions and α_1 and α_2 are class $\mathcal{K}$ functions. If there exists class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 that satisfy:

$$(Id + \rho_2) \circ \alpha_2 \circ (Id + \rho_1) \circ \alpha_1(s) \leq s, \quad s \geq 0$$

then $X_1 = 0$, $X_2 = 0$ is a finite-time stable equilibrium of system (7).

3.2 Proposed target average system

In the design of ESC, one seeks an average system that can be achieved using a judicious choice of dither signals. We consider the following system:

$$\begin{aligned}\dot{x} &= -\gamma(\xi)\xi \\ \dot{\xi} &= -K\gamma(\xi - \nabla h(x))(\xi - \nabla h(x))\end{aligned} \tag{8}$$

where K is a controller gain to be assigned. As in Garg and Panagou (2018b), the function $\gamma(\xi)$ is given by:

$$\gamma(z) = \frac{c_1}{\|z\|^{\alpha_1}} + \frac{c_2}{\|z\|^{\alpha_2}}$$

where σ is a small positive constant, $\alpha_1 = \dfrac{q_1 - 2}{q_1 - 1}$ and $\alpha_2 = \dfrac{q_2 - 2}{q_2 - 1}$ for $q_1 \in (2, \infty)$ and $q_2 \in (1, 2)$.

The function $X_1(x, \xi) = \gamma(\xi)\xi$ is not locally Lipschitz continuous at $\xi = 0$ but it is continuous everywhere. Similarly, the function $X_2(x, \xi) = \gamma(\xi - \nabla h(x))(\xi - \nabla h(x))$ is also not locally Lipschitz continuous for $\forall x$ and $\forall \xi$ such that $\xi = \nabla h(x)$ but it is continuous at this point. It is locally Lipschitz everywhere else.

We define the state-space transformation $z = \xi - \nabla h(x)$ and rewrite the dynamics as:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= -K\gamma(z)z + \nabla^2 h(x)\gamma(z + \nabla h(x)))(z + \nabla h(x))) \\ \frac{\mathrm{d}\nabla h}{\mathrm{d}t} &= -\gamma(z + \nabla h(x))\nabla^2 h(x)(z + \nabla h(x))).\end{aligned} \tag{9}$$

This system is such that it has a unique equilibrium at the point $z = 0$ and $\nabla h(x) = 0 \Rightarrow x = x^*$. It is also continuous everywhere and locally Lipschitz continuous away from the equilibrium.

Next we proceed to the stability analysis of system (8).

Theorem 3. Consider the nonlinear system (8). Let Assumption 1 be satisfied. Then the optimum x^* is a finite-time stable equilibrium of the system.

Proof: We first consider the function $V_1 = \dfrac{1}{2}z^2$. Its derivative along the trajectories of the system yields:

$$\dot{V}_1 = -K\gamma(z)z^2 + \gamma(z + \nabla h(x))\nabla^2 h(x)(z + \nabla h(x))z.$$

The function $\phi(z, \nabla h(x)) = \gamma(z + \nabla h(x))(z + \nabla h(x))$ is continuous. It is such that:

$$\begin{aligned}|\phi(z, \nabla h(x))| \leq& \frac{c_1|z + \nabla h(x)|}{|z + \nabla h(x)|^{\alpha_1}} + c_2\frac{|z + \nabla h(x)|}{|z + \nabla h(x)|^{\alpha_2}} \\ \leq& c_1|z + \nabla h(x)|^{1-\alpha_1} + c_2|z + \nabla h(x)|^{1-\alpha_2}.\end{aligned}$$

By the triangle inequality, one obtains:

$$\begin{aligned}|\phi(z, \nabla h(x))| \leq c_1|z|^{1-\alpha_1} + c_1|\nabla h(x)|^{1-\alpha_1} \\ + c_2|z|^{1-\alpha_2} + c_2|\nabla h(x)|^{1-\alpha_2}\end{aligned}$$

or,

$$\begin{aligned}|\phi(z, \nabla h(x))| \leq c_1\frac{|z|}{|z|^{\alpha_1}} + c_1\frac{|\nabla h(x)|}{|\nabla h(x)|^{\alpha_1}} \\ + c_2\frac{|z|}{|z|^{\alpha_2}} + c_2\frac{|\nabla h(x)|}{|\nabla h(x)|^{\alpha_2}}.\end{aligned}$$

By definition, the last inequality can be written as:

$$|\phi(z, \nabla h(x))| \leq \gamma(z)|z| + \gamma(\nabla h(x))|\nabla h(x)|.$$

Upon substitution, it follows that $\dot{V}_1$ fulfills the following inequality:

$$\begin{aligned}\dot{V}_1 \leq& -(K - \nabla^2 h(x))\gamma(z)z^2 + \gamma(\nabla h(x))|\nabla h(x)||z| \\ \leq& -(K - \nabla^2 h(x))(|z|^{2-\alpha_1} + |z|^{2-\alpha_2}) \\ &+ (|\nabla h(x)|^{1-\alpha_1} + |\nabla h(x)|^{1-\alpha_2})|z|\end{aligned}$$

Next, we choose K such that $(K - \nabla^2 h(x)) \geq \alpha$:

$$\begin{aligned}\dot{V}_1 \leq& -\alpha(|z|^{2-\alpha_1} + |z|^{2-\alpha_2}) \\ &+ (|\nabla h(x)|^{1-\alpha_1} + |\nabla h(x)|^{1-\alpha_2})|z|\end{aligned}$$

Let $c \in (0, 1)$, we can rearrange the last inequality as:

$$\begin{aligned}\dot{V}_1 \leq& -(1 - c)\alpha(|z|^{2-\alpha_1} + |z|^{2-\alpha_2}) \\ &\alpha c|z|\bigg(|z|^{1-\alpha_1} + |z|^{1-\alpha_2} \\ &- \frac{1}{c\alpha}(|\nabla h(x)|^{1-\alpha_1} + |\nabla h(x)|^{1-\alpha_2})\bigg)\end{aligned}$$

Therefore, one obtains:

$$\dot{V}_1 \leq -(1 - c)\alpha(c_1|z|^{2-\alpha_1} + c_2|z|^{2-\alpha_2}), \text{ if} |z| \geq \frac{1}{c\alpha}|\nabla h(x)|.$$

Using the definition of V_1, we finally get:

$$\dot{V}_1 \leq -(1 - c)\alpha\left(c_1 V_1^{1-\frac{\alpha_1}{2}} + c_2 V_1^{1-\frac{\alpha_2}{2}}\right)$$

if $|z| \geq \dfrac{1}{c\alpha}|\nabla h(x)|$.

Following Definition 6, it follows that V_1 is an $FTISS$ Lyapunov function and therefore the z dynamics are $FTISS$ with input $\nabla h(x)$.

For the gradient dynamics $\nabla h(x)$, we consider the Lyapunov function $V_2 = \frac{1}{2}(\nabla h(x))^2$. Its rate of change is given by:

$$\dot{V}_2 = -\nabla^2 h(x)\gamma(z + \nabla h(x))(z + \nabla h(x))\nabla h(x).$$

As above, we write:

$$\begin{aligned}\dot{V}_2 = -\nabla^2 h(x)\bigg(c_1\frac{z + \nabla h(x)}{|z + \nabla h(x)|^{\alpha_1}} \\ + c_2\frac{z + \nabla h(x)}{|z + \nabla h(x)|^{\alpha_2}}\bigg)\nabla h(x).\end{aligned}$$

or,

$$\begin{aligned}\dot{V}_2 = -\nabla^2 h(x)\bigg(c_1\frac{z\nabla h(x) + \nabla h(x)^2}{|z + \nabla h(x)|^{\alpha_1}} \\ + c_2\frac{z\nabla h(x) + \nabla h(x)^2}{|z + \nabla h(x)|^{\alpha_2}}\bigg).\end{aligned}$$

We consider this equation evaluated on the set $\Omega_h = \{(x, z) \,|\, |\nabla h(x)| \geq |z|\}$. We first write the following inequality:

$$\begin{aligned}\dot{V}_2 \leq& -c_1\nabla^2 h(x)\frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z + \nabla h(x)|^{\alpha_1}} \\ &- c_2\nabla^2 h(x)\frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z + \nabla h(x)|^{\alpha_2}}.\end{aligned}$$

We readily see that $\dot{V}_2 \leq 0$ on Ω_h. Moreover, the second term on the right hand side is negative on Ω_h. Using the triangle inequality on $|z + \nabla h(x)|$, we get that:

$$|z + \nabla h(x)| \leq |z| + |\nabla h(x)|$$

As a result, we can write the inequality as:

$$\dot{V}_2 \leq -c_1\nabla^2 h(x)\frac{|\nabla h(x)|^2 - |z||\nabla h(x)|}{|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1}}$$

We introduce the parameter $c \in (0, 1)$ and rewrite the last inequality as:

$$\dot{V}_2 \leq -c_1\nabla^2 h(x)\frac{(1-c)\nabla h(x)^2 + |\nabla h(x)|(c|\nabla h(x)| - |z|)}{|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1}}$$

On the set Ω_h, one can write that

$$|z|^{\alpha_1} + |\nabla h(x)|^{\alpha_1} \leq 2|\nabla h(x)|^{\alpha_1}.$$

It is easy to see that $\dot{V}_2$ is negative definite for $|\nabla h(x)| \geq \frac{1}{c}|z|$. Defining the set, $\Omega_h^c = \{(x,z) \mid |\nabla h(x)| \geq \frac{1}{c}|z|\}$. Since $c \in (0,1)$ is follows that $\Omega_h \subset \Omega_h^c$. It then follows that:

$$\dot{V}_2 \leq -c_1\nabla^2 h(x)\frac{(1-c)|\nabla h(x)|^2}{2|\nabla h(x)|^{\alpha_1}}$$

for all (x,z) such that $|\nabla h(x)| \geq \frac{1}{c}|z|$. Using the definition of V_2, we can write, as above:

$$\dot{V}_2 \leq -(1-c)\left(c_1 V_2^{1-\frac{\alpha_1}{2}}\right), \text{ if } V_2 \geq \frac{1}{c^2}V_1.$$

As a result, we conclude that the gradient dynamics are $FTISS$ with z as an input.

From the previous development, it follows that V_1 satisfies the following:

$$\dot{V}_1 \leq -(1-c)\alpha\left(c_1 V_1^{1-\frac{\alpha_1}{2}} + c_2 V_1^{1-\frac{\alpha_2}{2}}\right), \text{ if } V_1 \geq \frac{1}{c^2\alpha^2}V_2.$$

As a result, we can view the finite-time systems as the interconnection of two $FTISS$ nonlinear systems. We can apply the small gain theorem from Hong et al. (2010) to conclude that the system has a finite-time stable equilibrium at the optimum $z = 0$, $\nabla h(x) = 0$ for any $\alpha = K - \nabla^2 h(x) > 1$. Thus it follows that if the Hessian $\nabla^2 h(x)$ is globally bounded then there exists a K such that the system is globally FT stable. Otherwise, for any (arbitrary) compact set in the state space on which the Hessian is bounded, there exists a K such that the optimum is a semi-global finite-time stable equilibrium of the closed-loop system. This completes the proof. ∎

3.3 Proposed Finite-time ESC

The proposed finite-time ESC approach is given by:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -\gamma(\xi)\xi \\ \frac{\mathrm{d}\xi}{\mathrm{d}t} &= -\gamma(\xi-\delta)K(\xi-\delta).\end{aligned} \tag{10}$$

where $\delta = \frac{2}{a}h(x + a\sin(\omega t))\sin(\omega t)$. The right hand side of this time-varying nonlinear system is continuous with respect to x and t. As a result, we guarantee the existence of at least one Carathéodory solution which may not be unique.

A formal average of this system is given by:

$$\begin{aligned}\frac{\mathrm{d}x^a}{\mathrm{d}t} &= -\gamma(\xi^a)\xi^a \\ \frac{\mathrm{d}\xi^a}{\mathrm{d}t} &= -\frac{K}{T}\int_0^T\left(\frac{c_1(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_1}}\right. \\ &\qquad \left. + \frac{c_2(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_2}}\right)dt.\end{aligned} \tag{11}$$

The term $\xi - \delta(x,t)$ can be expanded in the following manner:

$$\begin{aligned}\xi - \delta(x,t) = &\frac{1}{a}\Big(a\xi - 2h(x)\sin(\omega t) - 2\nabla h(x)\sin^2(\omega t)a \\ &- a^2 R(t,x,a,\omega)\Big)\end{aligned}$$

where $R(t,x,a,\omega)$ is a function of higher order derivatives of $h(x)$, higher powers of the sinusoidal signals and the amplitude. This can be rewritten as:

$$\begin{aligned}\xi - \delta(x,t) = &\frac{1}{a}\Big(-2h(x)\sin(\omega t) - \nabla h(x)\sin(2\omega t)a \\ &+ (\xi - \nabla h(x))a - a^2 R(t,x,a,\omega)\Big)\end{aligned}$$

Let us assume that the amplitude is picked small enough such that the last term is negligible:

$$\begin{aligned}\xi - \delta(x,t) \approx &\frac{1}{a}\Big(-2h(x)\sin(\omega t) - \nabla h(x)\sin(2\omega t)a \\ &+ (\xi - \nabla h(x))a\Big)\end{aligned}$$

As a result, we obtain:

$$\frac{1}{T}\int_0^T(\xi - \delta(x,t))dt \approx (\xi - \nabla h(x)).$$

In addition, it is also easy to compute that:

$$\frac{1}{T}\int_0^T|\xi - \delta(x,t)|dt \approx |\xi - \nabla h(x)|.$$

Since there are no analytical expressions of the right hand side of (11), we cannot provide a suitable closed form expression for the resulting averaged system. In this study, we propose to consider the stability of the averaged system (11) directly. In the following, it is shown that the averaged system meets stability conditions of the target averaged system presented in the previous section.

We consider the same change of coordinates to $z^a = \xi^a - \nabla h(x^a)$ and $\nabla h(x^a)$ and write the average dynamics as follows:

$$\begin{aligned}\frac{\mathrm{d}\nabla h(x^a)}{\mathrm{d}t} &= -\nabla^2 h(x^a)\gamma(z^a + \nabla h(x^a)(z^a + \nabla h(x^a)) \\ \frac{\mathrm{d}z^a}{\mathrm{d}t} &= -\frac{K}{T}\int_0^T\left(\frac{c_1(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_1}}\right. \\ &\qquad \left. + \frac{c_2(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_2}}\right)dt \\ &\qquad + \nabla^2 h(x^a)\gamma(z^a + \nabla h(x^a))(z^a + \nabla h(x^a))\end{aligned} \tag{12}$$

Then we pose the Lyapunov function candidates, $V_1^a = \frac{1}{2}|z^a|^2$ and $V_2^a = \frac{1}{2}|\nabla h(x^a)|^2$. The time derivative of V_1^a is given by:

$$\begin{aligned}\dot{V}_1^a = &z^a\left(-\frac{K}{T}\int_0^T\left(\frac{c_1(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_1}}\right.\right. \\ &\left.\left. + \frac{c_2(\xi^a - \delta(x^a,t))}{|\xi^a - \delta(x^a,t)|^{\alpha_2}}\right)dt\right) \\ &+ z^a\nabla^2 h(x^a) \\ &\times\left(K\gamma(z^a + \nabla h(x^a))(z^a + \nabla h(x^a))\right)\end{aligned}$$

We can then bring the variable z^a inside the integral term:

$$\begin{aligned}\dot{V}_1^a =& -K\bigg(\frac{1}{T}\int_0^T z^a\bigg(\frac{c_1(\xi^a-\delta(x^a,t))}{|\xi^a-\delta(x^a,t)|^{\alpha_1}} \\ &+\frac{c_2(\xi^a-\delta(x^a,t))}{|\xi^a-\delta(x^a,t)|^{\alpha_2}}\bigg)dt\bigg) \\ &+z^a\nabla^2 h(x^a) \\ &\times\Big(K\gamma(z^a+\nabla h(x^a))(z^a+\nabla h(x^a))\Big)\end{aligned} \tag{13}$$

The first term in the right hand side is clearly negative. However, we must confirm that the averaged system possesses the $FTISS$ property demonstrated for the target system in the previous section.

Let us denote $\theta(t)=\xi^a-\delta(x^a,t)$. We consider the first term in (13):

$$\Phi_1=\frac{1}{T}z^a\int_0^T\left(\frac{c_1\theta(t)}{|\theta(t)|^{\alpha_1}}\right)dt,$$

then

$$\Phi_1=\left(\frac{1}{T}\int_0^T\theta(t)dt\right)\left(\frac{1}{T}\int_0^T\frac{c_1\theta(t)}{|\theta(t)|^{\alpha_1}}dt\right).$$

Next we rewrite the following expression:

$$\Phi_1'=\Phi_1\left|\int_0^T\theta(t)dt\right|^{\alpha_1}$$

By Jensen's inequality, it follows that for any $0<\alpha<1$ (or $\alpha<0$), one can write:

$$\left|\int_0^T\theta(t)dt\right|^{\alpha}\geq\int_0^T|\theta(t)|^{\alpha}dt.$$

.

As result, we obtain:

$$\begin{aligned}\Phi_1'&\geq\Phi_1\int_0^T|\theta(t)|^{\alpha_1}dt \\ &=c_1\left(\frac{1}{T}\int_0^T\theta(t)dt\right)\frac{1}{T}\int_0^T\theta(t)\frac{\int_0^T|\theta(t)|^{\alpha_1}dt}{|\theta(t)|^{\alpha_1}}dt\end{aligned}\tag{14}$$

It is straightforward to show that the following inequality holds:

$$\frac{|\theta(t)|}{|\theta(t)|^{\alpha_1}}\int_0^T|\theta(t)|^{\alpha_1}dt\geq|\theta(t)|.$$

One can then rewrite (14) to obtain:

$$\begin{aligned}&c_1\left(\frac{1}{T}\int_0^T\theta(t)dt\right)\frac{1}{T}\int_0^T\left[\frac{\theta(t)}{|\theta(t)|}\frac{|\theta(t)|}{|\theta(t)|^{\alpha_1}}\int_0^T|\theta(t)|^{\alpha_1}dt\right]dt \\ &=c_1\left(\frac{1}{T}\int_0^T\theta(t)dt\right)\frac{1}{T}\int_0^T\rho_1(t)\theta(t)dt\end{aligned}$$

where $\rho(t)\geq 1\ \forall t\geq 0$.

We arrive at the following result:

$$\Phi_1'\geq\left(\frac{1}{T}\int_0^T\theta(t)dt\right)^2$$

which yields:

$$\Phi_1\geq\frac{\left(\frac{1}{T}\int_0^T\theta(t)dt\right)^2}{\left|\frac{1}{T}\int_0^T\theta(t)dt\right|^{\alpha_1}}=\frac{c_1|z^a|^2}{T^{\alpha_1}|z^a|^{\alpha_1}}.$$

The second term in (13) can be handled using a similar approach. As a result with obtain:

$$\begin{aligned}\dot{V}_1^a\leq& -K\left(c_1\frac{|z^a|^2}{T^{\alpha_1}|z^a|^{\alpha_1}}+c_2\frac{T^{\alpha_2}|z^a|^2}{|z^a|^{\alpha_2}}\right) \\ &+z^a\nabla^2h(x^a) \\ &\times\Big(K\gamma(z^a+\nabla h(x^a))(z^a+\nabla h(x^a))\Big)\end{aligned}\tag{15}$$

As before, we consider the candidate Lyapunov function $V_2^a=\frac{1}{2}\nabla h(x^a)^2$. Its time derivative is given by:

$$\begin{aligned}\dot{V}_2^a=&-\nabla^2h(x^a)\gamma(z^a+\nabla h(x^a)) \\ &\times(z^a+\nabla h(x^a))\nabla h(x^a).\end{aligned}$$

Repeating as in the previous section, we can use V_1^a and V_2^a to demonstrate that the averaged system is FT stable for any $K>K^*$.

Having established that the averaged system achieves the performance of the proposed target system, we must prove that the trajectories of the ESC system approach the trajectories of the averaged system. Since the right hand side of the dynamics are not Lipschitz, but only continuous, the application of standard averaging results that rely on Lipschitz properties is not suitable.

Many suitable averaging results have been proposed in the classical literature. In this study, we consider the classical Krasnosel'skii-Krein theorem Krasnosel'skii and Krein (1955) (generalized by Plotnikova for differential inclusions Plotnikova (2005)) to demonstrate the closeness of solution of the nominal system and the averaged system over a compact set $D\subset\mathbb{R}^2$ as $a\to 0$.

The theorem can be stated as follows.

Theorem 4. Consider the nonlinear system $\dot{X}=f(t,X,\epsilon)$ where

(1) the map $f(t,X,\epsilon)$ is continuous in t and x on $\mathbb{R}_{\geq 0}\times\mathbb{R}^n$,
(2) there exists a positive constant $L>0$ and a compact set $D\subset\mathbb{R}^n$ such that $\|f(t,X,\epsilon)\|\leq L$ for $t\in\mathbb{R}_{\geq 0}$, $x\in D$ and $\epsilon\in[0,\epsilon^*]$,
(3) the averaged system

$$\dot{X}^a=\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t,X^a,0)dt$$

exists with solutions defined on the set D.

Then, for any $\epsilon\leq\epsilon^*$, there exists constants δ and T such that:

$$\|X(t)-X^a\|\leq\delta$$

for $t\in[0,T]$.

For the analysis of the proposed finite-time ESC, the Krasnosel'skii-Krein theorem can be applied as follows.

Consider the state, $X=[x-x^*,\xi]^T$, and the corresponding averaged variables $X^a=[x^a-x^*,\xi^a]^T$. Consider the system's dynamics:

$$\begin{aligned}\frac{d\tilde{x}}{dt}&=-\gamma(\xi)\xi \\ \frac{d\xi}{dt}&=-\gamma(\xi-\delta(\tilde{x}+x^*,t))K(\xi-\delta(\tilde{x}+x^*,t)).\end{aligned}\tag{16}$$

and the corresponding average:

$$\begin{aligned} \frac{d\tilde{x}^a}{dt} &= -\gamma(\xi^a)\xi^a \\ \frac{d\xi^a}{dt} &= -\frac{K}{T}\int_0^T \left(\frac{c_1(\xi^a - \delta(\tilde{x}^a + x^*, t))}{|\xi^a - \delta(\tilde{x}^a + x^*, t)|^{\alpha_1}} \right. \\ &\qquad \left. + \frac{c_2(\xi^a - \delta(\tilde{x}^a + x^*, t))}{|\xi^a - \delta(\tilde{x}^a + x^*, t)|^{\alpha_2}} \right) dt. \end{aligned} \tag{17}$$

By the analysis provided above, the averaged system has a finite-time stable equilibrium at the origin $X = 0$. Furthermore, the solutions of (17) exists and they can be contained in a compact set $D \in \mathbb{R}^2$. Consider the nonlinear system (16). By the smoothness of the cost function $h(x)$ and the periodicity of the dither signal, it follows that the right hand side of the system can be bounded on a compact $D \in \mathbb{R}^2$ uniformly in t. The continuity and the boundedness of the right hand side of (16) over a compact set D over which the solutions of the averaged system exists enables the application of the Krasnosel'skii-Krein theorem to guarantee that for any $a \in (0, a^*)$ there exists a T and a δ such that:

$$\|X(t) - X^a(t)\| \leq \delta, \text{ for } t \in [0, T].$$

Using the finite-time stability property of the averaged system (in particular, the corresponding generalized $\mathcal{K}_\infty$ function) and the averaging result for small amplitude signals, one can apply the approach in the proof of Theorem 1 in Teel et al. (2003) to show that there exists a generalized class $\mathcal{K}_\infty$ function, β_X and a constant, c_X, such that:

$$\|X(t)\| \leq \beta_X(\|X(t_0)\|, t) + c_X$$

for $X(t_0) \in D$.

4. SIMULATION STUDY

We consider the minimization of the cost function: $y = 1 + 50(x-1)^2$. The finite extremum seeking controller is implemented with tuning parameters are: $a = 0.1$, $q_1 = 3$, $q_2 = 1.5$, $c_1 = 1$, $c_2 = 1$, $k = 1$, $K = 250$ and $\omega = 300$. The initial conditions are given by: $x(0) = 0$ and $\xi(0) = 0.01$. The simulation results are shown in Figure 1. The results demonstrate that the ESC brings the system to the unknown optimum $y^* = 1$ in finite-time.

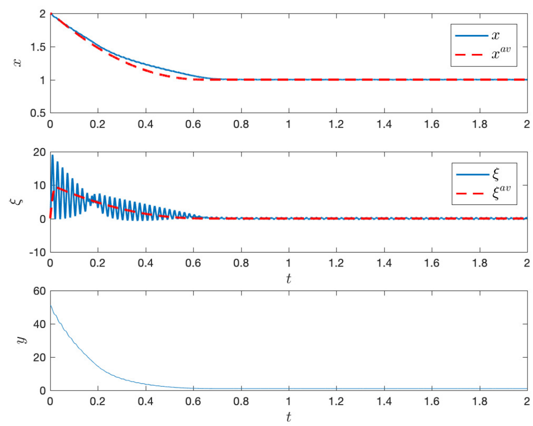

Fig. 1. Performance of the Finite-time ESC. The graph shows the decision variable x and its average x^a , the auxiliary variable ξ and the cost function y.

5. CONCLUSION

In this study, we proposed an ESC design that achieves finite-time convergence to the unknown optimum. In the analysis of the proposed ESC, it shown that the resulting averaged system has a finite-time stable equilibrium at the unknown optimum of the measured cost function. Using classical averaging results for dynamical systems with continuous right hand sides, it is shown that the optimum is practically asymptotically stable equilibrium of the ESC system. In future work, we will consider the development of Newton seeking techniques for multivariable problems. We will also consider the class of static maps subject to actuator limitations such as saturation and quantization.

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