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Switched Control Design with Guaranteed Cost for Uncertain Nonlinear Systems Subject to Actuator Saturation ⋆

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Abstract: This manuscript proposes a robust switched controller design with minimization of an upper bound of a quadratic performance index (guaranteed cost) related to the system output for a class of uncertain nonlinear systems with actuator saturation described by Takagi-Sugeno (TS) fuzzy models. The switched control eliminates the necessity of finding or estimating the membership functions, which can be uncertain or complex to obtain. In most practical implementations, systems and actuators have physical limitations. Therefore, in order to approximate the theoretical switched controller design closer to its implementation, it will be considered that the system has an operating region and control signal saturation. The proposed switched controller design will be implemented in a bench active suspension system considering actuator saturation with uncertain mass and actuator fault. A comparison will be shown between robust single-gain and switched controller with the same design parameters.

Keywords: Switched control; Uncertain nonlinear systems; Takagi-Sugeno fuzzy models; Actuator saturation; Linear matrix inequality (LMI).

1. INTRODUCTION

A wide range of nonlinear systems can be described in an operation region by TS fuzzy models, based on the non-linearity sector and the knowledge of the physical limits and uncertainties of the system (Alves et al., 2016; Souza et al., 2014; Tanaka and Wang, 2004). The idea of TS fuzzy models is to describe a nonlinear system through linear subsystems combined by membership functions (Santim et al., 2012; Tanaka and Wang, 2004). Fuzzy controllers based on parallel-distributed compensation (PDC) need to compute the membership functions to compose the control signal. In some cases, these functions may depend on uncertain parameters from the system, or even be complex to obtain (Alves et al., 2016; Souza et al., 2014).

The class of switched controllers studied here does not need the knowledge of the membership functions to compose the control signal. Furthermore it is found in the literature better performances using these switched controllers than only one state-feedback gain (Alves et al., 2016; Oliveira et al., 2018; Souza et al., 2014). In each time, these switched control laws select a state-feedback gain that minimizes the time derivative of a Lyapunov function (Souza et al., 2014, 2013).

It is usual to consider an appropriated guaranteed cost as a performance index and seek to minimize it (Caun et al., 2018; Deaecto et al., 2010). In this way, a guaranteed cost related to the system output will require the system output transient to be fast. In Deaecto et al. (2010), the guaranteed cost of the output was related to the dissipated energy in heating of DC-DC converters.

Therefore, based in the optimal quadratic regulator theory (Boyd et al., 1994; Caun et al., 2018) and switched controller design methodology, in this manuscript, it is proposed a new switched controller design that provides a good performance of the controlled systems minimizing the energy of the system output. In the proposed procedure, it is also considered that the switched control is subject to actuator saturation. The proposed switched controller design will be compared to the single-gain feedback controller with same design requirements. The controllers designed will be applied for controlling a bench active suspension, which is an uncertain nonlinear system and it is exactly described by TS fuzzy models (Oliveira et al., 2018; Santim et al., 2012).

Some notations will be used such as $x = x(t)$, $y = y(t)$, $u = u(t)$, $z = z(t)$, $\bar{\beta} = \beta^{-1}$, $\mathbb{K}_{n_r} = \{1, 2, \dots, n_r\}$, $n_r \in \mathbb{N}$, the set of natural numbers. I represents identity matrix with appropriate dimension. The function $\alpha = \alpha(z)$ is dependent on z vector, whose elements are premise variables that depend on the state vector and uncertain system parameters. The notation $He(A) = A + A^T$ is used for matrices and scalar numbers.

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2. SWITCHED CONTROL OF NONLINEAR SYSTEMS DESCRIBED BY TS FUZZY MODELS SUBJECT TO ACTUATOR SATURATION

2.1 Preliminaries

The TS fuzzy models is described by IF-THEN rules (Takagi and Sugeno, 1985). Such models relate locally the input and output of a nonlinear system.

$$\text{Rule } i : \text{If } z_1 \text{ is } \mathcal{M}_1^i \text{ and } \cdots \text{ and } z_{n_z} \text{ is } \mathcal{M}_{n_z}^i, \qquad \text{Then } \begin{cases} \dot{x} = A_i x + B_i u, \\ y = Cx, \end{cases} \tag{1}$$

where $i \in \mathbb{K}_{n_r}$, n_r is the number of the IF-THEN rules from the TS fuzzy models, and $j \in \mathbb{K}_{n_z}$, n_z is the number of system state variables and uncertainties or nonlinearities of the system. $\mathcal{M}_j^i$ is the fuzzy set j of the rule i, $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$ and $C \in \mathbb{R}^{n_y \times n_x}$ are the matrices from linear local models, $z \in \mathbb{R}^{n_z}$ are the vector of premise variables, $x \in \mathbb{R}^{n_x}$ is the state vector and $u \in \mathbb{R}^{n_u}$ is the control input. Each local model is given by the linear system $\dot{x} = A_i x + B_i u$, $y = Cx$.

The TS fuzzy models consists of the combination of local linear models using the membership functions. In Tanaka and Wang (2004), the fuzzy system is

$$\begin{cases} \dot{x} = \dfrac{\sum_{i=1}^{n_r} \omega_i(z)(A_i x + B_i u)}{\sum_{i=1}^{n_r} \omega_i(z)}, \\ y = Cx, \end{cases} \tag{2}$$

where $\omega_i(z) = \prod_{j=1}^{n_z} \mathcal{M}_j^i(z_j)$, $\sum_{i=1}^{n_r} \omega_i(z) > 0$ and $\omega_i(z) \geq 0$, for all $i \in \mathbb{K}_{n_r}$. $\mathcal{M}_j^i(z_j)$ is the weight of the fuzzy set associated with the premise variable z_j. The normalized weight represents a membership function $\alpha_i(z)$, $i \in \mathbb{K}_{n_r}$, and it is

$$\alpha_i(z) = \frac{\omega_i(z)}{\sum_{i=1}^{n_r} \omega_i(z)}, \tag{3}$$

where $\sum_{i=1}^{n_r} \alpha_i(z) = 1$ and $\alpha_i(z) \geq 0$ for all $i \in \mathbb{K}_{n_r}$. From (2) and (3), the nonlinear system described by TS fuzzy model is

$$\begin{cases} \dot{x} = A(\alpha)x + B(\alpha)u, \\ y = Cx. \end{cases} \tag{4}$$

where $(A(\alpha), B(\alpha)) = \sum_{i=1}^{n_r} \alpha_i(z)(A_i, B_i)$.

Important to note that the description by TS fuzzy model is valid for an appropriate system operation region. Therefore it must be ensured that the system state does not leave the operation region. The design procedure assures the TS fuzzy models (4) will remain in the operation region $\mathcal{X}$ for all $t \geq 0$ when the initial conditions $x(0)$ are in the region $\mathcal{E}(P, \beta)$ within the operation region $\mathcal{X}$, that is, $\mathcal{E}(P, \beta) \subset \mathcal{X}$, and the origin of closed-loop system is an asymptotically stable equilibrium point (Klug et al., 2015).

Consider $N = \begin{bmatrix} N_1^T & N_2^T & \dots & N_{n_\phi}^T \end{bmatrix}^T \in \mathbb{R}^{n_\phi \times n_x}$, a real constant $\beta > 0$ and $\phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{n_\phi} \end{bmatrix}^T \in \mathbb{R}^{n_\phi}$ all known and a symmetric definite positive matrix $P = P^T \in \mathbb{R}^{n_x \times n_x}$. The operation region $\mathcal{X}$ (Klug et al., 2015) in the state space and attraction region $\mathcal{E}(P, \beta)$ (Hu et al., 2002) are defined as follows

$$\mathcal{X} \triangleq \left\{ x \in \mathbb{R}^{n_x} : |N_h x| \leq \phi_h, h \in \mathbb{K}_{n_\phi} \right\}. \tag{5}$$

$$\mathcal{E}(P, \beta) \triangleq \left\{ x \in \mathbb{R}^{n_x} : x^T P x \leq \beta \right\}, \tag{6}$$

Consider $H = \begin{bmatrix} H_{k_1}^T & H_{k_2}^T & \dots & H_{k_{n_u}}^T \end{bmatrix}^T \in \mathbb{R}^{n_u \times n_x}$ and $\rho = \begin{bmatrix} \rho_1 & \rho_2 & \dots & \rho_{n_u} \end{bmatrix}^T \in \mathbb{R}^{n_u}$ a known vector, $\rho_l > 0$, $l \in \mathbb{K}_{n_u}$. The set $\mathcal{L}(H_k)$ (Hu et al., 2002) is

$$\mathcal{L}(H_k) \triangleq \left\{ x \in \mathbb{R}^{n_x} : |H_{k_l} x| \leq \rho_l,\ l \in \mathbb{K}_{n_u}, k \in \mathbb{K}_{n_r} \right\}. \tag{7}$$

The matrix set $\mathcal{D}$ (Cao and Lin, 2003) is defined such that each element is called $D_s \in \mathbb{R}^{n_u \times n_u}$, $s \in \mathbb{K}_{2^{n_u}}$ and D_s^- denotes the element of $\mathcal{D}$ associated with D_s, such that $D_s^- = I - D_s$. There are 2^{n_u} elements in $\mathcal{D}$ set

$$\mathcal{D} \triangleq \left\{ D_s \in \mathbb{R}^{n_u \times n_u} : d_{ii} = 0 \text{ or } 1 \text{ and } d_{ij} = 0\ \forall\, i \neq j \right\}, \tag{8}$$

Lemma 1. (Alves et al., 2016; Hu et al., 2002) Let the set $\mathcal{E}(P, \beta)$ (6) and the set $\mathcal{L}(H_k)$ (7). The constraint $\mathcal{E}(P, \beta) \subset \mathcal{L}(H_k)$ is enforced if the conditions

$$\begin{bmatrix} \rho_l^2 \beta^{-1} & G_{k_l} \\ G_{k_l}^T & X \end{bmatrix} \geq 0, \tag{9}$$

are feasible for all $k \in \mathbb{K}_{n_r}$ and $l \in \mathbb{K}_{n_u}$. $G_{k_l} = H_{k_l} X$, where $X = P^{-1}$.

Proof. The proof can be found in Alves et al. (2016).

2.2 Switched Control and Actuator Saturation

This section introduces a switched controller design subject to actuator saturation, based in (Alves et al., 2016), applicable to a class of nonlinear systems described by TS fuzzy models (4). The presented controller selects the feedback gain, from a set of gains, that minimizes the time derivative of quadratic Lyapunov function.

The switched control law returns the switching index $\sigma(t)$ to select a gain, which belongs to the set of gains $\{K_i \in \mathbb{R}^{n_u \times n_x}, i \in \mathbb{K}_{n_r}\}$. The switched control law with switching law is defined as follows

$$\begin{aligned} u = u_\sigma &= -K_\sigma x, \\ \sigma(t) &= \arg^* \min_{j \in \mathbb{K}_r} \ \{x^T Q_j x\}. \end{aligned} \tag{10}$$

where $\sigma(t) \in \mathbb{K}_{n_r}$ is the lowest value of the j index that results in the minimum value of $x^T Q_j x$. $Q_j \in \mathbb{R}^{n_x \times n_x}, j \in \mathbb{K}_{n_r}$, are auxiliary matrices.

Now, consider a nonlinear system subject to actuator saturation whose dynamic equation is

$$\begin{cases} \dot{x} = A(\alpha)x + B(\alpha)\text{sat}(u_)), \\ y = Cx, \end{cases} \tag{11}$$

where $\text{sat}(u) \in \mathbb{R}^{n_u}$ is the amplitude-bounded control input, such that

$$\text{sat}(u) = \begin{bmatrix} \text{sat}(u)_1 \\ \vdots \\ \text{sat}(u)_{n_u} \end{bmatrix}, \text{sat}(u)_l = \begin{cases} -\rho_l, & \text{if } u_l < -\rho_l, \\ u_m, & \text{if } |u|_l \leq \rho_l, \\ \rho_l, & \text{if } u_l > \rho_l, \end{cases} \tag{12}$$

where $l \in \mathbb{K}_{n_u}$, $u = u_\sigma$ for switched control and ρ_l is a known positive constant.

For $x \in \mathcal{L}(H_j)$, $j \in \mathbb{K}_{n_r}$, then $x \in \mathcal{L}(H_\sigma)$. According to Cao and Lin (2003), it follows that $\text{sat}(u_\sigma) = \text{sat}(-K_\sigma x) \in \{\text{co}\}\{D_s(-K_\sigma x) + D_s^-(H_\sigma x)\}$, where $\{\text{co}\}$ means convex combination.

2.3 Main Results

The main goal of this paper is, for system (11), to design the switched control law (10) in order to minimize the upper bound of output signal energy (guaranteed cost), as shown below

$$J = \int_0^\infty y^T R y \, dt = \int_0^\infty x^T C^T R C x \, dt. \tag{13}$$

where $R \in \mathbb{R}^{n_y \times n_y}$ is a definite positive matrix.

The performance index (13) considers the initial condition $x(0)$ belonging to the convex set

$$x(0) = \sum_{k=1}^{n_{x_0}} \lambda_k x_k(0), \quad \sum_{k=1}^{n_x} \lambda_k = 1, \tag{14}$$

$\lambda_k \geq 0$, for all $k \in \mathbb{K}_{n_{x_0}}$, n_{x_0} is the number of the vertices from the initial condition polytope. The minimization of J (13) requires that the system output transient be fast.

Conditions for control of uncertain nonlinear system subject to actuator saturation controlled by switched control law (10) with the origin of the system (11) asymptotically stable respecting the minimization of performance index J and decay rate γ are proposed in Theorem 1.

Theorem 1. Consider an uncertain nonlinear system subject to actuator saturation described by TS fuzzy model (11) in an operation region $\mathcal{X}$ (5) and the performance index J (13). Let $\rho \in \mathbb{R}^{n_u}$, $\phi \in \mathbb{R}^{n_\phi}$ and $N \in \mathbb{R}^{n_\phi \times n_x}$ known as well the range of the uncertain parameters. Suppose the existence of symmetric matrices $X > 0$, $\bar{Q}_i$ and $\bar{Z}_i \in \mathbb{R}^{n_x \times n_x}$, matrices G_j and $M_i \in \mathbb{R}^{n_u \times n_x}$ and scalars $\bar{\beta} > 0$ and $\gamma \geq 0$, such that the conditions

$$\underset{X, M_i, G_j, \bar{Q}_i, \bar{Z}_i}{maximize} \quad \bar{\beta}$$

subject to

$$He\Big(B_i\big[-D_s M_j + D_s^- G_j\big]\Big) - \bar{Z}_i - \bar{Q}_j \leq 0, \tag{15}$$

$$\begin{bmatrix} He(A_i X) + 2\gamma X + \bar{Q}_i + \bar{Z}_i & XC^T \\ CX & -R^{-1} \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} \rho_l^2 \bar{\beta} & G_{j_l} \\ G_{j_l}^T & X \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \phi_h^2 \bar{\beta} & N_h X \\ X N_h^T & X \end{bmatrix} \geq 0, \tag{17}$$

$$\begin{bmatrix} \bar{\beta} & \bar{\beta} x_k^T(0) \\ x_k(0)\bar{\beta} & X \end{bmatrix} \geq 0, \tag{18}$$

are feasible for all i and $j \in \mathbb{K}_{n_r}$, $l \in \mathbb{K}_{n_u}$, $h \in \mathbb{K}_{n_\phi}$, $k \in \mathbb{K}_{n_0}$, $s \in \mathbb{K}_{2^{n_u}}$, $D_s \in \mathcal{D}$ and $D_s^- = I - D_s \in \mathcal{D}$. Then the switched control (10) which $K_i = M_i X^{-1}$ and $Q_i = X^{-1}\bar{Q}_i X^{-1}$ for all $i \in \mathbb{K}_{n_r}$ applied in the system (11) makes the origin an asymptotically stable equilibrium point with decay rate greater than or equal to γ ensuring the performance index $J < \beta$ for all x_0 belonging to convex set (14).

Proof. Consider as Lyapunov function candidate $V_\sigma(x) = x^T P x$, such that $P = P^T \in \mathbb{R}^{n_x \times n_x}$. From (10) and (11), we have

$$\begin{aligned} \dot{V}_\sigma(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= He\big(x^T P A(\alpha) x + x^T P B(\alpha) \mathrm{sat}(-K_\sigma x)\big). \end{aligned} \tag{19}$$

If $x \in \mathcal{L}(H_k)$, for all $k \in \mathbb{K}_{n_r}$, then $x \in \mathcal{L}(H_\sigma)$ and $\mathrm{sat}(-K_\sigma x)$ can be rewritten as follows (Cao and Lin, 2003)

$$\mathrm{sat}(-K_\sigma x) = \sum_{s=1}^{2^{n_u}} \lambda_s \big[D_s(-K_\sigma x) + D_s^-(H_\sigma x)\big], \tag{20}$$

where $\sigma = \sigma(t) \in \mathbb{K}_{n_r}$, $D_s \in \mathcal{D}$, $D_s^- = I_{n_u} - D_s \in \mathcal{D}$, and $\lambda_s \geq 0$, $\sum_{s=1}^{2^{n_u}} \lambda_s = 1$. Thus, from (19) and (20)

$$\begin{aligned} \dot{V}_\sigma(x) &= x^T\{A(\alpha)^T P + P A(\alpha)\}x \\ &+ \textstyle\sum_{s=1}^{2^{n_u}} He\big(x^T P B(\alpha) \lambda_s [D_s(-K_\sigma x) + D_s^-(H_\sigma x)]\big). \end{aligned} \tag{21}$$

Supposing that the conditions of Theorem 1 are satisfied. Applying the Schur complement in (16) we have

$$X A_i^T + A_i X + 2\gamma X + \bar{Q}_j + \bar{Z}_i + X C^T R C X < 0. \tag{22}$$

Pre and post multiplying (22) by $P = X^{-1}$, doing the variable changes $G_j X^{-1} = H_j$, $M_j X^{-1} = K_j$, $Z_i = X^{-1}\bar{Z}_i X^{-1}$, $Q_i = X^{-1}\bar{Q}_i X^{-1}$, multiplying by α_i and summing from 1 to n_r, it follows that

$$He\big(PA(\alpha)\big) + 2\gamma P + Q(\alpha) + Z(\alpha) + C^T R C < 0. \tag{23}$$

Pre and post multiplying (15) by $P = X^{-1}$, doing the suitable variable changes, multiplying by α_i, summing from 1 to n_r and considering $j = \sigma$, we have

$$He\big(PB(\alpha)[-D_s K_\sigma + D_s^- H_\sigma]\big) - Z(\alpha) - Q_\sigma \leq 0. \tag{24}$$

Multiplying (24) by λ_s, $\lambda_s \geq 0$, $\sum_{s=1}^{2^{n_u}} \lambda_s = 1$, we have

$$\sum_{s=1}^{2^{n_u}} \lambda_s He\big(PB(\alpha)[-D_s K_\sigma + D_s^- H_\sigma]\big) \leq Z(\alpha) + Q_\sigma. \tag{25}$$

From switched control law (10) and knowing that the minimum of a set of real numbers is less than or equal to the convex combination of the set elements (Alves et al., 2016)

$$x^T \bar{Q}_\sigma x = \min_{i \in \mathbb{K}_{n_r}} \{x^T \bar{Q}_i x\} \leq \sum_{i=1}^{n_r} \alpha_i(z) x^T \bar{Q}_i x = x^T \bar{Q}(\alpha) x. \tag{26}$$

From (21), (25) and (26), one has

$$\dot{V}_\sigma(x) \leq x^T\Big\{A(\alpha)^T P + P A(\alpha) + Z(\alpha) + Q(\alpha)\Big\}x. \tag{27}$$

Considering (23) and (27), it follows that

$$\dot{V}_\sigma(x) + 2\gamma x^T P x < -x^T C^T R C x. \tag{28}$$

From (28), $\dot{V}_\sigma(x) + 2\gamma x^T P x < 0$, there is a sufficient condition for the asymptotic stability of the origin of the system (10) and (11) with decay rate greater than or equal to γ (Boyd et al., 1994). Furthermore, knowing that $P > 0$, $\dot{V}_\sigma(x) < -x^T C^T R C x$. Integrating both sides from 0 to ∞ and considering (13), we have

$$J < V_\sigma(0) = x^T(0) P x(0). \tag{29}$$

Pre and post multiplying (18) by $\mathrm{diag}\{\bar{\beta}^{-1}, \mathrm{I}\}$, multiplying by λ_k, summing from $k = 1$ to n_{x_0}, knowing that $\beta = \bar{\beta}^{-1}$ and applying Schur complement, one has

$$\beta \geq x^T(0) P x(0) = V_\sigma(0) > J. \tag{30}$$

Then, from (29) and (30), $J < \beta$ for all $x(0)$ (14).

From Lemma 1, the first constraint of (17) ensures $\mathcal{E}(P, \beta) \subset \mathcal{L}(H_k)$. Then, for all $x(0) \in \mathcal{E}(P, \beta)$, $x(0) \in$

$\mathcal{L}(H_k)$ and the description of saturation function as a convex combination (20) can be used (Alves et al., 2016; Cao and Lin, 2003). Following analogous steps from the proof of the Lemma 1, the second constraint of (17) ensures $\mathcal{E}(P,\beta) \subset \mathcal{X}$ and the uncertain nonlinear system subject to actuator saturation can be exactly described by (11) (Alves et al., 2016). As (15) and (16) ensure that $\mathcal{E}(P,\beta)$ is a positively invariant set (Blanchini, 1999) for the system in closed loop (10) and (11), then these conditions are sufficient for all trajectory started with $x(0) \in \mathcal{E}(P,\beta)$ to remain in $\mathcal{X}$ for all $t > 0$ (Klug et al., 2015), because $V(x) \leq \beta$, for $t \geq 0$ and the proof is concluded.

Usually, controllers may have high gains and these gains can make practical implementations impossible because of high control signal. In view of this problem, high gain values will be avoided by inserting a constraint that imposes $\mathcal{E}(P,\xi) \subset \mathcal{L}(K_j)$. The Theorem 2 is used to decrease high gains in the implementations

Theorem 2. (Hu et al., 2002) Let the set $\mathcal{E}(P,\xi)$ (6), $X = P^{-1}$ and $\xi > 0$. The constraint $\mathcal{E}(P,\xi) \subset \mathcal{L}(K_j)$ is imposed if the following conditions

$$\begin{bmatrix} \rho_l^2\xi^{-1} & M_{j_l} \\ M_{j_l}^T & X \end{bmatrix} \geq 0, \tag{31}$$

is feasible for all $j \in \mathbb{K}_{n_r}$ and $l \in \mathbb{K}_{n_u}$. Note that $\mathcal{L}(K_j)$ is defined as in $\mathcal{L}(H_k)$ (7).

Proof. The proof is similar to the proof from Lemma 1.

Conditions for control of uncertain nonlinear system subject to actuator saturation (11) controlled by single-gain feedback $\text{sat}(u) = \text{sat}(-Kx)$ are proposed in Theorem 3. The origin of the system is locally asymptotically stable respecting the performance index J and decay rate γ.

Theorem 3. Consider an uncertain nonlinear system subject to actuator saturation described by TS fuzzy model (11) in an operation region $\mathcal{X}$ (5) and the performance index J (13). Let $\rho \in \mathbb{R}^{n_u}$, $\phi \in \mathbb{R}^{n_\phi}$ and $N \in \mathbb{R}^{n_\phi \times n_x}$ known as well the range of the uncertain parameters. Suppose the existence of symmetric matrix $X > 0 \in \mathbb{R}^{n_x \times n_x}$, matrices G and $M \in \mathbb{R}^{n_u \times n_x}$ and scalars $\bar{\beta} > 0$ and $\gamma \geq 0$, such that the conditions

$$\underset{X,M,G}{\textit{maximize}} \quad \bar{\beta}$$

subject to

$$\begin{bmatrix} \nu & XC^T \\ CX & -R^{-1} \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \rho_l^2\bar{\beta} & G_l \\ G_l^T & X \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \phi_h^2\bar{\beta} & N_h X \\ XN_h^T & X \end{bmatrix} \geq 0, \tag{33}$$

and (18), $\nu = He\big(A_i X\big) + He\Big(B_i\big[-D_s M + D_s^- G\big]\Big) + 2\gamma X$, are feasible for all $i \in \mathbb{K}_{n_r}$, $l \in \mathbb{K}_{n_u}$, $h \in \mathbb{K}_{n_\phi}$, $k \in \mathbb{K}_{n_0}$, $s \in \mathbb{K}_{2^{n_u}}$, $D_s \in \mathcal{D}$ and $D_s^- = I_m - D_s \in \mathcal{D}$. Then the single-gain control law $u = -Kx$, which $K = MX^{-1}$, applied in the system (11) makes the origin an stable asymptotically equilibrium point with decay rate greater than or equal to γ ensuring the performance index $J < \beta$ for all x_0 belonging to convex set (14).

Proof. Similar to the proof from Theorem 1.

3. EXPERIMENTAL RESULTS

3.1 Active Suspension System with Actuator Fault

The purpose of the active suspension is to decrease oscillations to mitigate the passenger's discomfort. Consider the bench active suspension system manufactured by Quanser® (Quanser, 2010), that was acquired with resources from FAPESP project (2011/17610-0). The schematic model, Fig. 1, and the real system, Fig. 2, used in implementation are shown below.

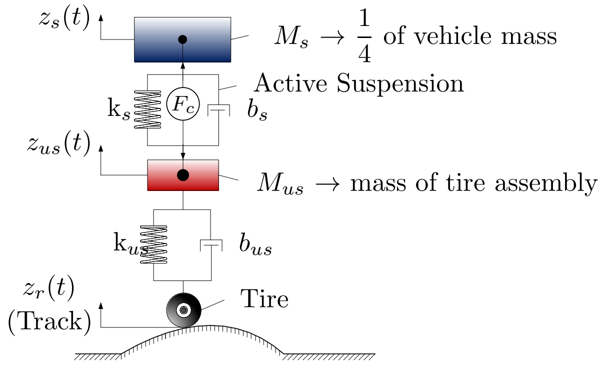

Fig. 1. Schematic of an active suspension system.

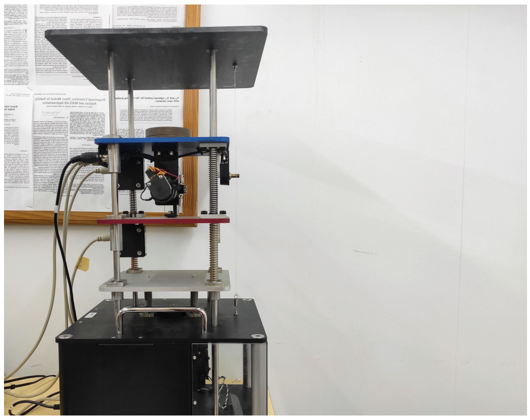

Fig. 2. Bench active suspension system.

The system consists of two masses, denoted by M_s and M_{us}. The mass M_s represents 1/4 of total vehicle mass and is supported by the spring k_s and by the damper b_s. The mass M_{us} corresponds to the tire mass and is supported by the spring k_{us} and by the damper b_{us}. The vibrations caused by irregularities on the road $z_r(t)$ can be attenuated by the active suspension system represented by a motor connected between the masses M_s and M_{us} which imposes the force F_c (Oliveira et al., 2018).

The spring stiffness has nonlinear behavior near the spring ends. Hence, it is adopted a mathematical model presented in Oliveira et al. (2018) for the nonlinearity stiffness

$$k_{us}(z_{us} - z_r, \Delta k_{us}) = k_{us_0}(1 + \Delta k_{us}|z_{us} - z_r|). \tag{34}$$

Furthermore, it is considered an actuator fault resulting in power loss. The power loss is represented by $k_{fault}(t)$. Supposing a control signal fault from controller to actuator, it fallows that

$$u_{fault} = k_{fault}(t)u, \quad u = F_c(t). \tag{35}$$

Thus, the actuator fault can be considered as parametric uncertainty. In the physical model of the active suspension system there is a mass M_s that can assume values between 1.455 and 2.45 kg. Hence, the M_s mass may be uncertain and belongs to interval $1.455 \leq M_s \leq 2.45$ kg.

Therefore, based on the modeling presented by Quanser (2010), considering (34) and (35), the dynamic model of active suspension system is represented by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\dfrac{k_s}{M_s} & -\dfrac{b_s}{M_s} & 0 & \dfrac{b_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_s}{M_{us}} & \dfrac{b_s}{M_{us}} & f_{43}(z) & -\dfrac{(b_s+b_{us})}{M_{us}} \end{bmatrix} x + \begin{bmatrix} 0 \\ \dfrac{k_{fault}}{M_s} \\ 0 \\ -\dfrac{k_{fault}}{M_s} \end{bmatrix} u, \quad (36)$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x, \; f_{43}(z) = -\frac{k_{us_0}(1+\Delta k_{us}|z_{us}-z_r|)}{M_{us}},$$

where $x = [z_s(t) - z_{us}(t) \;\; \dot{z}_s(t) \;\; z_{us}(t) - z_r(t) \;\; \dot{z}_{us}(t)]^T$ and $z = [x^T \; \Delta k_{us} \; k_{fault} \; M_s]^T$.

The Table 1 shows the values of the system parameters (Oliveira et al., 2018; Quanser, 2010).

Table 1. Active suspension parameters.

Parameters	Symbol	Value
Mass of 1/4 of the vehicle (kg)	M_s	$1.455 - 2.45$
Mass of the tire set (kg)	M_{us}	1
Spring Stiffness constant (N/m)	k_s	900
Spring Stiffness constant (N/m)	k_{us_0}	2500
Damping coefficient (Ns/m)	b_s	7.5
Damping coefficient (Ns/m)	b_{us}	5
Parameter of the spring (m^{-1})	Δk_{us_0}	5.71

Then, to find the local models, the methodology proposed in (Santim et al., 2012) is used and allows stabilization for a set of operation points. Considering that the actuator fault can decrease from 0% to 30% of actuator power, then $0.7 \leq k_{fault} \leq 1$. Due the physics of spring length, the state variables x_1 and x_3 are limited in the interval $0.035 \leq x_1, x_3 \leq 0.035$ m. The domain $\mathfrak{D}$ of the premise variables is

$$\begin{aligned} \mathfrak{D} = \{ z = [x^T \; \Delta k_{us} \; k_{fault} \; M_s]^T \in \mathbb{R}^7 : \\ -0.035 \leq x_1, x_3 \leq 0.035, \quad 0 \leq \Delta k_{us} \leq 5.71, \\ 0.7 \leq k_{fault}(t) \leq 1, \quad 1.455 \leq M_s \leq 2.45\}. \end{aligned} \quad (37)$$

The maximum and minimum values of the function $f_{43}(z)$, in domain $\mathfrak{D}$, are

$$-3000 \leq f_{43}(z) \leq -2500. \quad (38)$$

From (36)-(38) and Table 1, the local models are obtained

$$A_1{=}A_3{=}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.34 & -3.06 & 0 & 3.06 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -3000 & -12.5 \end{bmatrix}, B_1{=}B_2{=}\begin{bmatrix} 0 \\ 0.2857 \\ 0 \\ -0.7 \end{bmatrix},$$

$$A_2{=}A_4{=}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.34 & -3.06 & 0 & 3.06 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, B_3{=}B_4{=}\begin{bmatrix} 0 \\ 0.4082 \\ 0 \\ -1 \end{bmatrix}, \quad (39)$$

$$A_5{=}A_7{=}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.55 & -5.15 & 0 & 5.15 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -3000 & -12.5 \end{bmatrix}, B_5{=}B_6{=}\begin{bmatrix} 0 \\ 0.4811 \\ 0 \\ -0.7 \end{bmatrix},$$

$$A_6{=}A_8{=}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.55 & -5.15 & 0 & 5.15 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, B_7{=}B_8{=}\begin{bmatrix} 0 \\ 0.6873 \\ 0 \\ -1 \end{bmatrix}.$$

3.2 Simulation and Practical Implementation

A numerical comparison between switched and single-gain control about β (upper limit of guaranteed cost) is shown with actuator fault. Following, a practical implementation in the bench active suspension system Fig. 2 is also presented.

Firstly, in Fig. 3, the comparison between Theorems 1 and 3 with norm constraint from Theorem 2 is done considering the parameters $\rho = 39.2$, $\phi = [0.035 \;\; 0.035]^T$, $\xi^{-1} = 50000$, $R = I_{2\times 2}$ and $\gamma = 0$. The convex initial conditions set $x(0)$ (14) is formed considering a convex combination of $-0.02 \leq x_1(0), x_3(0) \leq 0.02$ and $x_2(0) = x_4(0) = 0$.

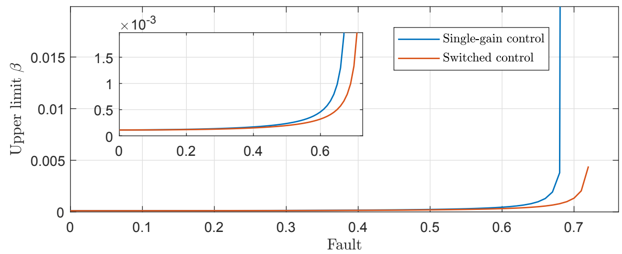

Fig. 3. Upper limit β for single-gain and switched control.

Note that the upper limit β of performance index J (13) for switched controller is smaller than that one for single-gain controller in view of increasing actuator fault. The problem is unfeasible for values greater than 68% fault for single-gain control and 72% fault for switched control.

The switched controller from Theorem 1 considering 30% of actuator fault and the same parameters of previous example is implemented. The optimal solution obtained is $\beta = 1.3007 \times 10^{-4}$ and controller gains are

$$\begin{aligned} K_1 &= [1094.582 \;\; 89.234 \;\; 165.869 \;\; -42.256], \\ K_2 &= [1042.260 \;\; 89.071 \;\; -18.835 \;\; -43.725], \\ K_3 &= [997.004 \;\; 87.989 \;\; -49.455 \;\; -44.913], \\ K_4 &= [896.700 \;\; 85.966 \;\; -385.574 \;\; -49.333], \\ K_5 &= [712.390 \;\; 85.543 \;\; -73.171 \;\; -47.118], \\ K_6 &= [712.390 \;\; 85.543 \;\; -73.171- \;\; 47.118], \\ K_7 &= [654.378 \;\; 83.233 \;\; -250.106 \;\; -49.362], \\ K_8 &= [654.378 \;\; 83.233 \;\; -250.106 \;\; -49.362]. \end{aligned} \quad (40)$$

In implementation is considered $M_s = 2.45$ kg and implementation time 18 seconds. From [0, 6) seconds the system is in open-loop, from [6, 12) seconds the system is in closed-loop with switched control (10) and gains (40) without actuator fault, and from [12, 18] seconds a fault equal to 30% occurs in the control signal. The reference $z_r(t)$ produces a square wave signal with 0.02 m amplitude, 1/3 Hz frequency and 50% pulse width. The relative plate position is shown in the Fig. 4 as well the control signal u and switching index $\sigma(t)$.

The practical implementation decreases the oscillations caused by the road surface $z_r(t)$. The open-loop system is stable but presents wide amplitude oscillations which produces discomfort for vehicle passengers and high level of mechanical stress. The switched controller design by Theorem 1 and 2 reduces the oscillations and establishment time of the plates z_s and z_{us} even considering an

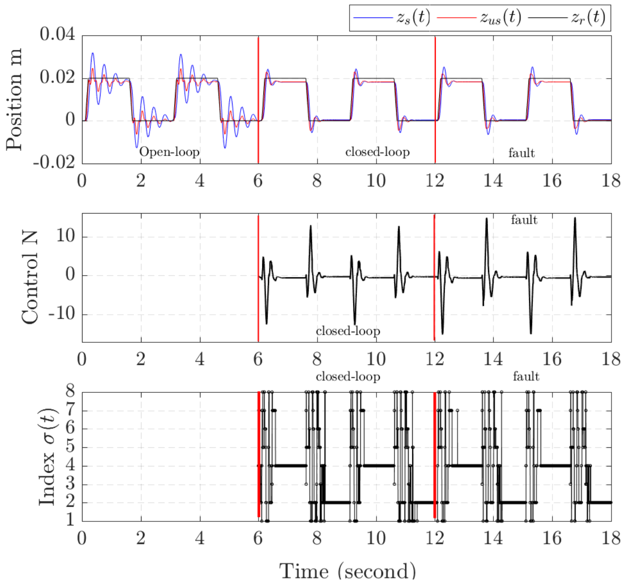

Fig. 4. Implementation of switched control designed using Theorem 1 and constraint of the controller gains (31).

actuator fault, in this case 30% fault. The relative position of the plates are the system output, then the output energy of the system decreases, that is, the cost guaranteed function. It is important to note that the saturation of control signal did not occur but it is considered in control design ensuring stability and performance if it happens.

4. CONCLUSION

The proposed methodology in this manuscript with minimization of upper limit of the quadratic index related to energy of system output allows the control of a class of uncertain nonlinear systems without knowing the membership function. The switched controller can achieve better guaranteed costs compared to the single-gain feedback controller for the same scenario. The practical results for switched controller show that there is a valid and implementable control strategy, which leads to a good performance for the active suspension system even with actuator fault decreasing the oscillations felt by passengers. Another performance index can be included to the design of switched controllers in future works.

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