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Lyapunov design of least-squares model-reference adaptive control

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Abstract: A Lyapunov design of a least-squares model-reference adaptive control (LS-MRAC) algorithm is presented. The plants considered are continuous with relative degree one. A Monopoli multiplier, originally proposed to extend the MRAC algorithm to the case of relative degree two, is introduced. As a result, fast convergence of the tracking error is achieved and, moreover, the Lyapunov analysis shows that a quadratic term depending on the parametric error belongs to $\mathcal{L}_2$, which improves the stability properties of the system. This is the key feature that allows a more powerful LS algorithm to be employed in the update law. The resulting LS-MRAC seems to be a missing algorithm in the literature. Simulation results illustrate the improvement in the transient behavior as well as in the parameter convergence attained with the proposed adaptive schemes.

Keywords: Adaptive control, model-reference adaptive control, stability, transient performance, least-squares adaptive law.

1. INTRODUCTION

The normalized least-squares (LS) algorithm was introduced in the literature on continuous time adaptive control in (Goodwin and Mayne, 1987). Recognized to have a vastly superior convergence rate, the idea was pursued in other works. In (Sastry and Bodson, 1989) a slight variant of the algorithm is presented. For the analysis, the identification structure is separated from the control structure. An error equation is obtained in the so called *linear form* with respect to the parameter error which allows the use of well known identification approaches. The same idea can be found in (Ioannou and Sun, 1996). To assure stability of the closed loop connection of the identification and controller structures, a projection is introduced in the update law to make the high frequency gain estimate bounded away from zero. The algorithm however is sensitive to the initial conditions and a large transient tracking error may be observed.

The composite adaptive controller presented in (Slotine and Li, 1991) also employs a least-squares update law. However, its implementation assumes that the derivative of the output error is available.

Here, the availability of the output error derivative is circumvented by employing a Monopoli's multiplier, originally introduced to generalize the MRAC for plants with relative degree 2. Two remarkable properties are achieved with this modified MRAC: (1) The tracking error dynamics is strikingly improved. The convergence rate is shown to depend explicitly on the adaptation gain and on the level of the excitation signal, a result that is not observed in the conventional MRAC algorithm. (2) The Lyapunov analysis allows to conclude that a quadratic $\tilde{\theta}$-term belongs to $\mathcal{L}_2$, where $\tilde{\theta}$ is the parameter error. In the conventional MRAC design, it is only shown that the tracking error has this property. This result is the key for the introduction of a least-squares update law in the design, which is referred here as LS-MRAC.

Both algorithms, the modified MRAC and the LS-MRAC are shown to be at least uniformly asymptotically stable. Only the case of plants with relative degree equal one is considered in this note. Simulation results are presented to illustrate the improvement achieved in the tracking dynamics and parameter convergence of the algorithms.

2. STRUCTURE OF THE CONTROLLER

Consider a linear plant with unknown parameters given by (Tao, 2003)

$$y = P(s)u\,, \quad P(s) = k_p \frac{N_p(s)}{D_p(s)}\,, \tag{1}$$

where u is the control signal and y is the plant output. Also consider a *reference model* described by

$$y_m = M(s)r\,, \quad M(s) = k_m \frac{N_m(s)}{D_m(s)}\,, \tag{2}$$

where r is a reference input signal.

The objective of the controller, as usually stated in the literature, is to find a control $u(t)$ that stabilizes the closed loop system and such that the *output error* (or *tracking error*)

$$e_0 = y - y_m\,, \tag{3}$$

tends to zero asymptotically for arbitrary initial conditions and arbitrary piece–wise continuous uniformly bounded reference signals r. Here, in particular, good convergence properties of the controller parameters are also aimed.

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The following *basic assumptions* summarize the prior available information regarding $P(s)$:

(1) the order of the plant n is known,
(2) $n^* = degree[D_p(s)] - degree[N_p(s)] = 1$,
(3) $N_p(s)$ is Hurwitz, i.e., $P(s)$ is minimum phase, and
(4) the sign of the high frequency gain k_p is known.

In view of the above assumption (2), and for simplicity, $M(s)$ is chosen as

$$M(s) = \frac{k_m}{s + a_m}\,, \tag{4}$$

and, without loss of generality, $k_m > 0$. As usual, *state variable filters* (SVF's)

$$\dot{v}_1 = \Lambda v_1 + gu\,, \qquad \dot{v}_2 = \Lambda v_2 + gy\,, \tag{5}$$

where Λ is chosen such that $\det(sI - \Lambda) = N_m(s)$, and v_1, $v_2 \in \mathbb{R}^{n-1}$, are used to form the *regressor vector*

$$\omega = [v_1^T \quad y \quad v_2^T \quad r]^T \in \mathbb{R}^{2n}\,. \tag{6}$$

Assumptions (1)–(2) assure the existence and uniqueness of a constant parameter vector $\theta^* = [\theta_1^{*T} \quad \theta_n^* \quad \theta_2^{*T} \quad \theta_{2n}^*]^T$ such that the transfer function of the closed–loop system with $u = \theta^{*T}\omega$ matches $M(s)$ exactly, i.e., $y = P(s)u = P(s)\theta^{*T}\omega = M(s)r$.

3. THE ERROR EQUATION

Define the vectors $\bar{\omega} = [v_1^T \quad y \quad v_2^T]^T \in \mathbb{R}^{2n-1}$ and $\bar{\theta}^* = [\theta_1^{*T} \quad \theta_n^* \quad \theta_2^{*T}]^T \in \mathbb{R}^{2n-1}$ so that the matching control can be written as $u = \bar{\theta}^{*T}\bar{\omega} + \theta_{2n}^* r$.

The plant (1), with the filters (5), can be rewriting as (Slotine and Li, 1991)

$$\begin{aligned} y &= P(s)\big[u - \theta^{*T}\omega + \theta^{*T}\omega\big] \\ &= P(s)\big[\theta_1^{*T}v_1 + \theta_n^* y + \theta_2^{*T}v_2 + \theta_{2n}^* r + u - \theta^{*T}\omega\big] \\ &= P(s)\big[\bar{\theta}^{*T}\bar{\omega} + \theta_{2n}^*\big(r + k^*u - k^*\theta^{*T}\omega\big)\big] \\ &= M(s)r + M(s)k^*[u - \theta^{*T}\omega]\,, \end{aligned} \tag{7}$$

where $k^* = (\theta_{2n}^*)^{-1} = k_p/k_m$. Then, from the above, one has that the error equation of the system is given by

$$e_0 = M(s)k^*[u - \theta^{*T}\omega]\,, \tag{8}$$

irrespective of how u is defined. An exponentially decaying term, due to the initial conditions, will be omitted hereafter whenever unessential for the theoretical development. Recalling that the order of the system (7) is $3n-2$ and defining the error vector $e \in \mathbb{R}^{3n-2}$, one has the following non–minimal state space realization of (8)

$$\dot{e} = A_m e + B_m k^*[u - \theta^{*T}\omega]\,, \tag{9}$$

$$e_0 = C_m e\,. \tag{10}$$

4. REVIEW OF LYAPUNOV DESIGN

The Lyapunov approach to the MRAC design is relatively simple and elegant (Tao, 2003). The control law is given by

$$u = \theta^T\omega\,, \tag{11}$$

where the controller parameter $\theta \in \mathbb{R}^{2n}$ should be adaptively adjusted to ensure desired performance. Defining the *parameter error* $\tilde{\theta} = \theta - \theta^*$, the error equation can be rewritten as

$$e_0 = M(s)k^*\left[\tilde{\theta}^T\omega\right]. \tag{12}$$

Since the reference model $M(s)$ is chosen *strictly positive real* (SPR), this been fundamental for the stability analysis, from the Popov–Kalman–Yakubovich (PKY) Lemma (Krstić et al., 1995), there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that the matrices $\{A_m, B_m, C_m\}$ of the state space realization (9) satisfies

$$A_m^T P + PA_m = -2Q\,, \tag{13}$$

$$PB_m = C_m^T\,. \tag{14}$$

The following Lyapunov function is considered

$$2V_1(e, \tilde{\theta}) = e^T Pe + \gamma^{-1}|k^*|\tilde{\theta}^T\tilde{\theta}\,, \tag{15}$$

where γ is a positive constant. Then, using (13) and (14) the derivative of V_1 along (9) is given by

$$\dot{V}_1 = -e^T Qe + \gamma^{-1}|k^*|\tilde{\theta}^T\left[\gamma\,\mathrm{sign}(k^*)\omega e_0 + \dot{\tilde{\theta}}\right]. \tag{16}$$

By choosing the update law

$$\dot{\tilde{\theta}} = \dot{\theta} = -\gamma\mathrm{sign}(k^*)\omega e_0\,, \tag{17}$$

the $\tilde{\theta}$-term is cancelled and (16) reduces to

$$\dot{V}_1 = -e^T Qe \leq 0\,. \tag{18}$$

This guarantees $e \in \mathcal{L}_\infty$ and $\tilde{\theta} \in \mathcal{L}_\infty$. Since the reference signal $r \in \mathcal{L}_\infty$, then $y \in \mathcal{L}_\infty$. Because ω constitutes a part of the system (9), it follows that $\omega \in \mathcal{L}_\infty$ and $u \in \mathcal{L}_\infty$. Therefore, the update law (17) assures global uniform stability of the system.

The convergence properties of the signals are concluded as follows. Observing that $V_1(t)$ is monotone non–increasing along the trajectories of (9) and (17), bounded above by $V(0)$ and below by 0, one can conclude that $e \in \mathcal{L}_2$. From (9), it follows that $\dot{e} \in \mathcal{L}_\infty$ and, consequently, e is uniformly continuous. Then, using the Barbalat's Lemma, it follows that $e(t) \to 0$ as $t \to \infty$ (Tao, 2003).

From (12), one can see that if the control mismatch $\tilde{u} = \tilde{\theta}^T\omega \approx 0$ then $e_0 \approx 0$. This condition is achieved when $\tilde{\theta} \approx 0$ (good identification) or, more often, when $\tilde{\theta}$ and ω are close to be orthogonal. To assure that $\tilde{\theta}(t) \to 0$, a *persistent excitation* condition is required (Tao, 2003). However, even with persistent excitation, the quality of the adaptation transient is not uniform and the convergence of $e(t)$ or $\tilde{\theta}(t)$ to zero may be very slow (Hsu and Costa, 1987) (Narendra, 1994).

A high adaptation gain γ may not necessarily result in an improvement of the convergence behaviour. This is illustrated by a simulation result in the sequel. By the other hand, it is usually accepted that γ should be kept small for robustness purposes.

5. MODIFIED MRAC DESIGN

In this section the Lyapunov design of the MRAC is modified in order to improve its transiente behavior. The idea comes from (Costa, 1999; Pinto and Costa, 2008) where a lead filter is employed to obtain a derivative of e_0. Here, the use of a lead filter is avoided by introducing a Monopoli's multiplier (Ioannou and Sun, 1996) in the design.

The control law is now defined as

$$u = L(s)\theta^T\xi = \theta^T\omega + \dot{\theta}^T\xi\,, \tag{19}$$

where $L(s) = s + \lambda$, $\lambda > 0$, and

$$\xi = L^{-1}(s)\omega\,. \tag{20}$$

With this modification in the control law, the error equation becomes

$$\begin{aligned} e_0 &= M(s)k^*\big[L(s)\theta^T\xi - \theta^{*T}\omega\big] \\ &= M(s)L(s)k^*\big[\tilde{\theta}^T\xi\big]\,. \end{aligned} \tag{21}$$

Notice that $M(s)L(s)$ can be decomposed as

$$M(s)L(s) = \frac{k_m(s+\lambda)}{s+a_m} = \frac{k_m(\lambda - a_m)}{s+a_m} + k_m\,, \tag{22}$$

where, for $\lambda > a_m$, $\frac{k_m(\lambda-a_m)}{s+a_m}$ is a SPR transfer function.

A non-minimal realization of (21) is given by

$$\dot{e} = A_m e + B_m'[k^*\tilde{\theta}^T\xi]\,, \tag{23}$$

$$e_0 = C_m e + k_m[k^*\tilde{\theta}^T\xi]\,, \tag{24}$$

where $B_m' = (\lambda - a_m)B_m$ and $\{A_m, B_m', C_m\}$ satisfies the PKY lemma,

$$A_m^T P' + P'A_m = -2Q'\,, \tag{25}$$

$$P'B_m' = C_m^T\,, \tag{26}$$

with $P' = (\lambda - a_m)^{-1}P$ and $Q' = (\lambda - a_m)^{-1}Q$ (P and Q from (13)).

Consider the partial Lyapunov function (the signal ξ is missing)

$$2V_2(e,\tilde{\theta}) = e^TP'e + \gamma^{-1}|k^*|\tilde{\theta}^T\tilde{\theta}\,, \tag{27}$$

where $\gamma > 0$. The derivative of V_2 along the trajectories of (23) is given by

$$2\dot{V}_2 = -2e^TQ'e + 2e^TP'B_m'k^*\tilde{\theta}^T\xi + 2\gamma^{-1}|k^*|\tilde{\theta}^T\dot{\theta}\,.$$

From (24) one has that

$$e^TP'B_m' = e^TC_m^T = e_0 - k_m[k^*\tilde{\theta}^T\xi]\,. \tag{28}$$

Therefore,

$$\begin{aligned} \dot{V}_2 &= -e^TQ'e + \big(e_0 - k_mk^*\tilde{\theta}^T\xi\big)k^*\tilde{\theta}^T\xi + \gamma^{-1}|k^*|\tilde{\theta}^T\dot{\theta} \\ &= -e^TQ'e - k_m\big(k^*\tilde{\theta}^T\xi\big)^2 + \\ &\qquad + \gamma^{-1}|k^*|\tilde{\theta}^T\big[\gamma\,\mathrm{sign}(k^*)\xi e_0 + \dot{\theta}\big]\,. \end{aligned} \tag{29}$$

By choosing the update law

$$\dot{\theta} = -\gamma\,\mathrm{sign}(k^*)\xi e_0\,, \tag{30}$$

the derivative (29) reduces to

$$\dot{V}_2 = -e^TQ'e - k_m\big(k^*\tilde{\theta}^T\xi\big)^2 \le 0\,. \tag{31}$$

Compared with (18), the derivative now has a non-positive $\tilde{\theta}^T\xi$-term. Using similar arguments as in the previous analysis, one can conclude that $e \in \mathcal{L}_\infty$, $\theta \in \mathcal{L}_\infty$, $\omega \in \mathcal{L}_\infty$, and, moreover, $e \in \mathcal{L}_2$, $\tilde{\theta}^T\xi \in \mathcal{L}_2$.

To prove the boundedness of ξ, a fictitious regressor signal ω_m, obtained from state variable filters applied to the reference model, is employed to define the error (Ioannou and Sun, 1996) (Hsu et al., 2006)

$$\tilde{\xi} = \xi - \xi_m\,,$$

where $\xi_m = L^{-1}\omega_m \in \mathcal{L}_\infty$. This error signal $\tilde{\xi}$ can be expressed as the output of a stable and proper filter with inputs e_0 and r, i.e.,

$$\tilde{\xi} = f(s)e_0 + g(s)r = \tilde{\xi}_0 + \tilde{\xi}_r\,.$$

Obviously, the term $\tilde{\xi}_r \in \mathcal{L}_\infty$. For the term $\tilde{\xi}_0$, one can write the following state space realization

$$\dot{\varepsilon} = A_0\varepsilon + B_0e_0\,, \tag{32}$$

$$\tilde{\xi}_0 = C_0\varepsilon\,. \tag{33}$$

Because this filter is stable, there exist $P_0 = P_0 > 0$ and $Q_0 = Q_0 > 0$ such that $A_0^TP_0 + P_0A_0 = -2Q_0$.

Now consider the true Lyapunov function

$$2V_3(e,\tilde{\theta},\varepsilon) = 2V_2(e,\tilde{\theta}) + \alpha\varepsilon^TP_0\varepsilon\,, \tag{34}$$

where $\alpha > 0$. From (31) and (32), the derivative of V_3 is given by

$$\dot{V}_3 = -e^TQ'e - k_m\big(k^*\tilde{\theta}^T\xi\big)^2 - \alpha\big[\varepsilon^TQ_0\varepsilon - \varepsilon^TP_0B_0e_0\big]\,.$$

Using (24),

$$\begin{aligned} \dot{V}_3 &= -e^TQ'e - k_m\big(k^*\tilde{\theta}^T\xi\big)^2 - \alpha\varepsilon^TQ_0\varepsilon + \\ &\qquad + \alpha\varepsilon^TP_0B_0C_me + \alpha k_m\varepsilon^TP_0B_0\big(k^*\tilde{\theta}^T\xi\big)\,, \end{aligned} \tag{35}$$

which is made negative semidefinite by choosing α sufficiently small (using Schur's complement). With this, one concludes that $\varepsilon \in \mathcal{L}_\infty$, $\tilde{\xi}_0 \in \mathcal{L}_\infty$, $\tilde{\xi} \in \mathcal{L}_\infty$, and $\xi \in \mathcal{L}_\infty$. As a consequence, $\dot{e} \in \mathcal{L}_\infty$, $\dot{\theta} \in \mathcal{L}_\infty$, and $\dot{\xi} \in \mathcal{L}_\infty$.

Therefore, the modified algorithm guarantees that the system is at least globally uniformly stable. Using the same arguments as in the analysis of the previous section, one has that $\lim_{t\to\infty} e(t) = 0$ and, furthermore, $\lim_{t\to\infty} \tilde{\theta}^T\xi(t) = 0$. This was expected since the transfer function $M(s)L(s)$ in (21) has relative degree zero. From (21),

$$\begin{aligned} \dot{e}_0 &= -a_me_0 + k^*\dot{\tilde{\theta}}^T\xi + k^*\tilde{\theta}^T\dot{\xi} + \lambda k^*\tilde{\theta}^T\xi \\ &= -a_me_0 - k^*\big[\gamma\,\mathrm{sign}(k^*)\xi e_0\big]^T\xi + k^*\tilde{\theta}^T\big(\dot{\xi} + \lambda\xi\big) \\ &= -\big(a_m + \gamma\,|k^*|\xi^T\xi\big)e_0 + k^*\tilde{\theta}^T\omega\,, \end{aligned} \tag{36}$$

which shows that the rate of convergence of e_0 now depends explicitly on the adaptation gain γ. The term $\gamma\,|k^*|\xi^T\xi \ge 0$ is a time-varying gain feedback that can be increased by selecting a large γ and a large level of excitation $r(t)$.

The control mismatch now is $\tilde{u} = \tilde{\theta}^T\omega + \dot{\theta}^T\xi$. This means that it is possible to attain $\tilde{u} \approx 0$ and, consequently, $e_0 \approx 0$ even with large $\tilde{\theta}^T\omega$. In other words, $e_0 \approx 0$ when $\tilde{\theta}^T\omega \approx -\dot{\theta}^T\xi$. This is an extra degree of freedom for the control action, which is illustrated by simulation results in section 7.

The convergence of $\tilde{\theta}$ to zero, of course, depends on a persistent excitation condition of the system. Table 1 summarizes the algorithm.

6. LEAST-SQUARES MRAC DESIGN

The term $-k_m\big(k^*\tilde{\theta}^T\xi\big)^2$ obtained in (31) is the key to allow a more powerful least-squares algorithm to be employed.

Output error	$e_0 = y - y_m$
SVF	$\dot v_1 = \Lambda\, v_1 + g\, u$ $\dot v_2 = \Lambda\, v_2 + g\, y$ Λ s.t. $\det(sI-\Lambda) = N_m(s)$
Regressor	$\omega^T = \begin{bmatrix} v_1^T & y & v_2^T & r \end{bmatrix}$
Filter	$\xi = L^{-1}(s)\,\omega$ $L(s) = s+\lambda\,, \quad \lambda > a_m$
Control	$u = \theta^T\omega + \dot\theta^T\xi$
Update law	$\dot\theta = -\gamma\,\mathrm{sign}(k^*)\xi e_0$

Table 1. Modified MRAC.

Consider the partial Lyapunov function

$$2V_4(e,\tilde\theta) = e^T P' e + \gamma^{-1}|k^*|\tilde\theta^T R^{-1}(t)\tilde\theta\,, \qquad (37)$$

where $R(t)$ is a *covariance matrix* with $R(0) = R^T(0) > 0$.

Deriving along (23) and using (25), (26), (28), and the fact that $\dot R^{-1} = -R^{-1}\dot R R^{-1}$, one obtains

$$\dot V_4 = -e^T Q' e - k_m\big(k^*\tilde\theta^T\xi\big)^2 + \\ + \gamma^{-1}|k^*|\tilde\theta^T R^{-1}\big[\gamma\,\mathrm{sign}(k^*)R\xi e_0 + \dot\theta\big] - \\ - \frac{1}{2}\gamma^{-1}|k^*|\tilde\theta^T R^{-1}\dot R R^{-1}\tilde\theta\,.$$

Choosing the update laws

$$\dot\theta = -\gamma\,\mathrm{sign}(k^*)R\xi e_0\,, \qquad (38)$$

$$\dot R = -R\xi\xi^T R\,, \qquad (39)$$

one gets

$$\dot V_4 = -e^T Q' e - k_m\big(k^*\tilde\theta^T\xi\big)^2 + \frac{1}{2}\gamma^{-1}|k^*|\tilde\theta^T\xi\xi^T\tilde\theta \\ = -e^T Q' e - \frac{1}{2}|k^*|\big(2k_m|k^*| - \gamma^{-1}\big)\big(\tilde\theta^T\xi\big)^2\,, \qquad (40)$$

which is semi-definite negative for $\gamma > \frac{1}{2|k_p|}$. This proves that $e\in\mathcal{L}_\infty$, $\tilde\theta^T R^{-1}\tilde\theta\in\mathcal{L}_\infty$, $\omega\in\mathcal{L}_\infty$, $e\in\mathcal{L}_2$, $\tilde\theta^T\xi\in\mathcal{L}_2$.

Boundedness of R and θ are obtained as in (Tao, 2003, p. 104).

Since $R(0) = R^T(0) > 0$, then

$$\dot R^{-1}(t) = -R^{-1}\dot R R^{-1} = \xi\xi^T.$$

This means that

$$R^{-1}(t) = R^{-1}(0) + \int_0^t \xi(\tau)\xi^T(\tau)d\tau \geq 0\,, \quad t\geq 0\,. \qquad (41)$$

Therefore, $R^{-1}(t) > R^{-1}(0)$, and so $R(t) > 0$, $\forall t\geq 0$, and $R\in\mathcal{L}_\infty$. From (37) and (41), one has

$$2V_4 = e^T P' e + \gamma^{-1}|k^*|\tilde\theta^T R^{-1}(0)\tilde\theta + \\ + \gamma^{-1}|k^*|\tilde\theta^T\Big(\int_0^t \xi(\tau)\xi^T(\tau)d\tau\Big)\tilde\theta\,.$$

Since $V_4\in\mathcal{L}_\infty$, then the term $\tilde\theta^T R^{-1}(0)\tilde\theta\in\mathcal{L}_\infty$ and, hence, $\tilde\theta\in\mathcal{L}_\infty$.

Boundedness of ξ and convergence of e_0 to zero are concluded using the same arguments as in the previous section. The derivative of the true Lyapunov function

$$2V_5(e,\tilde\theta,\varepsilon) = 2V_4(e,\tilde\theta) + \alpha\varepsilon^T P_0\varepsilon\,, \qquad (42)$$

can be made negative semidefinite with a sufficiently small $\alpha > 0$. Hence, $\xi\in\mathcal{L}_\infty$, $\dot e\in\mathcal{L}_\infty$, $\dot\theta\in\mathcal{L}_\infty$, and $\dot R\in\mathcal{L}_\infty$. It follows that $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty}\tilde\theta^T\xi(t) = 0$. From (21),

$$\dot e_0 = -\big(a_m + \gamma\,|k^*|\xi^T R\,\xi\big)e_0 + k^*\tilde\theta^T\omega\,,$$

which shows that the rate of convergence of e_0 depends on γ and R.

Output error	$e_0 = y - y_m$
SVF	$\dot v_1 = \Lambda\, v_1 + g\, u$ $\dot v_2 = \Lambda\, v_2 + g\, y$ Λ s.t. $\det(sI-\Lambda) = N_m(s)$
Regressor	$\omega^T = \begin{bmatrix} v_1^T & y & v_2^T & r \end{bmatrix}$
Filter	$\xi = L^{-1}(s)\,\omega$ $L(s) = s+\lambda\,, \quad \lambda > a_m$
Control	$u = \theta^T\omega + \dot\theta^T\xi$
Update laws	$\dot\theta = -\gamma\,\mathrm{sign}(k^*)R\xi e_0\,, \quad \gamma > \frac{1}{2\|k_p\|}$ $\dot R = -R\xi\xi^T R\,, \quad R(0) = R^T(0) > 0$

Table 2. Least-squares MRAC.

Table 2 summarizes the algorithm. The implementation requires a bound for the gain γ. Thus, besides the assumptions regarding $P(s)$ stated in section 2, the following a priori knowledge is also necessary:

(5) A lower bound for $|k_p|$, the unknown plant parameter, is known.

Remark. The adaptive algorithms based on least-squares presented in (Sastry and Bodson, 1989) and (Ioannou and Sun, 1996) consider the controller structure and the identification structure separately. Thus, to ensure stability of their closed loop connection, a projection is introduced which also requires a prior knowledge about a bound on $|k_p|$ to be implemented. Here, the necessity of this prior knowledge arises directly from the Lyapunov stability analysis.

7. SIMULATION RESULTS

The improved transient and convergence behavior attained by the proposed algorithms are illustrated by some simulation results. The simulations are carried out with a fourth order plant

$$P(s) = \frac{0.3(s+2)^3}{s^4}\,. \qquad (43)$$

The reference model is

$$M(s) = \frac{1}{s+1}\,. \qquad (44)$$

For this example, the matching parameter is $\theta^* = [-7.25,\ -9.25,\ -3,\ -13.33,\ 7.5,\ 25,\ 20.83,\ 3.33]$, $\|\theta^*\| = 38.1$, $k^* = 0.3$.

For all simulations

$$y(0) = 10\,.$$

All other initial conditions are zero.

Two reference signals are used in the simulations:

$$r_{sin}(t) = 3 + \sin(t) + \sin(3t) + \sin(5t) + \sin(7t)\,,$$
$$r_{sqw}(t) = 3 + 10\,\mathrm{sign}(\sin(0.1\pi t))\,.$$

7.1 Simulation 1 - Standard MRAC algorithm

The following data are used:

$$r(t) = r_{sin}(t)\,, \quad \gamma = 20\,.$$

The result of this simulation is shown in figure 1.

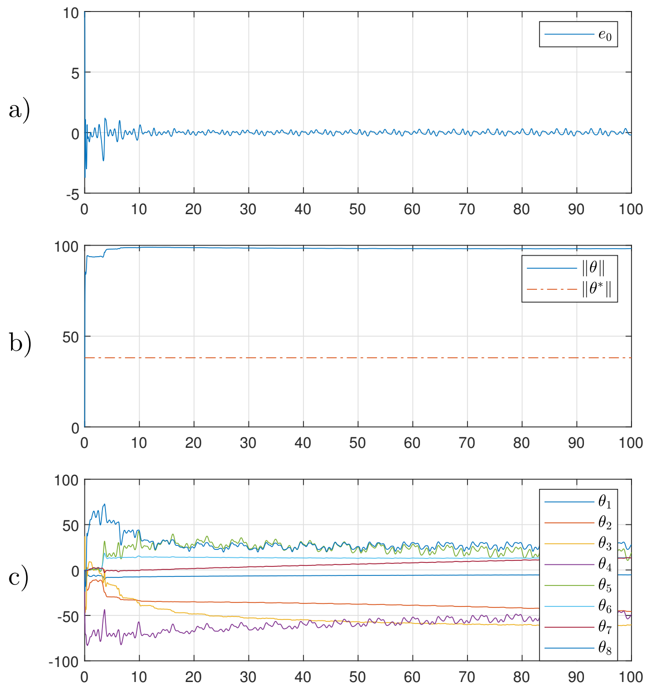

Fig. 1. Simulation result with the standard MRAC.

Figure 1(b) illustrates the slow convergence of the parameter which is representative for the conventional MRAC algorithm even in this case, with an excitation signal composed by 4 sinusoids of different frequencies. Due to the initial condition, $\|\theta\|$ jumps to a high value, far from $\|\theta^*\|$, and stay almost constant. Increasing γ also increases the $\|\theta\|$ jump.

Figure 1(c) shows the behaviour of all the components of the parameter vector θ. It is clear that the vector is quickly changing its orientation while keeping the norm practically constant.

The tracking error plotted in figure 1(a) exemplifies the fact that it can be relatively small even with a very large parameter error $\tilde{\theta}$. The tracking error is small because $\tilde{\theta}^T\omega \approx 0$.

7.2 Simulation 2 - Modified MRAC algorithm

The following data are used:

$$r(t) = r_{sin}(t)\,, \quad \lambda = -2\,, \quad \gamma = 100\,.$$

Figure 2 shows the result obtained with the modified MRAC. The improvement in the tracking error shown in figure 2(a) is remarkable. As expected, increasing the adaptation gain γ improves the tracking behavior.

Figure 2(b) shows the behavior of $\|\theta\|$. The initial jump is quite attenuated when compared with the previous simulation with the modified MRAC, however, the convergence may take a very long time as well even with reference signal used. Notice that γ here is higher than in the previous simulation.

Figure 2(d) shows that, after a short transient, the mismatch control $\tilde{u} = \tilde{\theta}^T\omega + \dot{\theta}^T\xi \approx 0$, that is, $\tilde{\theta}^T\omega \approx -\dot{\theta}^T\xi$. This illustrates the extra degree of freedom introduced in the control law which allows a small tracking error even with $\tilde{\theta}$ and $\tilde{\theta}^T\omega$ large.

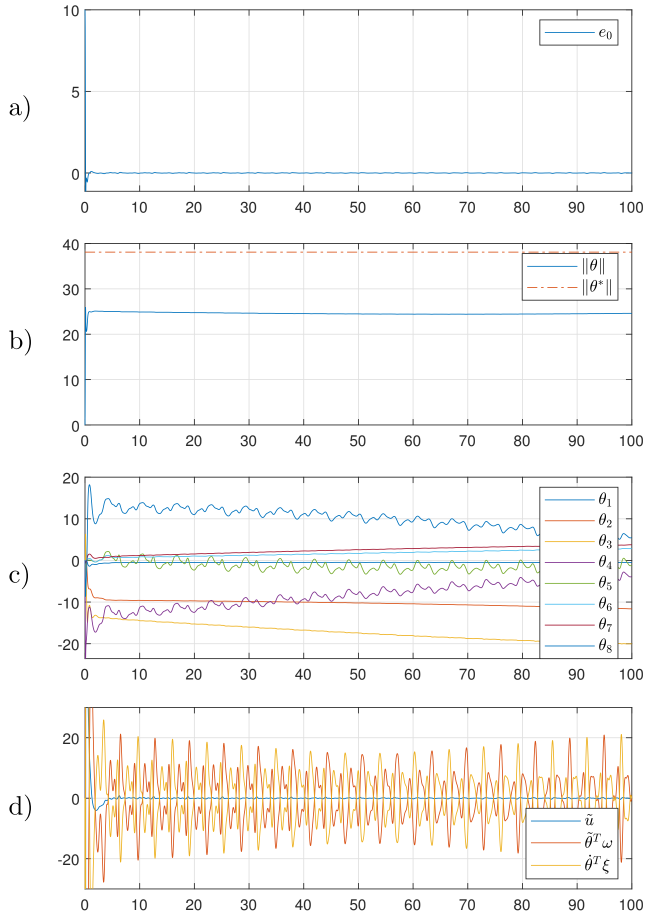

Fig. 2. Simulation result with the modified MRAC.

7.3 Simulation 3 - LS-MRAC algorithm

The following data are used:

$$r(t) = r_{sin}(t)\,, \quad \lambda = -2\,, \quad \gamma = 20\,, \quad R(0) = 20I\,.$$

Figure 3 shows the result obtained for this simulation. As expected, a notable parameter convergence improvement is obtained by introducing the LS algorithm. This can be clearly observed in figures 3(b) and (c). As the parameter converges so does the tracking error.

7.4 Simulation 4 - LS-MRAC algorithm

The following data are used:

$$r(t) = r_{sqw}(t)\,, \quad \lambda = -2\,, \quad \gamma = 20\,, \quad R(0) = 50I\,.$$

Figure 4 shows the result of another simulation with the LS-MRAC algorithm. Now the reference signal is a square wave. It can be observed that the convergence of the parameter is faster than that presented in figure 3 with a reference signal composed by 4 sinusoids. After $t = 40$, $\theta \approx \theta^*$.

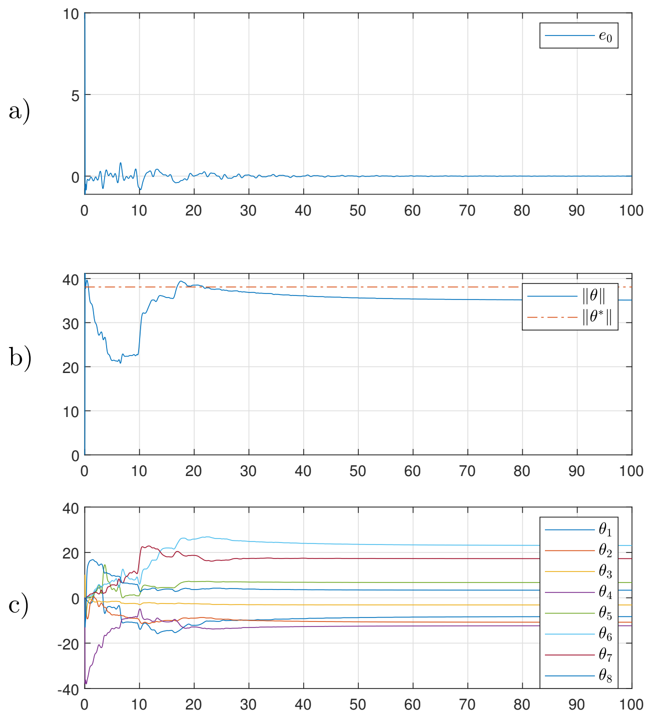

Fig. 3. Simulation result with the LS-MRAC.

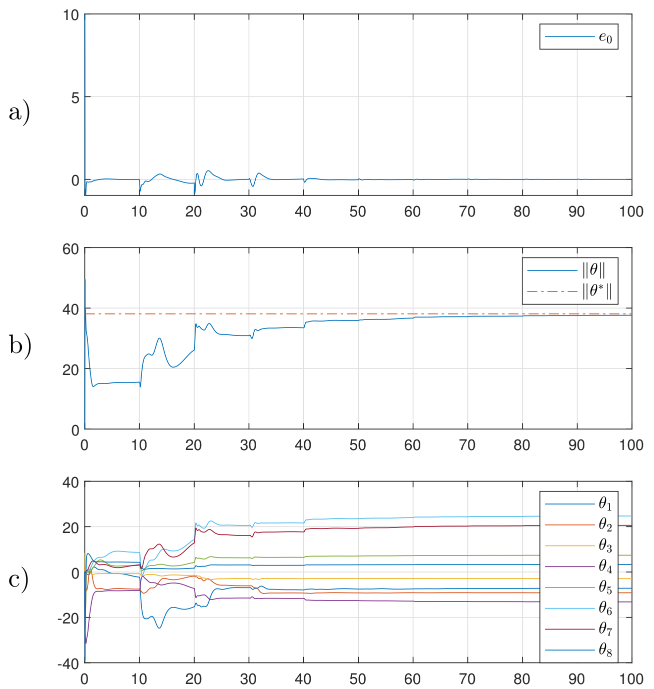

Fig. 4. Simulation result with the LS-MRAC and square wave $r(t)$.

8. CONCLUSION

Two algorithms are proposed and analysed. First a modified MRAC, which is derived from the conventional MRAC by introducing a Monopoli's multiplier. This modification substantially improves the behavior of the tracking error e_0. One reason for this result is the explicit dependence of the error dynamics on the adaptation gain γ (and also on the level of excitation signal). Another reason is the extra degree of freedom introduced in the control law, which allows to attain $\tilde{u} \approx 0$ (due to $\tilde{\theta}^T\omega \approx \dot{\theta}^T\xi$) by means of high adaptation gain γ.

The second algorithm is the LS-MRAC which is directly obtained from the previously modified MRAC by introducing a least-squares update law. Although more complex to implement, the LS-MRAC shows a much better parameter convergence property as expected. This is due to the covariance matrix R, which acts as a time-varying *directional adaptation gain* (Sastry and Bodson, 1989).

Extensive simulations have confirmed the improvement in the tracking and parameter convergence behavior of both algorithms.

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