Optimal Control of Joint Multi-Virus Infection and Information Spreading

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Abstract: Nowadays, epidemic models provide an appropriate tool to describe the propagation of biological viruses in human or animal populations, rumors and misinformation in social networks, and malware in both computer and ad hoc networks. It is common that there are multiple types of malware infecting a network of computing devices, and different messages can spread over the social network. Information spreading and virus propagation are interdependent processes. To capture their independencies, we integrate two epidemic models into one holistic framework, known as the modified Susceptible-Warned-Infected-Recovered-Susceptible (SWIRS) model. The first epidemic model describes the information spreading regarding the risk of malware attacks and possible preventive procedures. The second one describes the propagation of multiple viruses over the network of devices. To minimize the impact of the virus spreading and improve the protection of the networks, we consider an optimal control problem with two types of control strategies: information spreading among healthy nodes and the treatment of infected nodes. We obtain the structure of optimal control strategies and study the condition of epidemic outbreaks. The main results are extended to the case of the network of two connected clusters. Numerical examples are used to corroborate the theoretical findings.

Keywords: Network Security, Optimal Control, Epidemic Process, Information Spreading.

1. INTRODUCTION

Recent advances in information technologies have witnessed an exponential growth in the number of devices connected to the Internet and the rapid expansion of the use of social networks. The proliferation of devices creates opportunities to spread information more conveniently. Still, it has also created a large attack surface for the malware to exploit existing vulnerabilities of the devices and spread malicious codes over the Internet. The channels of malware spreading nowadays are not just limited to computer networks but also include mobile networks and online social networks. Moreover, the wide applications of networks generate an increasing amount of security threats. Computer virus or malware spread and attack a large number of nodes as the network connectivity increases. It can disrupt computer functionalities, collect sensitive, confidential information, and gain illegal access to private computer networks at a much larger scale. Therefore, it is critical to design preventive and effective treatment strategies.

The control of malware spreading can be considered as an optimal control problem that defines a trade-off solution between the cost of fast and periodic development of patches and the value of the recovery of the devices. At the same time, information propagation of the vulnerability of the computing devices and personal accounts in social networks, as well as the knowledge of the effective protection measures, can help raise the awareness of the security threats and their solutions to reduce the number of infected devices. Generally, multiple types of viruses co-exist at the same time. Hence, we model the malware spreading as Susceptible-Infected-Recovered-Susceptible (SIRS) dynamics in which the population of devices is grouped into several subpopulations, i.e., the susceptible (S), the infected (I) and recovered (R). In addition, a group of infected nodes is also divided into several subgroups. The SIRS dynamics describe the evolution of the population size that can be controlled using special patching and recovery. Spreading of information is also described by the modified SIR model, which includes susceptible (S), warned (W), and recovered (R) nodes. Here, warned nodes are informed of the necessity of protection of their accounts and devices from their neighbors.

The goal of the work is to combine the two epidemic processes in one model. One epidemic process describes the dissemination of the information, and the other one is the spreading of viruses. We consider a generalized Susceptible-Warned-Infected-Recovered-Susceptible (SWIRS) model, which extends the model for information

spreading by incorporating the SIRS model that describes the propagation of two types of malware. In the paper, we formulate a controlled SWIR and SWIRS model and show the structure of the optimal policies of spreading information about virus protection and optimal treatment. Moreover, we carry out a series of numerical simulations to corroborate the results.

Recent literature has seen a surge of interest in using optimal control and stability equilibrium analysis to study malware protection in computer networks, social networks, and ad-hoc networks (See Fedyanin (2011); Wu (2013); Sharma (2015); Zuzek (2015); Taynitskiy (2015, 2017); Farooq and Zhu (2019); Moon (2019); Huang and Zhu (2019a)). Moreover, the clusters of the population play an important role. Several waves of the viruses propagation might occur due to sequential propagation information from one cluster to another, even when a single cluster model might predict just a monotone spreading.

In this paper, we establish a control-theoretic model to design optimal quarantining and immunization strategies to mitigate the impact of epidemics on our society. The recent spreading of ransomware (e.g., CryptoLocker, CryptoDefense, or CryptoWall) has spread using spam emails to extort money from home users and businesses alike by locking files on a PC or network storage (See Luo (2009); Newman (2016)). Mean-field dynamical systems are used to model the underlying evolution of the host subpopulations. In Wang (2017), many variants of optimal control models of SIR-epidemics are investigated in the context of medical vaccination and health promotion campaigns. Previous studies have shown the application of epidemic frameworks to the models of network protection as in Mieghem (2009); Sahneh (2013); Vespignani (2015); Farooq and Zhu (2019); Taynitskiy (2017, 2018); Altman (2019). Many different research works have provided variants of epidemic models in computer security. Spreading information on social networks has been studied in Moore (2002).

The rest of the paper is organized as follows. Section 2 presents the controlled SWIRS mathematical model. In Section 2.1, we formulate the SWIRS model. Section 2.2 describes the optimal control problem and Section 2.3 presents the structure of optimal protection and information spreading policies. In Section 3, theoretical results are applied to the case of a clusterized population. Section 4 presents a series of numerical experiments. Section 5 concludes the paper.

2. DETERMINISTIC POPULATION MODEL

2.1 Model formulation

In this Section, we formulate a two-level modified SIRS model (Susceptible-Infected-Recovered-Susceptible) with two different types of viruses circulated in a population of size N. This auxiliary partitioning allows capturing two processes that occur in both computer and social networks. The first process is the propagation of information about harmful malware attacks and the protection of personal data, documents, projects, etc. We consider this spreading process as the first level hierarchy in the Susceptible-Warned-Infected-Recovered-Susceptible (SWIRS) model.

The second process, which corresponds to the physical propagation of antivirus software, is considered as the second level of the model, which is a modified Susceptible-Infected-Recovered-Susceptible (SIRS) model with two competitive viruses. Thereby, in contrast to classical SIRS models (Capasso (1993); Allen (2008)), where populations are divided into three groups: *Susceptible* (S), *Infected* (I), and *Recovered* (R), here the Infected subgroup is divided into two subgroups: a subgroup of nodes infected by the first type of virus V_1 and the subgroup infected by the second type V_2. Spreading information on the first level introduces a new group *Warned* (W) into consideration. This group consists of the nodes, which have received information about the potential risks of virus attack/spreading and methods of protection.
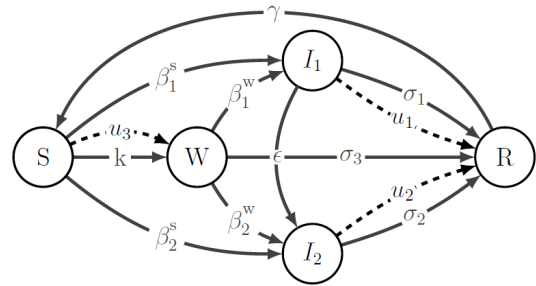

Fig. 1. The scheme of transitions between groups S, W, I_1, I_2, R.

We model the epidemic process as a system of nonlinear differential equations. The total number of nodes in the network during the entire process remains constant and equal to $n_S + n_W + n_{V_1} + n_{V_2} + n_R = N$. Let $S(t) = \frac{n_S(t)}{N}$, $W(t) = \frac{n_W(t)}{N}$, $I_1(t) = \frac{n_{V_1}(t)}{N}$, $I_2(t) = \frac{n_{V_2}(t)}{N}$, $R(t) = \frac{n_R(t)}{N}$ as a fraction of the *Susceptible*, the *Warned*, the *Infected*, and the *Recovered* nodes, respectively. At the beginning of the epidemic, at time $t = 0$, the majority of the individuals are in the Susceptible state, and a small fraction of individuals are infected by different types of virus. Hence, initial states are $S(0) = S^0 > 0$, $W(0) = W^0 \geq 0$, $I_1(0) = I_1^0 > 0$, $I_2(0) = I_2^0 > 0$ and $R(0) = R^0 = 1 - S^0 - W^0 - I_1^0 - I_2^0$.

Behavior of the system is described by a system of nonlinear differential equations:

$$
\begin{aligned}
dS/dt &= -kWS - \beta_1^S SI_1 - \beta_2^S SI_2 + \gamma R - u_3 S; \\
dW/dt &= kWS - \beta_1^W WI_1 - \beta_2^W WI_2 + u_3 S - \sigma_3 W; \\
dI_1/dt &= \beta_1^S SI_1 + \beta_1^W WI_1 - \varepsilon I_1 I_2 - \sigma_1 I_1 - u_1 I_1; \quad (1)\\
dI_2/dt &= \beta_2^S SI_2 + \beta_2^W WI_2 + \varepsilon I_1 I_2 - \sigma_2 I_2 - u_2 I_2; \\
dR/dt &= \sigma_1 I_1 + u_1 I_1 + \sigma_2 I_2 + u_2 I_2 + \sigma_3 W - \gamma R,
\end{aligned}
$$

where β_i^S are infection rates for susceptible nodes for virus V_i, $i = 1, 2$ and β_i^W are infection rates for the warned nodes. On the second level of the epidemic process, we can view a self-recovery rate σ_1 for virus V_1 or σ_2 for virus V_2 as the probability that infected nodes from subgroups I_1 or I_2 are recovered from the infection without incurring any costs on our system. On the first level, nodes that are informed of virus attacks have a recovery rate σ_3. Without loss of generality, we can say that the second virus V_2 is stronger than the first V_1, and with the probability ε virus V_2 can supersede the first virus in the node infected by the first virus.

The application of antivirus patches reduces the number of infected nodes. It can be interpreted as control parameters by $u_1(t)$ and $u_2(t)$ in (1), where u_i are the fractions of the infected under treatment, $u_1(t), u_2(t) \in [0,1]$, for all t. The warned nodes can avoid an epidemic by taking special quarantine measures. Control parameter $u_3(t)$ is the fraction of susceptible nodes that become warned of the virus spreading at time t.

2.2 Optimal Control of Epidemics

Let the objective function J be the sum of two functionals, which correspond to the two levels of the model. On the first level, functional J_1 describes the costs of the quarantine measures, i.e., the costs of disseminating information about the epidemics to susceptible nodes. On the second level, functional J_2 defines the cost of antivirus treatment and includes the costs incurred by infected nodes, costs of spreading antivirus, and the benefit from the recovered nodes.

At any given t, $f_1(I_1(t)), f_2(I_2(t))$ are infection costs; $L(W(t))$ is the utility of the warned nodes. Function $g(R(t))$ defines the benefit rate for recovered nodes; functions $h_1(u_1(t)), h_2(u_2(t))$ are costs for antivirus treatments and $h_3(u_3(t))$ is cost of information spreading. Here functions $f_i(I_i)$ are non-decreasing and twice-differentiable, convex functions, $f_i(0) = 0$, $f_i(I_i) > 0$ for $I_i > 0$, $i = 1, 2$, $g(R)$ and $L(W)$ are non-decreasing and differentiable functions, and $h_i(u_i(t))$ is twice-differentiable and increasing function in $u_i(t)$ such as $h_i(0) = 0$, $h_i(x) > 0$, $i = 1, 2, 3$, when $u_i > 0$. Also costs of information spreading are lower than costs for antivirus treatments $h_3(\cdot) < h_1(\cdot)$ and $h_3(\cdot) < h_2(\cdot)$.

The aggregated system costs over the time interval $[0, T]$ are defined as $J = J_1 + J_2$, where

$$J_1 = \int_0^T h_3(u_3(t)) - L(W(t))dt,$$

$$J_2 = \int_0^T \sum_{q=1}^2 \Big(f_q(I_q(t)) + h_q(I_q(t)) \Big) - g(R(t)). \tag{2}$$

and the optimal control problem is to minimize these costs, i.e., $\min_{\{u_1, u_2, u_3\}} J$.

By using Pontryagin's maximum principle (Pontryagin (1962)), we construct the optimal control $u(t) = (u_1(t), u_2(t), u_3(t))$ of the problem described above in Section 2. To simplify the presentation, we use short-hand notations S, I_1, u_1, etc. in place of $S(t), I_1(t), u_1(t)$, etc. Define the associated Hamiltonian H and the adjoint functions $\lambda_S(t), \lambda_W(t), \lambda_{I_1}(t), \lambda_{I_2}(t), \lambda_R(t)$ as follows:

$$
\begin{aligned}
H = {}& g(R) - f_1(I_1) - f_2(I_2) + L(W) + (\lambda_W - \lambda_S)kWS + \\
& (\lambda_{I_1} - \lambda_S)\beta_1^S SI_1 + (\lambda_{I_2} - \lambda_S)\beta_2^S SI_2 + \\
& (\lambda_{I_1} - \lambda_W)\beta_1^W WI_1 + (\lambda_{I_2} - \lambda_W)\beta_2^W WI_2 + \\
& (\lambda_{I_2} - \lambda_{I_1})\varepsilon I_1 I_2 + (\lambda_R - \lambda_{I_1})\sigma_1 I_1 + \\
& (\lambda_R - \lambda_{I_2})\sigma_2 I_2 + (\lambda_S - \lambda_R)\gamma R + \\
& (-h_1(u_1) + \varphi_1 u_1) + (-h_2(u_2) + \varphi_2 u_2) + \\
& (-h_3(u_3) + \varphi_3 u_3),
\end{aligned}
\tag{3}
$$

where functions $\varphi_i(t)$ are defined as follows:

$$
\begin{aligned}
\varphi_q(t) &= (\lambda_R(t) - \lambda_{I_q}(t))I_q(t), \ q \in \{1, 2\}, \\
\varphi_3(t) &= (\lambda_W(t) - \lambda_S(t))S(t).
\end{aligned}
\tag{4}
$$

The adjoint system is defined as follows:

$$
\begin{aligned}
\dot{\lambda}_S(t) = {}& (\lambda_S - \lambda_W)kW + (\lambda_S - \lambda_{I_1})\beta_1^S I_1 + \\
& (\lambda_S - \lambda_{I_2})\beta_2^S I_2 + (\lambda_S - \lambda_W)u_3; \\
\dot{\lambda}_W(t) = {}& -L'(W) + (\lambda_S - \lambda_W)kS + (\lambda_W - \lambda_{I_1})\beta_1^W I_1 + \\
& (\lambda_W - \lambda_{I_2})\beta_2^W I_2 + (\lambda_W - \lambda_R)\sigma_3; \\
\dot{\lambda}_{I_1}(t) = {}& f_1'(I_1) + (\lambda_S - \lambda_{I_1})\beta_1^S S + (\lambda_W - \lambda_{I_1})\beta_1^W W + \\
& (\lambda_{I_1} - \lambda_{I_2})\varepsilon I_2 + (\lambda_{I_1} - \lambda_R)(\sigma_1 + u_1); \\
\dot{\lambda}_{I_2}(t) = {}& f_2'(I_2) + (\lambda_S - \lambda_{I_2})\beta_2^S S + (\lambda_W - \lambda_{I_2})\beta_2^W W + \\
& (\lambda_{I_1} - \lambda_{I_2})\varepsilon I_1 + (\lambda_{I_2} - \lambda_R)(\sigma_2 + u_2); \\
\dot{\lambda}_R(t) = {}& -g'(R) + (\lambda_R - \lambda_S)\gamma,
\end{aligned}
\tag{5}
$$

with the transversality conditions given by

$$\lambda_S(T) = \lambda_W(T) = \lambda_{I_1}(T) = \lambda_{I_2}(T) = \lambda_R(T) = 0. \tag{6}$$

According to Pontryagin's maximum principle, there exist continuous and piece-wise continuously differentiable co-state functions $\lambda_r(t)$, $r \in \{S, W, I_1, I_2, R\}$ that satisfy (5) and (6) for $t \in [0, T]$ together with continuous functions $u_1^*(t)$, $u_2^*(t)$ and $u_3^*(t)$:

$$
\begin{aligned}
(u_1^*, u_2^*, u_3^*) &\in \\
\arg &\max_{u_1, u_2, u_3 \in [0,1]} H(\lambda, S, W, I_1, I_2, R, u_1, u_2, u_3).
\end{aligned}
\tag{7}
$$

2.3 Structure of Optimal Control

In this subsection, we construct the structure of the optimal control $u^*(t) = (u_1^*(t), u_2^*(t), u_3^*(t))$.

Proposition 1. The following statements hold for the optimal control problem described in Section 2:

- When $h_i(\cdot)$ are concave functions, then there exists $t_0 \in [0, T]$ such that for any $i = 1, 2, 3$:

$$
u_i^*(t) = \begin{cases} 1, & \text{for } 0 \le t \le t_0; \\ 0, & \text{for } t_0 < t \le T. \end{cases}
$$

- When $h_i(\cdot)$ are strictly convex functions, then there exist the time t_0, t_1, $0 < t_0 < t_1 < T$ such that for any $i = 1, 2, 3$ ($\alpha(t) \in (0, 1)$):

$$
u_i^*(t) = \begin{cases} 1, & 0 \le t \le t_0; \\ \alpha(t), & t_0 < t \le t_1; \\ 0, & t_1 < t \le T. \end{cases}
$$

To prove Proposition 1, we consider the following auxiliary lemma.

Lemma 1. Functions φ_i, $i = \overline{1, 3}$ are decreasing functions of t for $t \in [0, T]$.

We can divide this maximization problem into three sub-problems and find optimal control $u_1^*(t)$, $u_2^*(t)$ and $u_3^*(t)$, separately:

$$
\begin{aligned}
\max_{u_1}[-h_1(u_1) + \varphi_1 u_1] &+ \max_{u_2}[-h_2(u_2) + \varphi_2 u_2] + \\
&\max_{u_3}[-h_3(u_3) + \varphi_3 u_3].
\end{aligned}
\tag{8}
$$

We obtain the following derivatives:

$$\frac{\partial H}{\partial u_i} = -\dot{h}_i(u_i) + \psi_i = 0, \ i = \overline{1, 3}. \tag{9}$$

As $h_i(u_i)$ are increasing functions and $I_q \ge 0$ and $S \ge 0$, then the Hamiltonian reaches its maximum if $\psi_i = \dot{h}_i(u_i) \ge 0$, $i = 1, 2, 3$. We can find such u_i if and only if the following conditions are satisfied: $\lambda_R(t) - \lambda_{I_1}(t) \ge 0$,

$\lambda_R(t) - \lambda_{I_2}(t) \geq 0$ and $\lambda_W(t) - \lambda_S(t) \geq 0$. To complete the proof of proposition, we consider the auxiliary lemma.

Lemma 2. For all $t \in [0, T]$, we have $\lambda_R(t) - \lambda_{I_1}(t) \geq 0$, $\lambda_R(t) - \lambda_{I_2}(t) \geq 0$ and $\lambda_W(t) - \lambda_S(t) \geq 0$.

The proof of Lemma 2 consists of two parts: firstly, we consider the case when $t = T$ and show that derivatives of the functions $\lambda_R(t) - \lambda_{I_1}(t)$, $\lambda_R(t) - \lambda_{I_2}(t)$ and $\lambda_W(t) - \lambda_S(t)$ are non-positive; secondly, we prove by contradiction that on the whole interval $[0, T]$ these functions are non-negative. The complete proofs of Proposition 1, Lemma 1 and Lemma 2 follow the same technique as in Altman (2011); Taynitskiy (2018).

Functions $h_i(\cdot)$ are concave
Let $h_i(\cdot)$ be a concave functions ($h_i''(\cdot) < 0$), then according to (3) the Hamiltonian is a convex function of u_i, $i = \overline{1,3}$. There are two different options for $u_i \in [0,1]$ that maximimize the Hamiltonian. If $-h_i(0) + \varphi_i \cdot 0 > -h_i(1) + \varphi_i \cdot 1$ or $h_i(1) > \varphi_i$, then optimal control is $u_i = 0$ (see Fig. 2 (left)); otherwise $- u_i = 1$ (see Fig. 2 (right)).
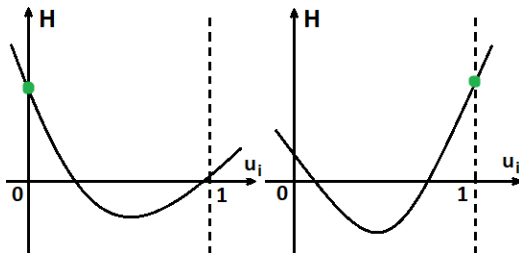

Fig. 2. Hamiltonian if functions $h_i(\cdot)$ are concave.

For $i = \overline{1,3}$, the optimal control parameters $u_i(t)$ are defined as follows:

$$u_i^*(t) = \begin{cases} 0, & \varphi_i(t) < h_i(1), \\ 1, & \varphi_i(t) \geq h_i(1). \end{cases} \quad (10)$$

Functions $h_i(\cdot)$ are strictly convex
Let $h_i(\cdot)$ be a strictly convex functions ($h_i''(\cdot) > 0$), then Hamiltonian is concave function. Consider the following derivative:

$$\frac{\partial}{\partial x}(-h_i(x) + \varphi_i x) \mid_{x=x_i} = 0, \quad (11)$$

where $x \in [0, 1]$, $u_i^*(t) = x_i$. There are three different types of points at which the Hamiltonian reaches its maximum (Fig. 3). To find them, we need to consider the derivatives of the Hamiltonian at $u_i = 0$ and $u_i = 1$. If the derivatives (11) at $u_i = 0$ are non-increasing ($-h_i'(0) + \varphi_i \leq 0$), then the value of the control that maximizes the Hamiltonian is less than 0, and according to our restrictions ($u_i \in [0, 1]$) optimal control will be equal to 0 (Fig. 3a). If the derivatives at $u_i = 1$ are increasing ($-h_i'(1) + \varphi_i > 0$), it means that the value of the control that maximizes the Hamiltonian is greater than 1. Hence the optimal control will be 1 (Fig. 3c); otherwise, we can find such value $u_i^* \in (0, 1)$ (see Fig. 3b):

$$u_i^*(t) = \begin{cases} 0, & \varphi_i \leq h_i'(0), \ i = 1, 2, 3; \\ h'^{-1}(\varphi_i), & h_i'(0) < \varphi_i \leq h_i'(1), \ i = 1, 2, 3. \\ 1, & h_i'(1) < \varphi_i, \ i = 1, 2, 3. \end{cases} \quad (12)$$

Functions $\varphi_i(t)$, $h_i'(t)$, $u_i^*(t)$ are continuous at all $t \in [0, T]$. In this case h_i is strictly convex and h_i' is strictly increasing
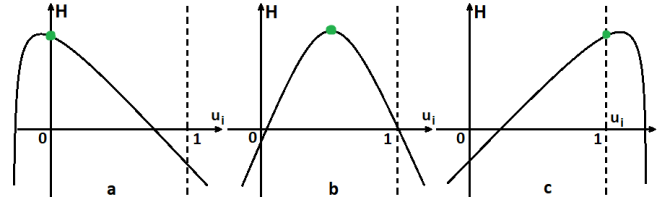

Fig. 3. Hamiltonian when functions $h_i(\cdot)$ are convex.

functions, so $h'(0) < h'(1)$. Thus there exist points t_0 and t_1 ($0 < t_0 < t_1 < T$) so that conditions (12) are satisfied, and according to φ_i are decreasing functions.

3. SWIRS MODEL ON META-POPULATION NETWORK

The clustering of the nodes in the network can be considered as a natural extension of the SWIRS model from Section 2. We assume that all nodes inside the one cluster follow the same behavioral rules. However, the infection can be transferred among clusters. For this reason, we consider a case of a network with N nodes, which can be divided into several clusters. Here, the matrix $A = \{a_{\tau\mu}\}$ is the adjacency matrix of the first level of SWIRS model, where information about possible consequences of malware attacks is spreading, and $B = \{b_{\tau\mu}\}$ is the adjacency matrix of the second level, where special antivirus patches are applied. Denote as $ka_{\tau\mu}$ the probability that a node from cluster τ of size N_τ and a node from a cluster μ of size N_μ change their states from S to W at every time instant. The probability that a susceptible node from cluster τ will be infected due to the contact with a node from a cluster μ, infected by virus $V_l, l \in \{1, 2\}$ is equal to $\beta_{V_l}^S b_{\tau\mu}$. A warned node from cluster τ will be infected by virus V_l, $l = \{1, 2\}$ through the contact with the node from a cluster μ with probability $\beta_{V_l}^W b_{\tau\mu}$.

Vector $X_j(t) = (S_j(t), W_j(t), I_{1j}(t), I_{2j}(t), R_j(t))$ defines the proportions distribution of being in each of the states for the cluster $j = 1, \dots, M$ at t. For any $t \in [0, T]$, the sum of the probabilities for any node j is equal to $S_j(t) + W_j(t) + I_{1j}(t) + I_{2j}(t) + R_j(t) = 1$. All other parameters in the system remain the same as in Section 3.1. This simultaneous process of information spreading and patching is described by a system of nonlinear differential equations:

$$dS_j(t)/dt = -kS_j(t)\sum_l a_{jl}W_l(t) - \beta_1^S S_j(t)\sum_l b_{jl}I_{1l}(t) -$$
$$\beta_2^S S_j(t)\sum_l b_{jl}I_{2l}(t) + \gamma R_j(t) - u_{3j}(t)S_j(t);$$

$$dW_j(t)/dt = kS_j(t)\sum_l a_{jl}W_l(t) - \beta_1^W W_j(t)\sum_l b_{jl}I_{1l}(t) -$$
$$\beta_2^W S_j(t)\sum_l b_{jl}I_{2l}(t) + u_{3j}(t)S_j(t) - \sigma_3 W_j(t);$$

$$dI_{1j}(t)/dt = \beta_1^S S_j(t)\sum_l b_{jl}I_{1l}(t) + \beta_1^W W_j(t)\sum_l b_{jl}I_{1l}(t) -$$
$$\varepsilon I_{1j}(t)\sum_l b_{jl}I_{2l}(t) - \sigma_1 I_{1j}(t) - u_{1j}(t)I_{1j}(t);$$

$$dI_{2j}(t)/dt = \beta_2^S S_j(t)\sum_l b_{jl}I_{2l}(t) + \beta_2^W W_j(t)\sum_l b_{jl}I_{2l}(t) +$$
$$\varepsilon I_{1j}(t)\sum_l b_{jl}I_{2l}(t) - \sigma_2 I_{2j}(t) - u_{2j}(t)I_{2j}(t);$$

$$(13)$$

$$dR_j(t)/dt = \sigma_1 I_{1j}(t) + u_{1j}(t)I_{1j}(t) + \sigma_2 I_{2j}(t) +$$
$$u_{2j}(t)I_{2j}(t) + \sigma_3 W_j(t) - \gamma R_j(t),$$

where $\sum_l$ defines the sum from 1 to M. Initial states are $S_j(0) > 0$, $W_j(0) \geq 0$, $I_{1j}(0) > 0$, $I_{2j}(0) > 0$, $R_j(0) = 1 - S_j(0) - W_j(0) - I_{1j}(0) - I_{2j}(0)$ for all clusters j.

The aggregated system costs on the time interval $[0, T]$ are defined as $J = J_1 + J_2$, where

$$J_1 = \int_0^T h_3 \Big(\sum_j (u_{3j}(t)) \Big) - L \Big(\sum_j W_j(t) \Big) dt,$$

$$J_2 = \int_0^T \sum_{q=1}^2 \Big(f_q \Big(\sum_j (I_{qj}(t)) \Big) + h_q \Big(\sum_j (I_{qj}(t)) \Big) \Big) - \quad (14)$$
$$g \Big(\sum_j R_j(t) \Big) dt.$$

and the optimal control problem is to minimize these costs, i.e., $\min_{\{u_{1j}, u_{2j}, u_{3j}\}} J$.

We focus on a case when both malware can cause extreme damages, and there is a need to lock down the entire system to prevent future destruction. To avoid this lockdown or other expensive security activity, we have to construct a constant control such that any malware will be instantly eliminated, even though the time when the viruses attack the system cannot be precisely identified. We assume that

$$\max\big(h_1(u_{1j}), h_2, (u_{2j}), h_3(u_{3j}), L(W_j), g(R_j)\big) <<$$
$$\min\big(f_1(I_{1j}), f_2(I_{2j})\big),$$
$$\forall j, u_{1j}, u_{2j}, u_{3j}, W_j, R_j, I_{1j}, I_{2j} > 0.$$

We have to define the condition for u which remains system in disease free state with minimum costs. We assume that $h_1(u_{1j}) = h_2(u_{2j}) = h_3(u_{3j}) = u$. The initial state of the system is the equilibrium point E_2 from the Section 3.2. (13) can be reformulated as:

$$\beta_q^S S_j^0 \sum_l b_{jl} I_{ql}^0 + \beta_q^W W_j^0 \sum_l b_{jl} I_{ql}^0 + (-1)^q \varepsilon I_{1j}^0 \sum_l b_{jl} I_{ql}^0 -$$
$$\sigma_q I_{qj}^0 - u_{qj}(0) I_{qj}^0 \leq 0, \ q \in \{1, 2\}. \quad (15)$$

It is assumed that viruses can infect only one node at one time moment, then the system can be transformed in the following way:

$$\beta_q^S S_j^0 b_{jm} + \beta_q^W W_j^0 b_{jm} - \sigma_q - u_{qj}(t) \leq 0, \ q = 1, 2, \quad (16)$$

where m is a node which was infected by a virus. Inequalities (16) can be rewritten as

$$u_{qj}(0) \geq (\beta_q^S S_j^0 + \beta_q^W W_j^0) b_{jm} - \sigma_1, \ q = 1, 2. \quad (17)$$

We find control strategies that maintain the disease free state in the the worst case of epidemics. This value provides an estimation on system costs when $h_j = u_j(t)$ on the time interval $[0, T]$. Summing the control parameters gives:

$$\sum_j (u_{1j}(0) + u_{2j}(0)) \geq$$

$$\Big((\beta_1^S + \beta_2^S) \sum_j S_j^0 + (\beta_1^W + \beta_2^W) \sum_j W_j^0 \Big) b_{jm} -$$
$$M(\sigma_1 + \sigma_2) = U, \quad (18)$$

where $u_{ij}(t)$ is the control of a type $i \in \{1, 2, 3\}$ in a cluster μ at time t. As a result, we obtain

$$J \to T \cdot \Big(\min(h_1(U), h_2(U)) - L(W(0)) - g(R(0)) \Big). \quad (19)$$

4. NUMERICAL EXPERIMENTS

In this section, we present numerical case studies to corroborate our results. For the experiments, we use the following costs functions: infection costs – $f_1(I_1(t)) = 30I_1(t)$ and $f_2(I_2(t)) = 40I_2(t)$; treatment costs – $h_1(u_1(t)) = 20u_1^2(t)$, $h_2(u_2(t)) = 25u_1^2(t)$; vaccination cost – $h_3(u_3(t)) = 10u_3^2(t)$; and utility functions are $L(W(t)) = 2W(t)$ and $g(R(t)) = 5R(t)$. The time interval in the first two experiments is equal to $[0,20]$.
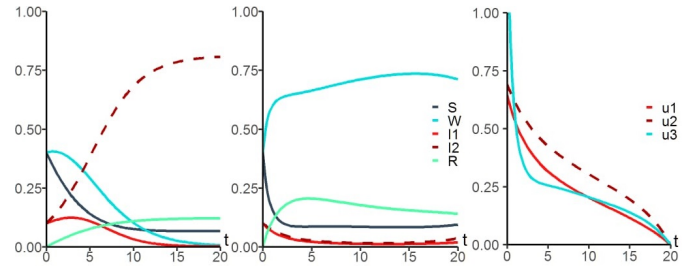
Fig. 4. Experiment I: Behavior of the system in the uncontrolled case (left), the controlled case (middle) and the structure of the optimal control(right). Parameters are: $k = 0.3$, $\beta_1^S = 0.35$, $\beta_2^S = 0.45$, $\beta_1^W = 0.25$, $\beta_2^W = 0.35$, $\sigma_1 = 0.05$, $\sigma_2 = 0.03$, $\sigma_3 = 0.01$, $\gamma = 0.2$, $\varepsilon = 0.5$).

Experiment I shows the behavior of the SWIRS-model in two different cases: controlled and uncontrolled ones (see Fig. 4). In the uncontrolled cases, at $T = 20$ the majority of nodes are infected by virus V_2 ($I_2(20) = 0.77$). The values of the two cost functionals are equal to $J_1 = -2.86$ and $J_2 = 10.41$. After the treatment and information dissemination about possible epidemic outbreaks, all infected nodes are cured. Here, all nodes are in the disease-free state ($S(20) = 0.29$, $W(20) = 0.23$, $R(20) = 0.48$) and the values of the two cost functionals are equal to $J_1 = -11.12$ and $J_2 = 0.58$, respectively. Comparing the aggregated costs in the uncontrolled case ($J_{uncntl} = 7.55$) and the controlled case ($J_{cntl} = -10.54$), we can see that information spreading and the applied treatment are beneficial.
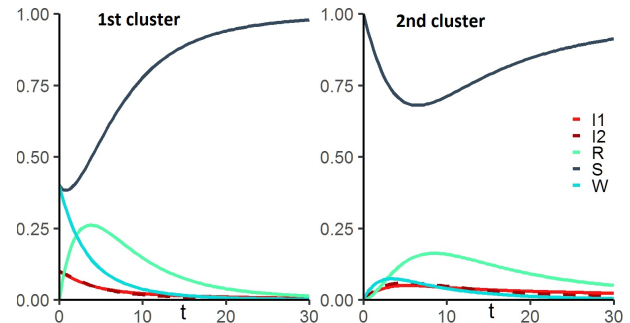
Fig. 5. Experiment II: Behavior of the system in two different clusters of the population. Parameters are: $k = 0.15$, $\beta_1^S = 0.25$, $\beta_2^S = 0.3$, $\beta_1^W = 0.2$, $\beta_2^W = 0.25$, $\sigma_1 = 0.3$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $\gamma = 0.3$, $\varepsilon = 0.5$).

Experiment II presents the SWIRS model on a metapopulation network. The behavior of the system (13) in

two different clusters is represented in Fig. 5. Matrices A, B indicate the strong connections between these clusters, hence the epidemic which has been started in the first cluster continue in the second one:

$$A = B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \qquad (20)$$

Initial parameters are $X_1(0) = (0.4, 0.4, 0.1, 0.1, 0)$ and $X_2(0) = (1, 0, 0, 0, 0)$. Final states are $X_1(30) = (0.97, 0, 0, 0, 0.03)$ and $X_2(30) = (0.9, 0, 0.02, 0.02, 0.06)$.

5. CONCLUSIONS

This paper presents a modified Susceptible-Warned-Infected-Recovered-Susceptible (SWIRS) model of simultaneous spreading of the virus protection information and the malware over a large population of nodes. We have obtained the structure of the optimal control as well as the properties of feasible controls for a special class of cost functions. Numerical examples have been used to corroborate the results. We would further explore the extension of the SWIRS model to an epidemic model over complex networks with different topologies and design the optimal control strategies in the meta-population SWIRS model.

6. ACKNOWLEDGEMENT

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