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Decentralised Interpolating Control: A Periodic Invariance Approach

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Abstract: This paper presents a decentralised periodic interpolating control (dpIC) scheme for the constrained control of interconnected systems, which employs *periodic invariance* and *vertex reachability of target sets*. Periodic invariance allows the state of the system to leave a candidate set temporarily but return into the set in a finite number of steps. We consider a periodic invariant set with low-complexity (e.g. rectangle, hexagon for planar systems) to replace the expensive controllable invariant outer set. This set is defined within the controllable stabilising region of each subsystem and a reachability problem is solved off-line for each vertex of the outer set to provide an admissible control sequence that steers the system state back into the original target set after a finite number of steps. dpIC is effectuated between such periodic invariant sets for each subsystem and the local maximal admissible inner set by means of an inexpensive linear programming problem, which is solved on-line at the beginning of each periodic control sequence. dpIC is demonstrated on the problem of stabilising a platoon of vehicles.

Keywords: Set Invariance; Periodic Invariant Sets; Reachability; Interpolating Control

1. INTRODUCTION

Interpolating control (IC) in (Nguyen et al., 2013; Nguyen, 2014) has been introduced as an alternative of the vertex control (Gutman and Cwikel, 1986) for constrained linear time-invariant (LTI) systems. Vertex control exploits only the border of the feasible area and convergence becomes slower as the system approaches the origin. IC overcomes this drawback of slow convergence by providing a smooth transition between a low-gain vertex controller and a high-gain constraint-admissible feedback control, which satisfies some user-defined performance (Nguyen et al., 2012). An explicit formulation and some geometrical properties have been presented in Nguyen et al. (2016), while different interpolating control strategies are discussed in Rossiter and Ding (2010); Nguyen et al. (2014b); Kheawhom and Bumroongsri (2015) Decentralised interpolating control (dIC) (Scialanga and Ampountolas, 2018a,b) is introduced to overcome the computational burden of large-scale systems, dIC approaches decompose the overall system into an interconnected system and solve constrained control problems via distributed interpolation in low-dimensional spaces. To this end, separable invariant sets are computed and the control is obtained as solution of a low-dimensional linear programming (LP) problem for each subsystem.

Both centralised and decentralised IC approaches rely on the availability of controllable invariant sets associated with different closed loop objectives (local performance or the enlargement of the stabilising set) and use the interpolation as a tool for constructing a smooth state feedback function. Broadly speaking, interpolating control asks for the computation of an approximation of the controllable area (Nguyen, 2014; Kheawhom and Bumroongsri, 2015), which algorithms might not converge converge in finite time (Borrelli et al., 2017).

To alleviate these shortcomings, the present paper presents a low-complexity decentralised periodic interpolating control (dpIC) scheme for the constrained control of interconnected systems, which employs *periodic invariant sets* and *vertex reachability of target sets. Periodic invariant sets* are computed for each subsystem and used to enlarge the stabilising region in case that *maximal controllable invariant sets* cannot be determined or are unknown during the design process. A periodic invariant set is defined in the local controllable area of the subsystem and a sequence of local controls that guarantee periodic invariance are computed as solution of a reachability problem, both procedures being tractable off-line. For the interpolation, an inexpensive LP problem is solved at the beginning of each periodic invariance sequence. The resulting constrained interpolating control guarantees stability of the interconnected system despite the unknown couplings. The proposed dpIC offers a fair approximation of controllable invariant sets in a low-dimensional low-complexity structure and provides an alternative to the decentralised IC (Scialanga and Ampountolas, 2018b,a) while overcoming its main limitation, that is, the availability of controllable invariant sets. Similar to the other decentralised approaches (Mayne et al., 2005; Riverso et al., 2013; Grancharova and Johansen, 2014), dpIC is meant for weakly coupled interconnected systems where couplings are treated as disturbances. While this requirement can lead to a degree of conservatism, we show that dpIC can be applied to real world applications.

2. PRELIMINARIES

2.1 System Dynamics and Constraints

Consider a discrete-time linear time-invariant interconnected system consisting of N subsystems,

$$\mathcal{S}_i : x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j\in\mathcal{M}_i} e_{ij}\bar{A}_{ij}x_j(k), \quad (1)$$

where the local state variable $x_i \in \mathbb{R}^{n_i}$ is regulated by a local control input $u_i \in \mathbb{R}^{m_i}$, $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. $A_i \in \mathbb{R}^{n_i\times n_i}$ and $B_i \in \mathbb{R}^{n_i\times m_i}$ are the state and control matrices; and, $\bar{A}_{ij} \in \mathbb{R}^{n_i\times n_j}$ is an interconnection (adjacency) matrix between subsystem i and j, where $\mathcal{M}_i$ is the set of neighbour subsystems to i for information exchange; $e_{ij} \in [0, 1]$ are weighting constants, modelling the strength of adjacent interconnections. The local variables are subject to local constraints,

$$\begin{cases} x_i(k) \in \mathcal{X}_i,\ \mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid F_{x_i}x_i \le g_{x_i}\}, \\ u_i(k) \in \mathcal{U}_i,\ \mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid F_{u_i}u_i \le g_{u_i}\}, \end{cases} \quad (2)$$

$\forall\, k \ge 0$, $i \in \mathcal{N}$, where F_{x_i}, F_{u_i} are constant matrices and g_{x_i}, g_{u_i} are constant vectors of appropriate dimension with positive elements. The decentralised constrained control problem of the interconnected system (1) is to design a controller that regulates each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, to the origin under bounded and locally unknown interconnections, while verifying the constraints (2). We assume that the couple (A_i, B_i), $i \in \mathcal{N}$, define a controllable system and there exists a local stabilising state-feedback controller $u_i(k) = K_i x_i(k)$, $\forall i \in \mathcal{N}$. Finally, the resulting closed-loop state matrix $A_i + B_i K_i$, $i \in \mathcal{N}$, is Hurwitz. The interconnected system (1) will be re-written as,

$$\mathcal{S}_i : x_i(k+1) = A_i x_i(k) + B_i u_i(k) + D_i w_i(k), \quad (3)$$

where the term $D_i w_i(k) = \sum_{j\in\mathcal{N}_i} e_{ij}\bar{A}_{ij}x_j(k)$, $i \in \mathcal{N}$, $w_i \in \mathbb{R}^{n-n_i}$ accounts for the interconnection uncertainty. From the constraints of the local states $x_j \in \mathcal{X}_j$, $j \in \mathcal{M}_i$, a set of local constraints for the interconnection variable w_i, $i \in \mathcal{N}$, is defined by polytopic sets,

$$w_i(k) \in \mathcal{W}_i = \{w_i \in \mathbb{R}^{n_i} \mid F_{w_i}w_i \le g_{w_i}\}, \quad i \in \mathcal{N} \quad (4)$$

with matrix F_{w_i} and vector g_{w_i} of appropriate dimensions.

2.2 Set Invariance and Periodic Invariance

This section provides definitions on set invariance and periodic invariance for the constrained LTI system (3) with constraints (2) and (4) as presented in Lee and Kouvaritakis (2006); Borrelli et al. (2017); Blanchini and Miani (2015); Olaru et al. (2014). These will be used in the rest of the paper for the dynamics of each subsystem $\mathcal{S}_i$, $\forall i \in \mathcal{N}$. Below, we omit the index i for clarity.

Definition 2.1. (Constraint-admissible Robust Invariance). Given the local controller $u_i(k) = K_i x_i(k)$, the set $\Omega \subseteq \mathcal{X}$ is a *robust positively constraint-admissible invariant set* with respect to $x(k+1) = A_K x(k) + Dw(k)$, where $A_K = A+BK$, and subject to the local constraints (2) and (4), if $\forall\, x(0) \in \Omega$ and $\forall\, w(k) \in \mathcal{W}$, the system evolution satisfies $x(k) \in \Omega$ and $Kx(k) \in \mathcal{U}$, $\forall\, k \ge 0$.

The largest Constraint-admissible Robust Invariant Set (CaRIS) for the system (3) in closed loop with a stabilizing controller $u_i(k) = K_i x_i(k)$, that respects constraints (2) can be determined under mild conditions. It can be characterised in polyhedral form as $\Omega = \{x \in \mathbb{R}^n : F_\Omega x \le g_\Omega\}$, where F_Ω is a constant matrix and g_Ω is a constant vector of appropriate dimensions.

Definition 2.2. (Robust Controllable Invariance). Given system (3) and the constraints (2), (4), the set $\Psi \subseteq \mathcal{X}$ is *robust controllable invariant*, if $\forall\, x(k) \in \Psi$, there exists an admissible control $u(k) \in \mathcal{U}$ such that $x(k+1) \in \Psi$, $\forall\, w(k) \in \mathcal{W}$, $\forall\, k \ge 0$.

The maximal robust controllable invariant set Ψ might not be finitely determined within the class of polyhedral sets (Borrelli et al. (2017)). However, in the sequel, a polyhedral approximation will be considered with the half-space representation given by $\Psi = \{x \in \mathbb{R}^n : F_\Psi x \le g_\Psi\}$, where F_Ψ is a constant matrix and g_Ψ is a constant vector of appropriate dimensions. The previous definition can be seen as a limit case of λ-contractiveness as the next definition highlights.

Definition 2.3. (Robust λ-contractive Set). Given a scalar $\lambda \in (0, 1]$, a set $\Psi \subseteq \mathcal{X}$ containing the origin is called *robust controllable λ-contractive* for (3) with respect to (2), (4), if for any $x(k) \in \Psi$ there exists $u \in \mathcal{U}$ such that $x(k+1) \in \lambda\Psi$, $\forall w(k) \in \mathcal{W}$.

If $\lambda = 1$, the set is the robust controllable invariant set.

Definition 2.4. (Robust Periodic Invariant Set). For a given $\lambda \in (0, 1]$ the set $S \subset \mathbb{R}^n$ containing the origin is called *robust controllable periodic λ-contractive* with respect to the system (3) and constraints (2)-(4) if there exists a positive number $p > 0$ such that for any $x(0) \in S$ there exists an admissible control sequence $u(\ell) \in \mathcal{U}$, $\ell = 0, \ldots, p-1$, such that $x(p) \in \lambda S$ holds. If $\lambda = 1$ the set is called *robust controllable periodic invariant.*

2.3 Interpolating Control (IC) principles

This section presents the interpolating control (IC) scheme introduced as improved vertex control (Gutman and Cwikel, 1986). IC relies on the smooth interpolation between a vertex controller and an optimal high-gain feedback controller (Nguyen, 2014). Fig. 1(a) depicts the idea behind the interpolating control method. The set Ψ depicted in yellow is denoted as *outer set* and the CaRIS Ω denoted as *inner set* and depicted in red. Any $x(k) \in \Psi$ can be decomposed as a convex combination,

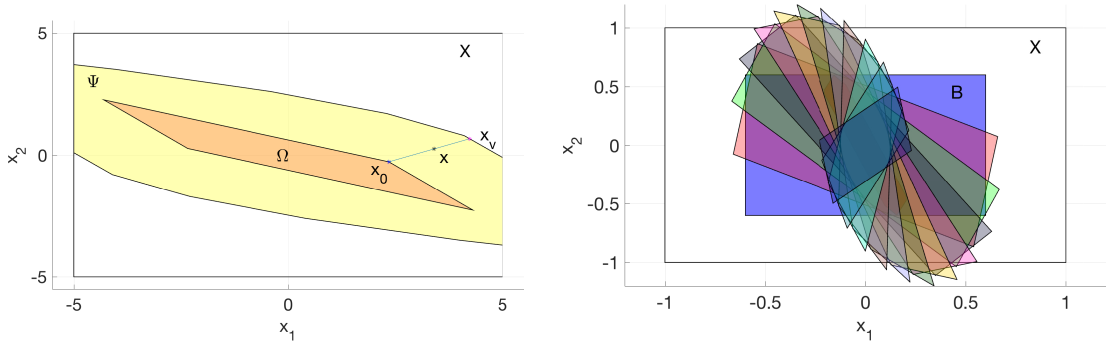
$$x(k) = s(k)\, x_v(k) + (1 - s(k))\, x_0(k), \quad (5)$$

where $x_v(k) \in \Psi$ and $x_o(k) \in \Omega$, and $s(k) \in [0, 1]$ plays the role of interpolating coefficient. At each time instant, given the coefficient $s(k)$, one can obtain the control:

$$u(k) = s(k)u_v(k) + (1 - s(k))u_0(k), \quad (6)$$

where $u_0(k) = K\, x_0(k)$ is an inner stabilising controller associated to the CaRIS and $u_v(k)$ is the vertex control applied to $x_v(k)$. The control (6) provides a smooth transition between the two controllers and convergence to the minimal robust invariant set. Consider the change of variables $r_0 = (1-s)x_0$ and $r_v = sx_v$. It follows $r_0 \in (1-s)\,\Omega$ and $r_v \in s\,\Psi$. The decomposition (5) can be rewritten as $r_0 = x - r_v$. To solve the interpolation problem, an LP problem is formulated (index k is omitted for clarify):

$$\min_{s, r_v} s, \quad \text{subject to:} \begin{cases} sg_\Omega - F_\Omega r_v \le g_\Omega - F_\Omega x, \\ -sg_\Psi + F_\Psi r_v \le 0, \\ 0 \le s \le 1, \end{cases} \quad (7)$$



(a) (b)

Fig. 1. (a) The current state x can be decomposed as a convex combination of $x_v \in \partial\Psi$ and $x_0 \in \partial\Omega$. (b) Periodic invariance idea.

where the zero in the second inequality is a vector of zeros with appropriate dimension. The solution of (7) is the interpolating coefficient s^* and the vector r_v^*. The original state can be recovered from $r_0^* = x - r_v^*$. The solution of (7) leads to an admissible control action (6) at each time step that stabilises the constrained system (Nguyen et al., 2013). Once the state enters Ω, the interpolation control is equivalent to the stabilising controller $u_0(k) = Kx_0(k)$.

3. PERIODIC INTERPOLATING CONTROL

3.1 Construction of Periodic Controlled Invariant Sets

The following arguments apply to each constrained interconnected LTI subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, in (3). Consider a low-complexity polytope (e.g. hyper-rectangle). This represents a candidate periodic invariant set for our control problem. Since it is not a traditional invariant set, it has to be defined within the controllable area of the constrained system (2), (3), (4). Additionally, it will be imposed to contain the *inner set* which is à priori available (as a robust controlled invariant set with respect to a linear stabilising controller). Although the set is defined in the controllable area, it cannot guarantee its invariance with respect to the evolution of the state. Since for this particular outer set the vertices are known beforehand, an optimisation problem, similar to the one presented in Nguyen et al. (2014a), can be formulated that guarantees by its feasibility that $\forall x \in \mathcal{P}$, the states re-enter $\mathcal{P}$ in a finite number of steps. This takes the form of a reachability problem that is solved off-line for each vertex of the outer set. It provides an admissible control sequence that steers the state of the system back into the original target set despite the unknown but bounded couplings between subsystems. Fig. 1(b) shows an initial rectangle $\mathcal{B}$ that verifies the state constraints (in white). The sequence of sets describes the evolution of the state $x \in \mathcal{B}$ for an admissible control sequence. Periodic invariance allows for the state vector to leave the set temporarily but return in a finite number of time steps, i.e., to leave the set for $k < p$, where p is the length of the period. In a finite number of steps the evolution of each $x \in \mathcal{B}$ is steered inside $\mathcal{B}$ with an admissible sequence of inputs computed off-line. The sequence of sets is plotted to show the periodic invariance idea and how the period length is determined.

Reachable sets describe the evolution of the system to target regions. The next section introduces a *reachability problem* for interconnected systems. It is solved *off-line* and allows for distributed periodic interpolating control.

p-step reachability problem for interconnected systems: Let v_{h_i}, $h_i = 1, \ldots, \Upsilon_i$, with $\Upsilon_i \geq n_i + 1$ be the vertices of the polytope. The reachability problem considers the action of the disturbances to the system at each time step of the period to determine the p_{h_i} control sequence $u_{h_i}^v = \left\{u_{h_i}^v(1), \ldots, u_{h_i}^v(p_{h_i})\right\}$ that steers the h_i-th vertex v_{h_i} of $\mathcal{P}_i$ back into the target set in a number of finite steps.

For (3), since interconnections are bounded, the reachability problem considers the worst case interconnections when computing the control sequence $u_{h_i}^v$, $i \in \mathcal{N}$. Define $\bar{w}_i$ as the worst case local interconnection for $\mathcal{S}_i$, $i \in \mathcal{N}$ (practically, $\bar{w}_i$ are the vertices of the set $\mathcal{W}_i$). The *reachability problem* associated to the problem (2), (3), (4) and polytope $\mathcal{P}_i$ reads (indices i and h_i are omitted for clarity):

$$\lambda\left(u^v(0), \ldots, u^v(p-1)\right) = \min_{u^v, \lambda} \lambda$$

subject to:

$$\begin{cases} A\,v + Bu^v(0) + \bar{w} \in \mathcal{X}, \\ \vdots \\ A^{p-1}\,v + A^{p-2}Bu^v(0) + A^{p-2}\bar{w} + Bu^v(p-2) + \bar{w} \in \mathcal{X} \\ A^p\,v + A^{p-1}Bu^v(0) + A^{p-1}\bar{w} + Bu^v(p-1) + \bar{w} \in \lambda\mathcal{P} \\ u^v(k) \in \mathcal{U}, \quad k = 0, \ldots, p-1, \quad 0 \leq \lambda < 1. \end{cases} \tag{8}$$

The reachability problem provides a suitable local control sequence that steers each vertex of the polytope $\mathcal{P}_i$, $i \in \mathcal{N}$, into the polytope in a contractive way, while verifying the system constraints and interconnections. For each subsystem $\mathcal{S}_i$, a period length is defined as:

$$p_i = \text{l.c.m.}\, p_{h_i}, \quad h_i = 1, \ldots, \Upsilon_i,$$

where l.c.m. stands for *least common multiple*. The inputs obtained from (8) are stored and then used to compute the decentralised pIC control while the state is outside Ω_i, $i \in \mathcal{N}$, as described in the next section.

3.2 Decentralised Periodic Interpolating Control

This section presents the proposed dpIC for the decentralised LTI interconnected system (3) subject to (2)–(4). In this approach, the weakly coupled interconnected systems are treated as disturbances. The first step is to introduce a local state feedback controller $u_i(k) = K_i x_i(k)$ for each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, and under the assumption it exists [1], compute the local CaRIS Ω_i which will play the role of the *inner set* for the constrained system. Then, a

[1] The existence is related to satifaction of input/state constraints by the minimal robust positive invariant set.

low-dimensional low-complexity outer set $\mathcal{P}_i$ is defined as a polytope. That is, the local robust periodic invariant set for the system $\mathcal{S}_i$, $i \in \mathcal{N}$ and will play the role of the *outer set*. A reachability problem formulation for decentralised control guarantees that, for every local initial state $x_i \in \mathcal{P}_i$, a sequence of p_i control steps, $i \in \mathcal{N}$, steers the state into the local target polytope in a contractive way, while verifying the state and control constraints.

Online decentralised periodic interpolating control

Consider the initial state $x_i(0) \in \mathbb{R}^{n_i}$ of a subsystem inside the corresponding outer set P_i (and target set of periodic control), $i \in \mathcal{N}$. A scaling factor $\lambda_i^* \in [0,1]$ can be computed such that the initial state is contained in the contractive set $\lambda_i^*\mathcal{P}_i$. λ_i^* can be considered as the smallest contractive factor such that $x_i(0) \in \lambda_i^*\mathcal{P}_i$, and can be obtained by solving the LP problem:

$$\lambda_i^* = \min_{\lambda_i} \lambda_i \quad \text{subject to:} \begin{cases} F_{\mathcal{P}_i}\, x_i \le \lambda_i\, g_{\mathcal{P}_i}, \\ 0 \le \lambda_i \le 1, \end{cases} \tag{9}$$

$i \in \mathcal{N}$, where $F_{\mathcal{P}_i}$ and $g_{\mathcal{P}_i}$ are the matrix and vector that define the half-space representation of $\mathcal{P}_i$. Then, $\lambda_i^*\mathcal{P}_i$ can be set as the target set for the periodic control sequence.

The state $x_i(0)$ can be decomposed as $x_i(0) = s_i(0)\, x_i^v(0) + (1 - s_i(0))\, x_i^0(0)$ by solving the LP problem (7) with $\Psi_i = \mathcal{P}_i$. The states x_i^v and x_i^0 lie on the border of $\mathcal{P}_i$ and Ω_i, respectively. Then, $x_i^v(0)$ can be written as a convex combination of the vertices of the outer set $\mathcal{P}_i$, i.e.,

$$x_i^v(0) = \sum_{h_i=1}^{\Upsilon_i} \alpha_{h_i}(0)\, v_{h_i}, \quad \alpha_{h_i} \ge 0, \quad \sum_{h_i=1}^{\Upsilon_i} \alpha_{h_i} = 1, \tag{10}$$

where α_{h_i}, $i = 1, \dots, \Upsilon_i$ are convexity coefficients in the unit simplex. The control action of each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$, at $k = 0$ is a convex combination of the state feedback control applied to the state $x_i^0(0)$ and the combination of the controls applied to the vertices v_{h_i}, as in the decomposition (10), i.e., $u_i(0) = s_i(0) \sum_{h=1}^{\Upsilon_i} \alpha_{h_i}(0)\, u_{h_i}^v(0) + (1 - s_i(0))K_i x_i^0(0)$, $i \in \mathcal{N}$, where $u_{h_i}^v(0)$ is the first element of the control sequence (8) applied to the vertex v_{h_i}. For the next $p_i - 1$ steps, consider the p_i-sequence that is available from the reachability problem (8), to obtain the control,

$$\begin{aligned} u_i(k) = s_i(0) &\sum_{h_i=1}^{\Upsilon_i} \alpha_{h_i}(0)\, u_{h_i}^v(k) \\ &+ (1 - s_i(0))K_i(A_i + B_iK_i)^k x_i^0(0), \end{aligned} \tag{11}$$

for $k = 0, \dots, p_i - 1$. The control (11) is applied to (3) for p_i steps or until the state reaches one of its target sets, i.e., either the admissible set Ω_i or the contractive set $\lambda_i^1\mathcal{P}_i$, where λ_i^1 is the scaling factor associated to the first periodic cycle. The control (11) guarantees that the initial state $x_i(0)$ enters the contractive polytope $\lambda_i^1\mathcal{P}_i$ in p_i steps maximum. After the state returns into the set, a new periodic sequence is computed. Note that in (11), the interpolating coefficient s_i and the coefficients α_{h_i}, $h_i = 1, \dots, \Upsilon_i$, in the convex combination (10) are kept constant, i.e. $s_i(k) = s_i(0)$ and $\alpha_{h_i}(k) = \alpha_{h_i}(0)$, $k = 1, \dots, p_i$, $h_i = 1, \dots, \Upsilon_i$.

The contractive factor λ_{i2} associated to each target set $\mathcal{P}_i$ is updated for the new state $x_i(\bar{k})$ by solving the LP problem (9), where $\bar{k}$ is the first time step of the periodic sequence. The current state would be inside $\lambda_{i2}\mathcal{P}_i$,

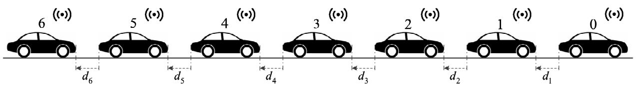


Fig. 2. Vehicle platooning (Liu and Zamani, 2019).

$\lambda_{i2} < \lambda_{i1}$, where $\mathcal{P}_i$ is the outer set of the periodic IC. After a new λ_i is obtained, a new interpolating decomposition $(s_i(\bar{k}), x_i^v(\bar{k}), x_i^0(\bar{k}))$ is computed between the outer set $\mathcal{P}_i$ and the inner set Ω_i based on (7). The outer state is defined as convex combination of some of the vertices of the polytope as in (10) with coefficients $\alpha_{h_i}(\bar{k})$, $h_i = 1, \dots, \Upsilon_i$. Similar to (11) applied to the initial local state, a sequence of pIC associated to the new local states is applied to the subsystems $\mathcal{S}_i$, $i \in \mathcal{N}$, i.e.,

$$\begin{aligned} u_i(\bar{k} + k) = s_i(\bar{k}) &\sum_{h_i=1}^{\Upsilon_i} \alpha_{h_i}(\bar{k})\, u_{h_i}^v(\bar{k} + k) \\ &+ (1 - s_i(\bar{k}))K_i(A_i + B_iK_i)^k x_i^0(\bar{k}), \end{aligned} \tag{12}$$

for $k = 0, \dots, p_i - 1$, where $s_i(\bar{k})$, $i \in \mathcal{N}$, are the new interpolating coefficients to be kept constant in the new periodic sequence.

To summarise, for each subsystem i a contractive factor λ_i is computed at the beginnning of the periodic cycle and (5) is obtained as solution of (7). The outer state is decomposed as in (10) and the periodic interpolating control (12) is applied to the state for p_i steps or until it reaches either the current $\lambda_i\mathcal{P}_i$ or the local CaRIS Ω_i. If the state enters the scaled target set, a new periodic sequence is computed. In case the state converges into CaRIS in less than p_i steps, the control action reduces to $u_i = K_i x_i$.

3.3 Recursive Feasibility and Asymptotic Stability

This section provides recursive feasibility and asymptotic stability theorems for the proposed decentralised pIC for each constrained subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$.

Theorem 1. The decentralised periodic interpolation problem (5), (6), (7), (12), (8) is p_i-step feasible for linear time invariant interconnected systems $\mathcal{S}_i$, $i \in \mathcal{N}$, and p-step feasible for the overall system (3), $\mathcal{S} = \bigcup_{i\in\mathcal{N}} \mathcal{S}_i$, where p is the overall period length defined as $p = \underset{i\in\mathcal{N}}{\text{l.c.m.}}\, p_i$, with constraints (2)–(4) for all states $x \in \mathcal{P} = \prod_{i\in\mathcal{N}} \mathcal{P}_i \subseteq \mathbb{R}^n$, where $x = \left[x_1^\intercal, \dots, x_N^\intercal\right]^\intercal$: $\forall x_i(k) \in \mathcal{P}_i \implies x_i(k + p) \in \mathcal{P}_i, \forall x(k) \in \mathcal{P} \implies x(k+p) \in \mathcal{P}$.

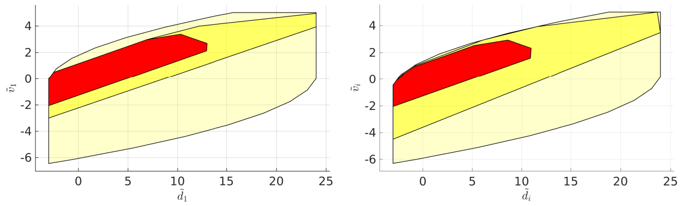
Proof. Proof is omitted due to space limitation.

Theorem 2. The decentralised periodic interpolating control (6), (7), (12), (8) guarantees asymptotic stability of the linear time invariant interconnected systems (3), $\mathcal{S}_i$, $i \in \mathcal{N}$, and the overall systems $\mathcal{S} = \bigcup_{i\in\mathcal{N}} \mathcal{S}_i$ with state constraints $\mathcal{X} = \prod_{i\in\mathcal{N}} \mathcal{X}_i$, control constraints $\mathcal{U} = \prod_{i\in\mathcal{N}} \mathcal{U}_i$, and coupling constraints $\mathcal{W} = \prod_{i\in\mathcal{N}} \mathcal{W}_i$ for any initial point $x(0) \in \mathcal{P}$, $\mathcal{P} = \bigcup_{i\in\mathcal{N}} \mathcal{P}_i$.

Proof. Proof is omitted due to space limitation.

4. APPLICATION TO VEHICLE PLATOONING

A platoon of $N + 1$ vehicles depicted in Fig. 2. The leader vehicle is marked with index 0 and the following vehicles have ordered increasing indices. Consider the simplified



(a) (b)

Fig. 3. Invariant set of subsystem $\mathcal{S}_1$ (left) and subsystems $\mathcal{S}_i$, $i = 2, \ldots, 6$ (right).

description of an interconnected platoon (Sadraddini and Belta, 2018) with six vehicles and one leader. The interconnected system $\mathcal{S}_i$ can be written as:

$$x_i(k+1) = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} x_i(k) + \begin{bmatrix}0\\1\end{bmatrix} u_i(k) + \begin{bmatrix}0 & \epsilon\\0 & 0\end{bmatrix} x_{i-1}(k) + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} w_i. \tag{13}$$

The state variable of each follower is defined as the local state variable $x_i = [d_i \quad v_i]^\mathsf{T}$, where d_i is the relative distance between the i-th vehicle and the $(i-1)$-th vehicle and the leader vehicle is denoted as 0-th vehicle; v_i is the vehicle speed; u_i is the local control input; the constant ϵ defines the interconnection degree with the preceding vehicle; w_i is the local disturbance present in the system.

We consider the interconnections as disturbances in addition to natural disturbances already present in the system. The control objective is to group together the vehicles in a platoon with steady-state speed and spacing, v_e and Δx_e, respectively, where $\Delta x_e = \Delta x_{min} + hv_e$, with h is a constant time headway, Δx_{min} is the minimum gap at zero speed. System (13) is subject to local constraints in order to guarantee stability and avoid collision:

$$3 \leq d_i \leq 30\,[\text{m}], \quad 0 \leq v_i \leq 20\,[\text{m/s}], \tag{14a}$$

$$-0.9 \leq u_i \leq 1, \quad -\lambda \begin{bmatrix}0.1\\2\end{bmatrix} \leq w_i \leq \lambda \begin{bmatrix}0.1\\2\end{bmatrix}\,[\text{m/s}], \tag{14b}$$

where $\lambda = 0.06$. The platoon aims to reach the speed $v_e = 15$ m/s, with time headway $h = 0.2$ s, and $\Delta x_e = 6$ m. In order to transform the control problem into a regulation to the origin problem, we introduce a change in the system variables: $\tilde{x}_i = d_i - \Delta x_e$ and $\tilde{v}_i = v_i - v_e$. The system matrices in (13) keep the same values but new constraints are defined as (for each $i \in \mathcal{N}$):

$$-3 \leq \tilde{d}_i \leq 24 \quad [\text{m}], \qquad -15 \leq \tilde{v}_i \leq 5 \quad [\text{m/s}]. \tag{15}$$

The control constraints u_i and disturbances w_i were unchanged. We simulate the constrained interconnected system (13), (14b), (15), composed of $N = 6$ followers and one leading vehicle. The weight ϵ is set to 0.03.

We compare the proposed dpIC with the dIC (Scialanga and Ampountolas, 2018b). For each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$ a high-gain feedback controller $u_i = K_i x_i$ is computed with weighting matrices $Q_i = \text{diag}([0.1, 0.1])$ and $R_i = 1$. Then, the positively invariant sets Ω_i are computed with respect to the constrained system and gain matrices $K_i = [0.2054 \ -0.7835]$, $i \in \mathcal{N}$; and the maximal controllable invariant sets Ψ_i, $i \in \mathcal{N}$ to implement the IC.

To implement dpIC and reduce the complexity of the outer controllable IS, consider a local low-complexity polyhedron $\mathcal{P}_i \subseteq \mathbb{R}^{n_i}$ with a smaller number of vertices compared with the outer set Ψ_i for each subsystem $\mathcal{S}_i$, $i \in \mathcal{N}$. Figs 3(a) and 3(b) depict the Ω_i (in red colour) and the maximal controllable set (in yellow colour). The sets in light yellow colour are the local polyhedrons that are user-defined. In reference to Fig. 3(a), the inner set has vertex representation with 7 vertices, the maximal controllable set has 18 vertices, while the low-complexity polyhedron is defined with 8 vertices. The high number of elements in the vertex representation of the outer set Ψ_1 explains how the local polyhedron $\mathcal{P}_1$ would lead to a reduction of complexity in the set representation and online computations. The low-complexity polyhedron $\mathcal{P}_1$ has period $p_1 = 1$ with reference to the reachability problem (8). Sets in Fig. 3(b) are the invariant sets for the subsystems $\mathcal{S}_i$, $i = 2, \ldots, 6$. $\mathcal{P}_i$, $i = 2, \ldots, 6$, have period length $p_i = 3$.Consider the following simulation scenario:

$$x_0 = [d_1 \ v_1 \ d_2 \ v_2 \ d_3 \ v_3 \ d_4 \ v_4 \ d_5 \ v_5 \ d_6 \ v_6]^\mathsf{T},$$

$$x_0 = [4 \ 12.6 \ 5 \ 11.5 \ 6.5 \ 13.3 \ 6.4 \ 11.8 \ 7 \ 13 \ 4 \ 12.4]^\mathsf{T}.$$

Fig. 4 shows that both decentralised control approaches stabilise the system around its target, $\Delta x_e = 6$ m and $v_e = 15$ m/s. Decentralised pIC provide similar control action and almost identical control evolution as decentralised IC without the necessity of computing the expensive maximal robust controllable invariant set. The online computations of the improved IC with periodic sets requires 0.512 CPU-secs compared with 1.291 CPU-secs that dIC needs for its online computations (CPU: Intel Core i7-3770S 3.1GHz; MATLAB 2016b). It is expected that the improvements introduced with the periodic IC should increase for large-scale interconnected systems. Fig. 5 depicts the interpolating coefficients for pIC and IC. As expected, all coefficients are non-increasing and positive functions. Decentralised pIC and IC steer the state variable into the stabilising inner set Ω_i, $i \in \mathcal{N}$, in less than 5 steps. Finally, these results underline that decentralised pIC provide a stabilising control action similar to the decentralised IC, while avoiding the computation of the (centralised) controllable invariant sets and with less on-line computational effort.

5. CONCLUSION

This work presented a novel low-complexity decentralised periodic interpolating control scheme for the constrained control of interconnected systems. Each subsystem is coupled with a low-complexity approximation of the controllable set and a reachability problem is solved off-line for each vertex of the outer set. This determines a sequence of admissible controls that steer the local state back into the original target set after a finite number of time steps (i.e. enforcing its periodic controlled invariance). For the interpolation, an LP problems is solved at the beginning of each periodic cycle. A numerical application showed that dpIC provides similar performance to previously proposed IC schemes, while it guarantees convergence and satisfaction of constraints, though it employs a naive rectangular representation of the controllable invariant set.

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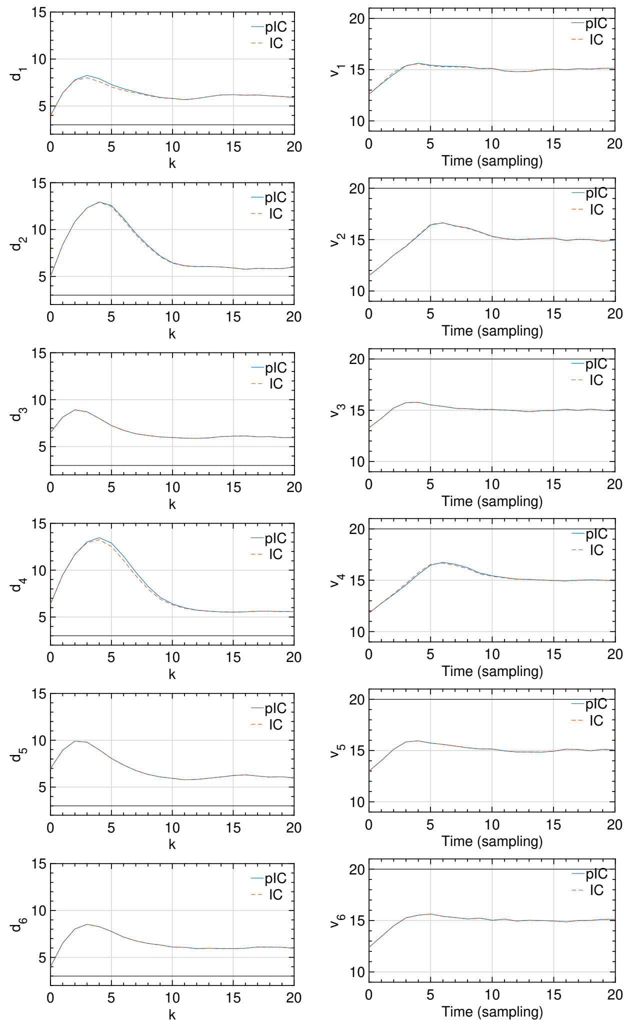

Fig. 4. State trajectories for dpIC and IC.

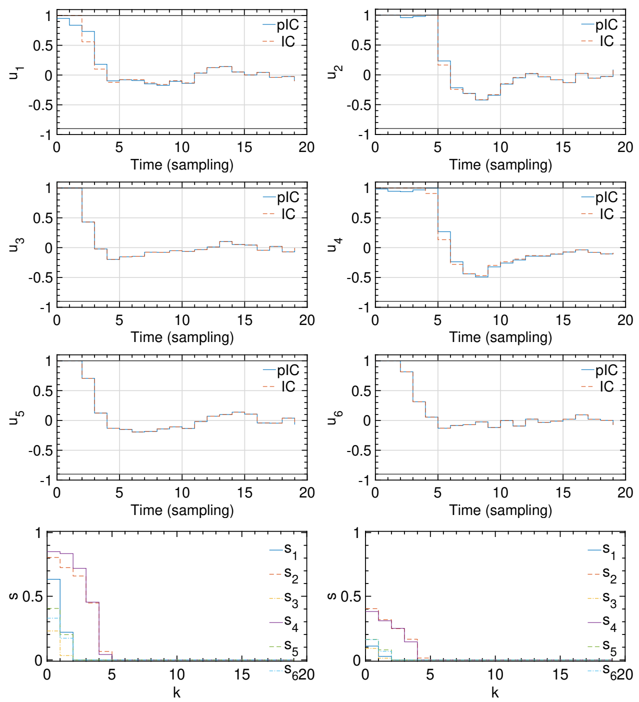

Fig. 5. Control & interpolating coefficients for IC and dpIC.

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