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Compositional Synthesis of Symbolic Models for Infinite Networks ⋆

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Abstract: In this paper, we provide a compositional method for the construction of symbolic models (a.k.a. finite abstractions) for infinite networks of discrete-time control systems. The concrete infinite network and its symbolic model are related by a so-called alternating simulation function which allows one to quantify the mismatch between the output behavior of the infinite interconnection of concrete subsystems and that of their symbolic models. We show that such an alternating simulation function can be obtained compositionally by assuming some small-gain type conditions and composing so-called local alternating simulation functions constructed for subsystems. Assuming certain stability property of concrete subsystems, we also provide a technique to synthesize their symbolic models together with their corresponding local alternating simulation functions. Finally, we apply our results to a traffic network divided into infinitely many cells.

Keywords: Compositionality, interconnected systems, small-gain condition, infinite networks, symbolic models.

1. INTRODUCTION

Large-scale networks appear in a wide variety of modern applications, including integrated circuits, traffic networks and transportation systems. In many applications, a system is considered as a finite but very large network with possibly unknown number of subsystems. Hence, it is reasonable to over-approximate such a system by an *infinite network* which is seen as an interconnection of infinitely many finite-dimensional subsystems. In that way, one ultimately aims at proposing methods which are *independent* of system size (i.e. scale-free) and specifically suitable for the original large-but-finite network.

The costs of incorrect configuration as well as safety and security concerns require automated and provably correct techniques for the verification and synthesis of complex systems. Moreover, emergent applications necessitate sophisticated control objectives, which go well beyond standard goals pursued in classic control theory. A promising direction to address these issues is to use automated controller synthesis based on symbolic models (a.k.a. finite abstractions). However, an efficient approach to the large-scale and possibly infinite networks is still missing. As the computational complexity of constructing symbolic models often scales exponentially with the dimension of the state space, a brute force approach to large-scale systems is not feasible.

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The dimensionality problem can be addressed by decomposing the overall system into a number of lower-dimensional subsystems for which individual abstractions can be efficiently computed. This methodology called compositional approach has received considerable attention; (see, e.g., Meyer et al. (2017); Pola et al. (2018); Swikir et al. (2018); Swikir and Zamani (2019c,b) and references therein). However, all compositional techniques for the construction of symbolic models introduced so far in the literature are tailored to networks composed of finitely many subsystems and can *not* be directly applied to networks consisting of an infinite number of components.

Related Work. Construction of symbolic models for infinite dimensional systems is already proposed in Pola et al. (2010); Girard (2014); Pola et al. (2015); Jagtap and Zamani (2020). In Pola et al. (2010), symbolic models are constructed for nonlinear continuous time-delay systems with known and constant delays. This work was extended in Pola et al. (2015) to the same class of systems with unknown and time-varying delays. The results in Girard (2014) provide a generic state-space discretization-free approach for computing symbolic models of finite or infinite dimensional incrementally stable systems. A state-space discretization-free approach was also introduced in Jagtap and Zamani (2020) for designing symbolic models for infinite dimensional stochastic systems, particularly, retarded jump-diffusion systems. While the results in Pola et al. (2010, 2015); Jagtap and Zamani (2020) deal with time-delay systems evolving over finite-dimensional state spaces, here we deal with an interconnection of infinitely many finite-dimensional subsystems evolving over infinite-dimensional state spaces. The result in Girard (2014) deals with a single incrementally stable infinite-dimensional system with finite-dimensional input space and the finite abstraction is based on input sequences which is not the case in this paper. In this work both state and input spaces of the interconnected system is infinite-dimensional and the construction of symbolic models is based on the discretization of both state and input spaces. Moreover, all the proposed results in Pola et al. (2010); Girard (2014); Pola et al. (2015); Jagtap and Zamani (2020) take a monolithic view of the systems while constructing finite abstractions. However, our main result provides a compositional approach on the construction of symbolic models of interconnected systems using those of subsystems.

This work proposes a compositional methodology for constructing symbolic models for a network composed of countably infinite number of finite-dimensional systems. To the best of our knowledge, this paper is the first attempt to provide a framework for synthesizing symbolic models for infinite networks. We first recall a notion of so-called alternating simulation function introduced in Swikir and Zamani (2019b) to relate two infinite networks. Alternating simulation functions provide upper bounds for the mismatch between the output behaviors of two infinite networks. Then, based on the recently developed small-gain theorem Dashkovskiy et al. (2019), we show that this alternating simulation function can be constructed by composing so-called local alternating simulation functions relating each finite-dimensional subsystems and their symbolic models. Moreover, we provide a technique to synthesize symbolic models together with their corresponding local alternating simulation functions for concrete finite-dimensional subsystems under some assumptions ensuring their incremental input-to-state stability.

We verify the effectiveness of our proposed technique by applying it to a model of a road traffic network containing infinitely many cells (subsystems). We construct symbolic models for the original subsystems and compositionally construct an alternating simulation function from the interconnection of infinitely many symbolic models to the interconnection of the concrete subsystems. We also design controllers compositionally maintaining the density of traffic between 10 and 25 vehicles per cell.

2. NOTATION AND PRELIMINARIES

2.1 Notation

We denote by $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ the sets of real numbers, non-negative integers, and positive integers, respectively. We denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a,b]$, (a,b), $[a,b)$, and $(a,b]$, respectively. For $a,b\in\mathbb{N}_0$ and $a\leq b$, we use $[a;b]$, $(a;b)$, $[a;b)$, and $(a;b]$ to denote the corresponding intervals in $\mathbb{N}_0$. Given any $a\in\mathbb{R}$, $|a|$ denotes the absolute value of a. Given any $\nu=(\nu_1,\cdots,\nu_n)\in\mathbb{R}^n$, the infinity norm of ν is defined by $|\nu|=\max_{1\leq i\leq n}|\nu_i|$. Elements of $\mathbb{R}^n$ are by default regarded as column vectors and we write $\nu^\top$ for the transpose of a vector $\nu\in\mathbb{R}^n$. Given a symmetric matrix A, $\lambda_{\max}(A)$, and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of A, respectively. By ℓ^∞ we denote the Banach space of all infinite uniformly bounded sequences $s:=(s_i)\in\ell^\infty, i\in\mathbb{N}$, where s_i denotes the ith position of a sequence $s\in\ell^\infty$. Moreover, ℓ^∞_+ denotes the positive cone in ℓ^∞ consisting of all vectors $s\in\ell^\infty$ with $s_i\geq 0, i\in\mathbb{N}$. For all $s,s'\in\ell^\infty$ we say that $s\leq s'$ if $s_i\leq s'_i$ for all $i\in\mathbb{N}$, and that $s\not\geq s$ if there is $i\in\mathbb{N}$ such that $s_i<s'_i$. The standard unit vectors in ℓ^∞ are denoted by e_i, $i\in\mathbb{N}$; i.e., e_i is the sequence of zeros with exception of position i, where the entry is 1. Given an operator $\Gamma:\ell^\infty_+\to\ell^\infty_+$, $k\geq 1\in\mathbb{N}$, we define $\Gamma^k(\cdot):=\Gamma^{k-1}\circ\Gamma(\cdot)$, where Γ^0 is the identity operator on ℓ^∞. We denote by $\mathcal{C}(A)$ the cardinality of a set A and by $\emptyset$ the empty set. For any set $S\subseteq\mathbb{R}^n$ which is a finite union of boxes, e.g., $S=\bigcup_{j=1}^M S_j$ for some finite number $M\in\mathbb{N}$, where $S_j=\prod_{i=1}^n[c_i^j,d_i^j]\subseteq\mathbb{R}^n$ with $c_i^j<d_i^j$, and a positive constant $\eta\leq span(S)$, where $span(S)=\min_{j=1,\ldots,M}\eta_{S_j}$ and $\eta_{S_j}=\min\{|d_1^j-c_1^j|,\ldots,|d_n^j-c_n^j|\}$, we define $[S]_\eta=\{a\in S\mid a_i=k_i\eta, k_i\in\mathbb{N}, i=1,\ldots,n\}$. The set $[S]_\eta$ will be used as a finite approximation of S with precision η. Note that $[S]_\eta\neq\emptyset$ for any $\eta\leq span(S)$. We use the notations $\mathcal{K}$ and $\mathcal{K}_\infty$ to denote different classes of comparison functions, as follows: $\mathcal{K}=\{\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}|\ \alpha \text{ is continuous, strictly increasing, and } \alpha(0)=0\}$; $\mathcal{K}_\infty=\{\alpha\in\mathcal{K}|\lim_{r\to\infty}\alpha(r)=\infty\}$. For $\alpha,\gamma\in\mathcal{K}_\infty$ we write $\alpha\leq\gamma$ if $\alpha(r)\leq\gamma(r)$, and, with abuse of the notation, $\alpha=c$ if $\alpha(r)=cr$ for all $r\geq 0$ and a given $c>0$. Finally, we denote by id the identity function over $\mathbb{R}_{\geq 0}$, that is $\mathsf{id}(r)=r,\forall r\in\mathbb{R}_{\geq 0}$.

2.2 Infinite Networks

First, we define discrete-time control subsystems which will be later interconnected to form an infinite network consisting of countably infinite number of discrete-time control subsystems.

Definition 2.1. A discrete-time control subsystem $\Sigma_i, i\in\mathbb{N}$, is a tuple

$$\Sigma_i=(X_i,W_i,U_i,f_i,Y_i,h_i), \tag{1}$$

where $X_i\subseteq\mathbb{R}^{n_i}$, $W_i\subseteq\mathbb{R}^{p_i}$, $U_i\subseteq\mathbb{R}^{m_i}$, $Y_i\subseteq\mathbb{R}^{q_i}$, are the state set, internal input set, external input set and output set, respectively. The set valued map $f_i:X_i\times W_i\times U_i\rightrightarrows X_i$ is called transition function and $h_i:X_i\to Y_i$ is the output map. The discrete-time control subsystem Σ_i is described by a difference inclusion of the form

$$\Sigma_i:\begin{cases}\mathbf{x}_i(k+1)\in f_i(\mathbf{x}_i(k),\omega_i(k),\nu_i(k)),\\ \mathbf{y}_i(k)=h_i(\mathbf{x}_i(k)),\end{cases} \tag{2}$$

where $\mathbf{x}_i:\mathbb{N}_0\to X_i$, $\mathbf{y}_i:\mathbb{N}_0\to Y_i$, $\omega_i:\mathbb{N}_0\to W_i$, and $\nu_i:\mathbb{N}_0\to U_i$ are the state signal, output signal, internal input signal, and external input signal, respectively.

For each $i\in\mathbb{N}$ let $\mathcal{N}_i$ and $\mathcal{M}_i$ be finite subsets of $\mathbb{N}$. Here, the index sets $\mathcal{N}_i$ and $\mathcal{M}_i$ enumerate the neighbors of Σ_i, i.e., those systems $\Sigma_j, j\in\mathcal{N}_i$, $\Sigma_{j'}, j'\in\mathcal{M}_i$ that affect or are affected by Σ_i, respectively. By definition we require that $i\notin\mathcal{N}_i\cup\mathcal{M}_i$, $\forall i\in\mathbb{N}$. Since $\mathcal{N}_i$ and $\mathcal{M}_i$ are finite subsets of $\mathbb{N}$, each Σ_i can have only a finite number of neighbors.

Formally, the input-output structure of each subsystem Σ_i, $i\in\mathbb{N}$, is given by

$$w_i=(w_{ij})_{j\in\mathcal{N}_i}\in W_i:=\prod_{j\in\mathcal{N}_i}W_{ij}, \tag{3}$$

$$y_i=(y_{ij})_{j\in(i\cup\mathcal{M}_i)}\in Y_i:=\prod_{j\in(i\cup\mathcal{M}_i)}Y_{ij}, \tag{4}$$

$$h_i(x_i)=(h_{ij}(x_i))_{j\in(i\cup\mathcal{M}_i)}, \tag{5}$$

with $w_{ij}\in W_{ij}$, $y_{ij}=h_{ij}(x_i)\in Y_{ij}$. The outputs y_{ii} are considered as external ones, whereas y_{ij}, $j\in\mathcal{M}_i$, are interpreted as internal ones which are used to construct interconnections between subsystems. The dimension of w_{ij} is assumed to be equal to that of y_{ji} for all $i\in\mathbb{N}$ and for all $j\in\mathcal{N}_i$.

If for all $x_i\in X_i, u_i\in U_i, w_i\in W_i$, $\mathcal{C}(f_i(x_i,u_i,w_i))\leq 1$ we will say the system S is deterministic, and nondeterministic otherwise. System Σ_i is called finite if X_i, U_i, W_i are finite sets and infinite otherwise. Furthermore, if for all $x_i \in X_i$ there exist $u_i \in U_i$ and $w_i \in W_i$ such that $\mathcal{C}(f_i(x_i, u_i, w_i)) \neq 0$ we say the system is non-blocking. In this paper, we assume that all subsystems are non-blocking.

Now, we provide a formal definition of the infinite network.

Definition 2.2. Consider discrete-time control subsystems

$$\Sigma_i = (X_i, W_i, U_i, f_i, Y_i, h_i),\ i \in \mathbb{N},$$

with input-output structure given by (3)-(5). The infinite network is then formally a tuple $\Sigma = (X, U, f, Y, h)$, where

$$X = \{x = (x_i)_{i\in\mathbb{N}} : x_i \in X_i,\ \|x\| := \sup_{i\in\mathbb{N}}\{|x_i|\} < \infty\},$$
$$U = \{u = (u_i)_{i\in\mathbb{N}} : u_i \in U_i,\ \|u\| := \sup_{i\in\mathbb{N}}\{|u_i|\} < \infty\},$$
$$f(x,u) = \{(x_i^+)_{i\in\mathbb{N}} | x_i^+ \in f_i(x_i, w_i, u_i)\},$$
$$Y = \prod_{i\in\mathbb{N}} Y_{ii}, h(x) = (h_{ii}(x_i))_{i\in\mathbb{N}}.$$

The infinite network is denoted by $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$, and described by

$$\Sigma : \begin{cases} \mathbf{x}(k+1) \in f(\mathbf{x}(k), \nu(k)), \\ \mathbf{y}(k) = h(\mathbf{x}(k)). \end{cases}$$

Moreover, the interconnection variables are constrained by

$$\forall i \in \mathbb{N}, \forall j \in \mathcal{N}_i, w_{ij} = y_{ji}, Y_{ji} \subseteq W_{ij}. \tag{6}$$

We also assume that $f(x,u) \in X$ for all pair $(x,u) \in X \times U$ to ensure the infinite network $\Sigma = (X, U, f, Y, h)$ is well-defined.

2.3 Alternating Simulation Functions

In the following, we introduce a notion of so-called alternating simulation functions, adapted from Swikir and Zamani (2019b), which quantitatively relates two infinite networks.

Definition 2.3. Consider infinite networks $\Sigma = (X, U, f, Y, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{f}, \hat{Y}, \hat{h})$, where $\hat{Y} \subseteq Y$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called an alternating simulation function from $\hat{\Sigma}$ to Σ if there exist $\alpha, \sigma \in \mathcal{K}_\infty$, where $\sigma \leq \mathsf{id}$, $\rho_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\varepsilon \in \mathbb{R}_{\geq 0}$ so that the following hold:

- For every $x \in X, \hat{x} \in \hat{X}$, one has
$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}). \tag{7}$$
- For every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$ there exists $u \in U$ such that for every $x^+ \in f(x,u)$ there exists $\hat{x}^+ \in \hat{f}(\hat{x}, \hat{u})$ so that
$$V(x^+, \hat{x}^+) \leq \max\{\sigma(V(x,\hat{x})), \rho_u(\|\hat{u}\|), \varepsilon\}. \tag{8}$$

The next result shows that the existence of an alternating simulation function for two infinite networks implies the existence of an approximate alternating simulation relation between them as defined in Pola and Tabuada (2009).

Proposition 2.4. Consider two infinite networks $\Sigma = (X, U, f, Y, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{f}, \hat{Y}, \hat{h})$, where $\hat{Y} \subseteq Y$. Assume V is an alternating simulation function from $\hat{\Sigma}$ to Σ as in Definition 2.3 and that there exists $v \in \mathbb{R}_{>0}$ such that $\|\hat{u}\| \leq v\ \forall \hat{u} \in \hat{U}$. Then, relation $R \subseteq X \times \hat{X}$ defined by $R = \left\{(x, \hat{x}) \in X \times \hat{X} | V(x,\hat{x}) \leq \max\{\rho_u(v), \varepsilon\}\right\}$, is an $\hat{\varepsilon}$-approximate alternating simulation relation from $\hat{\Sigma}$ to Σ with

$$\hat{\varepsilon} = \alpha^{-1}(\max\{\rho_u(v), \varepsilon\}). \tag{9}$$

The $\hat{\varepsilon}$-approximate alternating simulation relation guarantees that for each output behavior of Σ there exists one of $\hat{\Sigma}$ such that the distance between these output behaviors is uniformly bounded by $\hat{\varepsilon}$.

Remark 2.5. Since the input set in all practical applications is bounded, requiring the control inputs to be bounded is not restrictive at all. Moreover, under certain stability property of concrete subsystems (see Section 4), one can choose the function ρ_u in (9) to be identically zero which cancels the dependency to the size of control inputs in Proposition 2.4. $\diamond$

3. COMPOSITIONALITY RESULT

In this section, we provide a method for compositional construction of an alternating simulation function between two infinite networks $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ and $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ defined in Definition 2.2. Here, we assume that each subsystems $\Sigma_i = (X_i, W_i, U_i, f_i, Y_i, h_i)$ and $\hat{\Sigma}_i = (\hat{X}_i, \hat{W}_i, \hat{U}_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$ admits a local alternating simulation function as defined next.

Definition 3.1. Consider $\Sigma_i = (X_i, W_i, U_i, f_i, Y_i, h_i)$ and $\hat{\Sigma}_i = (\hat{X}_i, \hat{W}_i, \hat{U}_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$, for all $i \in \mathbb{N}$, where $\hat{W}_i \subseteq W_i$ and $\hat{Y}_i \subseteq Y_i$. A function $V_i : X_i \times \hat{X}_i \to R_{\geq 0}$ is called a local alternating simulation function from $\hat{\Sigma}_i$ to Σ_i if there exist $\underline{\alpha}_i, \overline{\alpha}_i, \sigma_i, \rho_{wi} \in \mathcal{K}_\infty$, where $\sigma_i < \mathsf{id}$, $\rho_{u_i} \in \mathcal{K}_\infty \cup \{0\}$, and some $\varepsilon_i \in \mathbb{R}_{\geq 0}$ so that the following hold:

- For every $x_i \in X_i, \hat{x}_i \in \hat{X}_i$, one has
$$\underline{\alpha}_i(|h_i(x_i) - \hat{h}_i(\hat{x}_i)|) \leq V_i(x_i, \hat{x}_i) \leq \overline{\alpha}_i(|x_i - \hat{x}_i|). \tag{10}$$
- For every $x_i \in X_i, \hat{x}_i \in \hat{X}_i, \hat{u}_i \in \hat{U}_i$ there exists $u_i \in U_i$ such that for every $w_i \in W_i, \hat{w}_i \in \hat{W}_i, x_i^+ \in f_i(x_i, w_i, u_i)$ there exists $\hat{x}_i^+ \in \hat{f}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ so that
$$V_i(x_i^+, \hat{x}_i^+) \leq \max\{\sigma_i(V_i(x_i, \hat{x}_i)), \rho_{wi}(|w_i - \hat{w}_i|), \rho_{ui}(|\hat{u}_i|), \varepsilon_i\}. \tag{11}$$

$\hat{\Sigma}_i$ is called an abstraction of Σ_i if there exists a local alternating simulation function from $\hat{\Sigma}_i$ to Σ_i. Additionally, if $\hat{\Sigma}_i$ is finite ($\hat{X}_i$, $\hat{U}_i$ and $\hat{W}_i$ are finite sets), $\hat{\Sigma}_i$ is called a symbolic model of Σ_i. Note that local alternating simulation functions of subsystems are mainly for constructing alternating simulation functions for the overall infinite networks and they are not directly used for deducing any approximate alternating simulation relation.

Note that the different quantifiers appeared before condition (8) in Definition 2.3 (condition (11) in Definition 3.1) capture the different role played by control inputs as well as nondeterminism in the system. We refer the interested readers to (Pola and Tabuada, 2009, Section 3.2) justifying in details the role of those quantifiers.

For functions σ_i, $\underline{\alpha}_i$, and ρ_{wi} associated with V_i, $\forall\, i \in \mathbb{N}$, given in Definition 3.1, we define $\forall i, j \in \mathbb{N}$

$$\gamma_{ij} := \begin{cases} \sigma_i & \text{if} \quad i = j, \\ \rho_{wi} \circ \underline{\alpha}_j^{-1} & \text{if} \quad j \in \mathcal{N}_i, \\ 0 & \text{if} \quad i \neq j, j \notin \mathcal{N}_i. \end{cases} \tag{12}$$

Correspondingly, we define an operator $\Gamma : \ell_+^\infty \to \ell_+^\infty$ by

$$\Gamma(s) = \big(\sup_{j\in\mathbb{N}}\{\gamma_{ij}(s_j)\}\big)_{i\in\mathbb{N}}, \quad s \in \ell_+^\infty. \tag{13}$$

Additionally, we assume that there exist $\tilde{\sigma}, \tilde{\rho}_w, \tilde{\alpha} \in \mathcal{K}_\infty$ such that $\sigma_i \leq \tilde{\sigma}, \rho_{wi} \leq \tilde{\rho}_w, \underline{\alpha}_i \geq \tilde{\alpha}$ for all $i \in \mathbb{N}$. This assumption guarantees that Γ is well-defined.

In order to establish the main compositionality results of the paper, we make the following small-gain type assumption, inspired by Dashkovskiy et al. (2019).

Assumption 3.2. Consider operator Γ defined in (13). Assume that $\sup_{j\in\mathbb{N}}\{\gamma_{ij}(s_j)\} > 0, \forall s_j > 0, \forall i,j \in \mathbb{N}$, Γ is continuous on ℓ_+^∞, $\lim_{k\to\infty} \Gamma^k(s) = 0, \forall s \in \ell_+^\infty$, and there exist positive constants c_1 and c_2 such that for all $i,j \in \mathbb{N}$ the operator $\Gamma_{i,j}(s) := \Gamma(s) + c_1 s_j e_i, \quad s \in \ell_+^\infty$ satisfies

$$\Gamma_{i,j}(s) \not\geq (1-c_2)s, \quad s \in \ell_+^\infty \backslash \{0\}. \tag{14}$$

Remark 3.3. If for any $b \geq 0$ the set of all functions $\{\gamma_{ij}, i,j \in \mathbb{N}\}$ is uniformly equicontinuous in $[0,b]$, the operator Γ defined in (13) is continuous. That is, for any $\beta_1 > 0$ there exists $\beta_2 > 0$ such that for any $r_1, r_2 \in [0,b]$ with $|r_1 - r_2| < \beta_2$ it follows that $|\gamma_{ij}(r_1) - \gamma_{ij}(r_2)| < \beta_1, \forall i,j \in \mathbb{N}$. We refer the interested readers to Dashkovskiy et al. (2019) for more details on regularity properties of the operator Γ. ⋄

Note that by using Lemma 4.5 in Dashkovskiy et al. (2019), the small-gain condition (14) implies that there exist a function $\delta := (\delta_i)_{i\in\mathbb{N}} : \mathbb{R}_{\geq 0} \to \ell_+^\infty$ with $\delta_i \in \mathcal{K}_\infty, i \in \mathbb{N}$, and $\epsilon \in (0,1)$ such that

$$\Gamma(\delta(r)) \leq (1-\epsilon)\delta(r), \quad r \in \mathbb{R}_{\geq 0}. \tag{15}$$

It follows from (15) that $\forall i \in \mathbb{N}$ and $\forall r \in \mathbb{R}_{\geq 0}$

$$\sup_{j\in\mathbb{N}}\{\gamma_{ij} \circ \delta_j(r)\} \leq (1-\epsilon)\delta_i(r) \leq \delta_i(r).$$

Applying δ_i^{-1} to both sides, one has

$$\delta_i^{-1}(\sup_{j\in\mathbb{N}}\{\gamma_{ij} \circ \delta_j(r)\}) = \sup_{j\in\mathbb{N}}\{\delta_i^{-1} \circ \gamma_{ij} \circ \delta_j(r)\} \leq r. \tag{16}$$

Since (16) holds for all $i \in \mathbb{N}$, one has

$$\sup_{i,j\in\mathbb{N}}\{\delta_i^{-1} \circ \gamma_{ij} \circ \delta_j\} \leq \mathsf{id}. \tag{17}$$

Now we have all the ingredients to formulate the main result of this paper. The next theorem provides a compositional approach to construct an alternating simulation function from $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ to $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ via local alternating simulation functions from $\hat{\Sigma}_i$ to Σ_i.

Theorem 3.4. Consider the network $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$. Assume that each Σ_i and its abstraction $\hat{\Sigma}_i$ admit a local alternating simulation function V_i as in Definition 3.1. Suppose Assumption 3.2 holds and there exist $\mathcal{K}_\infty$ functions $\underline{\delta}, \overline{\delta}, \hat{\alpha}, \overline{\rho}_u$, and constant $\overline{\varepsilon} \in \mathbb{R}_{\geq 0}$ such that $\underline{\delta} \leq \delta_i \leq \overline{\delta}$, $\overline{\alpha}_i \leq \hat{\alpha}$, $\rho_{ui} \leq \overline{\rho}_u$, $\varepsilon_i \leq \overline{\varepsilon}$, $\forall i \in \mathbb{N}$. Then, function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ defined as

$$V(x,\hat{x}) := \sup_{i\in\mathbb{N}}\{\delta_i^{-1}(V_i(x_i,\hat{x}_i))\} \tag{18}$$

is well-defined and it is also an alternating simulation function from $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ to $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$.

The proof follows by utilizing equation (17) and following similar argument to the one in Swikir and Zamani (2019a), and it is omitted due to lack of space.

Remark 3.5. If $\gamma_{ij} \leq \mathsf{id}$ for any $i,j \in \mathbb{N}$, inequality (17) holds with $\delta_i = \mathsf{id}$ for all $i \in \mathbb{N}$, and inequality (18) reduces to $V(x,\hat{x}) := \sup_{i\in\mathbb{N}}\{V_i(x_i,\hat{x}_i)\}$, and, consequently, the small-gain condition (14) is satisfied automatically. ⋄

Remark 3.6. Note that computing the symbolic model of the infinite network using those of its subsystems is not possible practically since it consumes infinite memory to store. However, our proposed compositional framework is still required even if controller synthesis problems can be solved compositionally using symbolic models of subsystems. In particular, if decentralized (or distributed) controllers exist for some types of specifications, one still needs to establish the compositional relation as in Theorem 3.4 to formally reason about the preservation and satisfaction of properties across related infinite networks. ⋄

Remark 3.7. In the context of stability analysis of infinite networks, condition (14) is used to show different stability proprieties (e.g., uniform global asymptotic stability or input-to-state stability) for the entire network by investigating stability criteria for subsystems. Moreover, condition (14) is also been shown to be tight and cannot be weakened in the context of stability verification of infinite networks. We refer interested readers to Dashkovskiy et al. (2019) for more details on the tightness analysis of small-gain condition (14). ⋄

4. CONSTRUCTION OF SYMBOLIC MODELS

In this section, we show how to construct a symbolic model $\hat{\Sigma}_i$ for a given finite-dimensional deterministic subsystem Σ_i together with the corresponding local alternating simulation function from $\hat{\Sigma}_i$ to Σ_i. Consider $\Sigma_i = (X_i, W_i, U_i, f_i, Y_i, h_i)$ as in Definition 2.1. Assume that there exists $\ell \in \mathcal{K}_\infty$ such that $\|h(x)-h(x')\| \leq \ell(\|x-x'\|)$ for all $x,x' \in X$. Additionally, let Σ_i be incrementally input-to-state stable (δ-ISS) Angeli (2002) as defined next.

Definition 4.1. System Σ_i is δ-ISS if there exist functions $\mathcal{V}_i : X_i \times X_i \to \mathbb{R}_{\geq 0}$, $\underline{\psi}_i, \overline{\psi}_i, \kappa_i, \rho_{wi}, \rho_{ui} \in \mathcal{K}_\infty$, with $\kappa_i < \mathsf{id}$ such that for all $x_i, x_i' \in X_i$, for all $w_i, w_i' \in W_i$, and for all $u_i, u_i' \in U_i$

$$\underline{\psi}_i(|x_i - x_i'|) \leq \mathcal{V}_i(x_i,x_i') \leq \overline{\psi}_i(|x_i - x_i'|), \tag{19}$$

$$\begin{aligned}&\mathcal{V}_i(f_i(x_i,w_i,u_i), f_i(x_i',w_i',u_i')) \\ &\quad \leq \kappa_i(\mathcal{V}_i(x_i,\hat{x}_i)) + \varrho_{wi}(|w_i - w_i'|) + \varrho_{ui}(|u_i - u_i'|).\end{aligned} \tag{20}$$

We say that $\mathcal{V}_i$ is δ-ISS Lyapunov function for system Σ_i if it satisfies (19) and (20). We refer the interested readers to Angeli (2002) for more details on incremental input-to-state stability.

Now, we construct a symbolic model $\hat{\Sigma}_i$ of a δ-ISS control system Σ_i as the following.

Definition 4.2. Let $\Sigma_i = (X_i, U_i, W_i, f_i, Y_i, h_i)$ be δ-ISS as in Definition 4.1, where X_i, U_i, W_i are assumed to be finite unions of boxes. One can construct a symbolic model $\hat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$ where:

- $\hat{X}_i = [X]_{\eta_i^x}$, where $0 < \eta_i^x \leq \mathit{span}(X_i)$ is the state set quantization parameter;
- $\hat{U}_i = [U_i]_{\eta_i^u}$, where $0 < \eta_i^u \leq \mathit{span}(U_i)$ is the external input set quantization parameter;
- $\hat{x}_i^+ \in \hat{f}_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)$ if and only if $|\hat{x}_i^+ - f_i(\hat{x}_i, \hat{w}_i, \hat{u}_i)| \leq \eta_i^x$;
- $\hat{Y}_i = \{h(\hat{x}) \mid \hat{x} \in \hat{X}\}$;
- $\hat{h}_i = h_i$;
- $\hat{W}_i$ is an appropriate finite internal input set satisfying $\hat{W}_i = \prod_{j\in\mathcal{N}_i} \hat{W}_{ij}$ and $\hat{Y}_{ji} \subseteq \hat{W}_{ij}$ $\forall i \in \mathbb{N}, \forall j \in \mathcal{N}_i$.

We impose the following assumptions on function $\mathcal{V}_i$ in Definition 4.1 which are used to prove the results later.

Assumption 4.3. There exists a $\mathcal{K}_\infty$ function γ_i such that

$$\forall x_i, y_i, z_i \in X_i,\ \mathcal{V}_i(x,y) \leq \mathcal{V}_i(x_i,z_i) + \gamma_i(|y_i - z_i|). \quad (21)$$

Note that condition (21) is not restrictive provided that one is interested to work on a compact subset of $X \times X$; see Zamani et al. (2014) for more details.

Now, we establish the relation between Σ_i and $\hat{\Sigma}_i$, introduced above, via the notion of local alternating simulation function as in Definition 3.1. The next theorem is adapted from Swikir and Zamani (2019b) and stated without a proof.

Theorem 4.4. Let Σ_i be an incrementally δ-ISS control system as in Definition 4.1 with δ-ISS Lyapunov function $\mathcal{V}_i$, and $\hat{\Sigma}_i$ be a symbolic model constructed as in Definition 4.2. Let Assumption 4.3 holds. Then $\mathcal{V}_i$ is an alternating simulation function from $\hat{\Sigma}_i$ to Σ_i.

In particular, $\mathcal{V}_i$ satisfies (10) and (11) with $\underline{\alpha}_i = (\ell_i \circ \underline{\psi}_i^{-1})^{-1}$, $\overline{\alpha}_i = \overline{\psi}_i$, $\sigma_i = \mathsf{id} - (\mathsf{id} - \varphi_i)(\mathsf{id} - \kappa_i)$, $\rho_{wi} = (\mathsf{id} + \lambda_i) \circ \kappa^{-1} \circ \varphi_i^{-1} \circ \chi_i \circ \varrho_{wi}$, $\rho_{ui} = 0$, $\varepsilon_i = (\mathsf{id} + \lambda_i^{-1}) \circ \kappa^{-1} \circ \varphi_i^{-1} \circ \chi_i \circ (\chi_i - \mathsf{id})^{-1} \circ \gamma_i(\eta_i^x)$, where $\lambda_i, \chi_i, \varphi_i$ are some arbitrarily chosen $\mathcal{K}_\infty$ functions with $\varphi_i < \mathsf{id},\ \chi_i > \mathsf{id}$.

Remark 4.5. For linear control systems $\big(i.e., x_i(k+1) = A_i x_i(k) + D_i w_i(k) + B_i u_i(k), y_i(k) = C_i x_i(k)\big)$, one can restrict the attention to δ-ISS Lyapunov functions of the form

$$\mathcal{V}_i(x_i, x_i') = \sqrt{(x_i - x_i')^\top Z_i (x_i - x_i')}, Z_i \succ 0. \quad (22)$$

It can be readily seen that such functions always satisfy Assumption 4.3 and (19), and inequality (20) is satisfied using the following linear matrix inequality

$$A_i^\top Z_i A_i \preceq \kappa_i Z_i, \quad (23)$$

in which Z_i can be computed by semi-definite programming, where $0 < \kappa_i < 1$. Consequently, it can be readily verified that ε_i in (11) is defined as $\varepsilon_i = c_i \lambda_{\max}(Z_i)$, for some $c_i > 0$ depending on the dimensions of Z_i. ⋄

Note that condition (23) is nothing more than asking matrix A being Hurwitz.

Remark 4.6. One can also verify that function $\mathcal{V}_i$ satisfying (21) (respectively (22) in Remark 4.5) is also an alternating simulation function from Σ_i to $\hat{\Sigma}_i$. In particular, $\mathcal{V}_i$ satisfies (10) and (11) with the same $\mathcal{K}_\infty$ functions $\underline{\alpha}_i, \overline{\alpha}_i, \sigma_i, \rho_{wi}, \rho_{ui}$ defined in Theorem 4.4, and $\varepsilon_i = (\mathsf{id} + \lambda_i^{-1}) \circ \kappa^{-1} \circ \varphi_i^{-1} \circ \chi_i \circ (\chi_i - \mathsf{id})^{-1} \circ (\varrho_{ui}(\eta_i^u) + \gamma_i(\eta_i^x))$. ⋄

Given Remark 4.6, it can be readily verified that function V defined in (18) is also an alternating simulation function from $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ to $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$, and, hence, $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ is a complete symbolic model Tabuada (2009) for $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$. In other words, there exists a controller enforcing the desired specifications on the symbolic model $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ *if and only if* there exists a controller enforcing the same specifications on the original infinite network $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$.

5. CASE STUDY: A ROAD TRAFFIC MODEL

In this case study, we apply our approach to a variant of the road traffic model from de Wit et al. (2012). We consider a traffic network divided into infinitely many cells, indexed by $i \in \mathbb{N}$. Each cell i represents a one-dimensional subsystem $\Sigma_i = (X_i, W_i, U_i, f_i, X_i, h_i)$ described by a difference equation of the following form

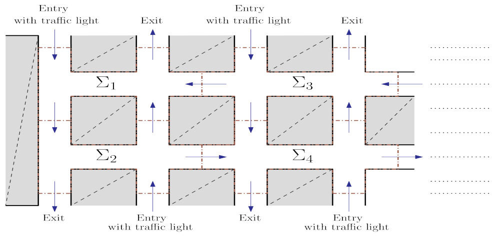

Fig. 1. Model of a road traffic network composed of infinitely many subsystems.

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = (1 - \dfrac{\tau v}{l} - e)\mathbf{x}_i(k) + d_i \omega_i(k) + b\nu_i(k), \\ \mathbf{y}_i(k) = h_i(\mathbf{x}_i(k)) = \mathbf{x}_i(k), \end{cases} \quad (24)$$

with the following structure

- $d_i = (\frac{1-e}{2})(\frac{\tau v}{l}, \frac{\tau v}{l})^\top, \omega_i = (\mathbf{y}_{i+1}, \mathbf{y}_{i+2})$ if $i \in S_1 := \{1 + 2c : c \in \mathbb{N}_0\}$;
- $d_i = (1-e)\frac{\tau v}{l}, \omega_i = \mathbf{y}_{i+1}$ if $i \in S_2 := \{2\}$;
- $d_i = (\frac{1-e}{2})(\frac{\tau v}{l}, \frac{\tau v}{l})^\top, \omega_i = (\mathbf{y}_{i-2}, \mathbf{y}_{i-1})$ if $i \in S_3 := \{4 + 2c : c \in \mathbb{N}_0\}$.

In (24), τ is the sampling time interval in hours, l is the length of a cell in kilometers (km), and v is the flow speed of the vehicles in kilometers per hour (km/h). The state of each subsystem Σ_i, i.e. $\mathbf{x}_i$, is the density of traffic, given in vehicles per cell, for each cell i of the network. The scalar b represents the number of vehicles that can enter the cells through entries which are controlled by $\nu_i(\cdot)$. In particular, $\nu_i(\cdot) = 1$ means green light and $\nu_i(\cdot) = 0$ means red light. Moreover, the constant $e \in (0,1)$ represents the percentage of vehicles that leave the cells using available exits. The infinite network and its cells are illustrated by Figure 1.

Let us first show that $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ is well-defined by showing that $\|f(x,u)\| < \infty$, where $f(x,u)$ is constructed as in Definition 2.2. Define $C_1 = |1 - \frac{\tau v}{l} - e|$, $C_2 = |(1-e)\frac{\tau v}{l}|$, $C_3 = |b|$, $C = \max_{1\leq i\leq 3}\{C_i\}$, then one has

$$\begin{aligned} \|f(x,u)\| &= \sup_{i\in\mathbb{N}}\{|f_i(x_i, w_i, u_i)|\} \\ &= \sup_{i\in\mathbb{N}}\{|(1 - \frac{\tau v}{l} - e)x_i + d_i w_i + b u_i|\} \\ &\leq C_1 \sup_{i\in\mathbb{N}}\{|x_i|\} + C_2 \sup_{i\in\mathbb{N}}\{|x_i|\} + C_3 \sup_{i\in\mathbb{N}}\{|u_i|\} \\ &\leq C(\sup_{i\in\mathbb{N}}\{|x_i|\} + \sup_{i\in\mathbb{N}}\{|x_i|\} + \sup_{i\in\mathbb{N}}\{|u_i|\}) \\ &= C(\|x\| + \|x\| + \|u\|\}) < \infty \end{aligned}$$

Hence, $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ is well-defined.

Fix $\tau = \frac{10}{60\times 60}, v = 60, l = 0.5$, and $e = 0.1$, then for any $i \in \mathbb{N}$, system Σ_i is δ-ISS, where conditions (19) and (20) are satisfied with $\mathcal{V}_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$, $\underline{\psi}_i = \overline{\psi}_i = \mathsf{id}$, $\kappa_i = (1 - (\frac{\tau v}{l} + e))\mathsf{id}$, $\varrho_{wi} = |(1-e)\frac{\tau v}{l}|\mathsf{id}$, and $\varrho_{ui} = 0$ with $u_i = \hat{u}_i$. Furthermore, (21) is satisfied with $\gamma_i = \mathsf{id}$. Consequently, $\mathcal{V}_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$ is an alternating simulation function from $\hat{\Sigma}_i = (\hat{X}_i, \hat{W}_i, \hat{U}_i, \hat{f}_i, \hat{X}_i, h_i)$, constructed as in Definition 4.2, to Σ_i satisfying condition (10) with $\underline{\alpha}_i = \overline{\alpha}_i = \mathsf{id}$ and condition (11) with $\sigma_i = 0.97\mathsf{id}$, $\rho_{wi} = 0.87\mathsf{id}$, $\rho_{ui} = 0$, $\varepsilon_i = 17\eta_i^x$, where η_i^x is the state set quantization parameter. Note that for the construction of symbolic models $\hat{\Sigma}_i$, we have chosen the finite set $\hat{W}_i = \hat{X}_{i+1} \times \hat{X}_{i+2}$ for all $i \in S_1$, $\hat{W}_i = \hat{X}_{i+1}$ for all $i \in S_2$, and $\hat{W}_i = \hat{X}_{i-2} \times \hat{X}_{i-1}$ for all $i \in S_3$. Moreover, it can be readily verified that $\gamma_{ij} < \mathsf{id}$. Therefore, by remark 3.5, $V(x, \hat{x}) := \sup_{i \in \mathbb{N}}\{|x_i - \hat{x}_i|\}$ is the alternating simulation function from $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ to $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ satisfying conditions (7) and (8) with $\alpha = \mathsf{id}$, $\sigma = 0.97\mathsf{id}$, $\rho_u = 0$, $\varepsilon = 17 \sup_{i\in\mathbb{N}}\{\eta_i^x\}$. In order to guarantee that ε is well-defined, one should choose η_i^x such that there exists $\eta^x \in \mathbb{R}_{>0}$ so that $\eta_i^x \leq \eta^x, \forall i \in \mathbb{N}$.

Now we show how to use the constructed symbolic models $\hat{\Sigma}_i$ to design a controller for Σ such that the density of traffic is maintained between 10 and 25 vehicles per cell (subsystems Σ_i). Based on assume-guarantee reasoning Henzinger et al. (1998), we compositionally synthesize controllers for symbolic models, and then refine them to the ones for concrete subsystems. In particular, we design local controllers $\hat{u}_i$ for $\hat{\Sigma}_i$ while assuming that the other subsystems $\hat{\Sigma}_{j,j\neq i}$ meet their specifications, and then refine $\hat{u}_i$ to u_i using $u_i = \hat{u}_i$. We leverage software tool `SCOTS` Rungger and Zamani (2016) for constructing symbolic models and controllers for Σ_i compositionally with $b = 5$, state quantization parameter $\eta_i = 0.1$ and the computation times are amounted to $0.016s$ and $9 \times 10^{-4}s$, respectively. Figure 2 shows trajectories of sample subsystem Σ_i starting from different initial conditions under input u_i. Finally, one can compute the mismatch between the output behavior of $\Sigma = \mathcal{I}(\Sigma_i)_{i\in\mathbb{N}}$ and its symbolic model $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_i)_{i\in\mathbb{N}}$ by utilizing Proposition 2.4. In particulate, using (9) and since $\alpha = \mathsf{id}, \rho_u = 0$, we have $\hat{\varepsilon} = \alpha^{-1}(\varepsilon) = \sup_{i\in\mathbb{N}}\{\varepsilon_i\} = 1.7$.

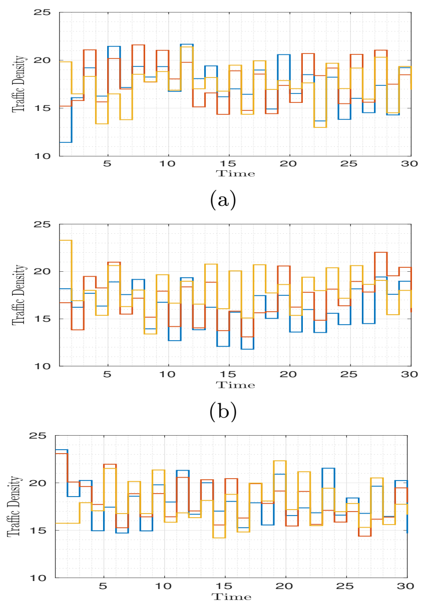

Fig. 2. Trajectories of sample subsystem Σ_i starting from different initial conditions with (a) $i \in S_1$, (b) $i \in S_2$, and (c) $i \in S_3$.

6. CONCLUSION

In this work, we proposed a compositional scheme for the construction of symbolic models of infinite networks consisting of infinitely many finite-dimensional discrete-time control systems. We used some small-gain type conditions in order to construct compositionally an alternating simulation function that is used to quantify the error between the output behavior of the infinite interconnection of discrete-time control subsystem and that of their symbolic models. Furthermore, under some assumptions ensuring incremental input-to-state stability of each concrete subsystems, we showed how to construct their symbolic models together with their corresponding alternating simulation functions.

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