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A Constraint-Tightening Approach to Nonlinear Stochastic Model Predictive Control under General Bounded Disturbances ⋆

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Abstract: This paper presents a nonlinear model predictive control strategy for stochastic systems with state- and input-dependent, finite-support disturbances subject to individual chance constraints. Our approach uses an online computed stochastic tube to ensure stability, constraint satisfaction, and recursive feasibility in the presence of stochastic uncertainties. The shape of the tube and the constraint backoff is based on an offline computed incremental Lyapunov function.

Keywords: Predictive control, Constrained control, Stochastic control, Nonlinear control

1. INTRODUCTION

Model Predictive Control (MPC) (Mayne, 2014) is a widely-used optimization-based control method, which is able to handle general nonlinear constrained systems. For nominal MPC schemes, which are assuming that an actual deterministic model of the system is available, rigorous theoretical guarantees (such as recursive feasibility, constraint satisfaction, and stability) are well established in the literature (Rawlings et al., 2017). Robust and stochastic MPC (RMPC and SMPC, respectively) have been developed to ensure these properties despite uncertainties in the model and/or external disturbances (Kouvaritakis and Cannon, 2016). While RMPC generally assumes that uncertainties lie in bounded sets, SMPC can additionally incorporate stochastic descriptions. This enables SMPC to enforce *chance constraints*, which are constraints that allow for a given probability of violation.

In many domains, stochastic models for complex phenomena, e.g., loads or failures in electrical power grids, are well-established, yet these phenomena often arise in already nonlinear control problems. In order to tackle such problems, we propose an SMPC framework for nonlinear systems with rigorous theoretical guarantees. Existing SMPC approaches for nonlinear systems (Schildbach et al., 2014) suffer from a tremendous amount of online computation. Our method on the other hand is able to consider nonlinear systems under general disturbances at the price of only a limited increase in online computational demand over nominal MPC scheme.

Related work

Mesbah (2016) summarizes the current state of the art of SMPC and notes that there is a lack of efficient algorithms for nonlinear systems that are able to consider general probabilistic uncertainty descriptions. In this work, we aim to provide such an algorithm in the tradition of tube-based approaches to SMPC, which are among the most efficient methods.

Tube-based solutions to propagate uncertainty were first proposed for RMPC (Chisci et al., 2001; Mayne et al., 2005) for linear systems. This has later been extended to nonlinear systems using class $\mathcal{K}$ functions or Lipschitz constants (Pin et al., 2009). However, such an approach is conservative, especially for long prediction horizons. This method was extended by Santos et al. (2019) to the stochastic case, where the constraint backoff for the chance constraints can be computed offline, since only constantly bounded disturbances are considered. In this article, we consider general uncertainty, which may also depend on the current state and input, and hence on the trajectories predicted in the online optimization.

An efficient method for online-tube-based RMPC was proposed by Köhler et al. (2019). Using sublevel sets of an incremental Lyapunov function (ILF) as the tube, the authors reduce the conservatism significantly compared to offline methods, while only requiring one additional scalar state and constraint over nominal MPC. Similarly, our method introduces just a single constraint for each chance constraint probability considered, enabling stochastic disturbances.

In a probabilistic setting, Wabersich and Zeilinger (2018) used ILFs to achieve safety in probability for reinforcement learning algorithms. The therein assumed uniformly bounded model error allows for simplifications that employ ideas from tubes for constantly bounded disturbances. We consider the more general problem, where the tube size is online adjusted on the predicted state and input trajectories.

Inspired by these results, we propose an extension of the computationally efficient framework by Köhler et al. (2019) to SMPC, which is additionally able to consider stochastic disturbances and chance constraints for systems under bounded state- and input-dependent disturbances.

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Notation

The quadratic norm with respect to a positive definite matrix $Q \succ 0$ is denoted by $\|x\|_Q^2 = x^\top Qx$. The minimal and maximal eigenvalue of Q are denoted by $\lambda_{\min}$ and $\lambda_{\max}$, respectively. The positive real numbers are $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} | r \geq 0\}$. $\mathcal{K}_\infty$ denotes the class of functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which are continuous, strictly increasing, unbounded and satisfy $\alpha(0) = 0$. The probability of an event $s \in \mathcal{S}$ is $\mathbb{P}[s \in \mathcal{S}]$ and the expected value of a random variable s is $\mathbb{E}[s]$. When conditioned on a time-index t, they are denoted by $\mathbb{P}_t$ and $\mathbb{E}_t$, respectively. If the time argument is not stated explicitly, x^+ denotes $x(t+1)$, while x is used for $x(t)$. A nominal prediction for time step $t+k$ based on the state at time t is denoted with index $k|t$, e.g., $x_{k|t}$.

2. PRELIMINARIES

2.1 Problem setup

We consider a nonlinear stochastic discrete-time system

$$x^+ = f(x,u) + d_w(x,u) \tag{1}$$

with time $t \in \mathbb{N}$, state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and bounded independent random variables $d_w(x,u)$ as disturbance. The d_w is chosen such that $\mathbb{E}_t[d(x,u)] = 0$. Further, for any given value of x and u $d_w(x,u)$ is identically distributed over time. The nominal prediction model is chosen by certainty-equivalence as

$$x^+ = f(x,u). \tag{2}$$

Firstly, we enforce hard state and input constraints

$$(x(t), u(t)) \in \mathcal{Z}_\mathrm{R} \tag{3}$$

with some compact nonlinear constraint set

$$\mathcal{Z}_\mathrm{R} = \left\{(x,u) \in \mathbb{R}^{n+m} \middle| g_j(x,u) \leq 0, j = 1, \dots, q_\mathrm{R}\right\} \subseteq \mathbb{R}^{n+m}.$$

Secondly, we impose nonlinear individual chance constraints (ICC) on the output at the next time step, i.e.,

$$\mathbb{P}_t[h_j(x(t+1), u(t+1)) \leq 0] \geq p_j, \quad j = 1, \dots, q_\mathrm{P}. \tag{4}$$

with a probability level $p_j \in (0,1)$. The set of all probability levels used by at least one of the ICCs is denoted as

$$\mathcal{P} := \left\{p_j \middle| j = 1, \dots, q_\mathrm{P}\right\}. \tag{5}$$

Instead of requiring the exact cumulative distribution function, we make use of a lower bound thereupon, which may be easier to obtain in practice.

Assumption 1. The random variable d_w (1) has a known probability distribution $p_w(x,u)$ with compact finite support $\mathcal{W}(x,u)$ for all $(x,u) \in \mathcal{Z}_\mathrm{R}$. Hence, for any $\varepsilon \in [0,1]$, there exists a scalar function $\hat{w}^\varepsilon : \mathcal{Z}_\mathrm{R} \to \mathbb{R}_{\geq 0}$ that satisfies

$$\mathbb{P}[\|d_w(x,u)\| \leq \hat{w}^\varepsilon(x,u)] \geq \varepsilon \tag{6}$$

with $\hat{w}^\varepsilon(x,u)$ finite for all x and u. Furthermore, $\hat{w}^\varepsilon$ satisfies the following monotonicity property:

$$\forall (x,u) \in \mathcal{Z}_\mathrm{R}, 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1 : \hat{w}^{\varepsilon_1}(x,u) \leq \hat{w}^{\varepsilon_2}(x,u). \tag{7}$$

This uncertainty description encompasses additive, multiplicative and more general nonlinear disturbances or unmodeled nonlinearities.

We assume that $f(0,0) = 0$ and that the constraints satisfy

$$0 \in \mathrm{int}(\mathcal{Z}_\mathrm{R} \cap \left\{(x,u) \in \mathbb{R}^{n+m} \middle| h_j(x,u) \leq 0, j = 1, \dots, q_\mathrm{P}\right\}),$$

since we consider the problem of stabilizing the origin. Further, the control objective is to minimize the open-loop cost J_N of the predicted state and input sequence, with

$$J_N(x_{\cdot|t}, u_{\cdot|t}) = \sum_{k=0}^{N-1} \ell(x_{k|t}, u_{k|t}) + V_f(x_{N|t}), \tag{8}$$

where the stage cost ℓ and terminal cost V_f (defined in Sec. 3.3) are positive definite.

2.2 Local incremental stabilizability

In order to describe 'how fast' the system can return to a nominal reference without the disturbances, we assume that the system is locally incrementally stabilizable.

Assumption 2. There exist a control law $\kappa : \mathbb{R}^n \times \mathcal{Z}_\mathrm{R} \to \mathbb{R}^m$, an incremental Lyapunov function (ILF) $V_\delta : \mathbb{R}^n \times \mathcal{Z}_\mathrm{R} \to \mathbb{R}_{\geq 0}$, which is continuous in the first argument and satisfies $V_\delta(z,z,v) = 0$ for all $(z,v) \in \mathcal{Z}_\mathrm{R}$, and parameters $c_{\delta,l}, c_{\delta,u}, \delta_\mathrm{loc}, \kappa_\mathrm{max} > 0, \rho \in (0,1)$, such that the following properties hold for all $(x,z,v) \in \mathbb{R}^n \times \mathcal{Z}_\mathrm{R}$ with $V_\delta(x,z,v) \leq \delta_\mathrm{loc}$, and all $(x^+, z^+, v^+) \in \mathbb{R}^n \times \mathcal{Z}_\mathrm{R}$:

$$c_{\delta,l} \|x - z\|^2 \leq V_\delta(x,z,v) \leq c_{\delta,u} \|x - z\|^2, \tag{9}$$

$$\|\kappa(x,z,v) - v\|^2 \leq \kappa_\mathrm{max} V_\delta(x,z,v), \tag{10}$$

$$V_\delta(x^+, z^+, v^+) \leq \rho^2 V_\delta(x,z,v), \tag{11}$$

with $x^+ = f(x, \kappa(x,z,v))$, and $z^+ = f(z,v)$.

The ILF will be used to construct the stochastic tube later on, yet we only require its existence and knowledge of the scalar parameters, but not the functions V_δ, κ themselves. In particular, we exploit the fact that the ILF provides an upper bound on the achievable contraction rate between two trajectories, e.g., between the predicted trajectory of the MPC scheme and the closed-loop trajectory.

The following assumptions enable us to compute scalar bounds that relate the nonlinear constraints (3) and (4) to the level sets of the ILF V_δ.

Assumption 3. The stage cost $\ell : \mathcal{Z}_\mathrm{R} \to R \geq 0$ satisfies

$$\ell(r) \geq \alpha_\ell(\|r\|), \tag{12}$$

$$\ell(\tilde{r}) - \ell(r) \leq \alpha_c(\|r\|), \quad \forall r \in \mathcal{Z}_\mathrm{R}, \tilde{r} \in \mathbb{R}^{n+m}, \tag{13}$$

with α_ℓ, $\alpha_c \in \mathcal{K}_\infty$, and $\forall \rho \in (0,1) : \sum_{k=0}^\infty \alpha_c(\rho^k c) \in \mathcal{K}_\infty$.

Assumption 4. There exist local Lipschitz constants L_i^R, L_j^P, such that

$$g_i(\tilde{r}) - g_i(r) \leq L_i^\mathrm{R} \|r - \tilde{r}\|, \quad i = 1, \dots, q_\mathrm{R}, \tag{14}$$

$$h_i(\tilde{r}) - h_i(r) \leq L_i^\mathrm{P} \|r - \tilde{r}\|, \quad i = 1, \dots, q_\mathrm{P}, \tag{15}$$

holds for all $r \in \mathcal{Z}_\mathrm{R}$ and all $\tilde{r} \in \mathbb{R}^{n+m}$ with $\|r - \tilde{r}\|^2 \leq \frac{\delta_\mathrm{loc}}{c_{\delta,l}}$.

These assumptions can be, for example, satisfied with a convex polytopic constraint set and a quadratic positive definite stage cost ℓ.

Proposition 5. Suppose that Ass. 2 – 4 hold, then there exists constants $c_i^\mathrm{R} \geq 0, i = 1, \dots, q_\mathrm{R}$, $c_j^\mathrm{P} \geq 0, j = 1, \dots, q_\mathrm{P}$, and a function $\alpha_u \in \mathcal{K}_\infty$ such that the following inequalities hold for all $(x,z,v) \in \mathbb{R}^n \times \mathcal{Z}_\mathrm{R}$ with $V_\delta(x,z,v) \leq c^2$ and any $c \in [0, \delta_\mathrm{loc}]$:

$$\ell(x, \kappa(x,z,v)) - \ell(z,v) \leq \alpha_u(c), \tag{16}$$

$$g_j(x, \kappa(x,z,v)) - g_j(z,v) \leq c_j^\mathrm{R} \cdot c, \tag{17}$$

$$h_j(x, \kappa(x,z,v)) - h_j(z,v) \leq c_j^\mathrm{P} \cdot c. \tag{18}$$

Proof. For the proof of the first part, i.e., (16) and (17), see Köhler et al. (2019, Prop. 1). Equation (18) is derived analogously to (17). □

This proposition will allow us to relate the constraints to the tube, we construct in the next sections.

2.3 Efficient uncertainty description

Additionally, for the tube construction, we need to consider how uncertainty propagation affects the ILF. A computationally efficient way, proposed by Köhler et al. (2019), is to describe the uncertainty in terms of the ILF. As we not only consider bounded set disturbances but also stochastic uncertainties, we need a revised construction.

Assumption 6. Consider the disturbance bound $\hat{w}$, the incrementally stabilizing feedback κ and the ILF V_δ from Ass. 1, and 2. For any $\varepsilon \in [0,1]$, there exists a function $w_\delta^\varepsilon : \mathcal{Z}_{\mathrm{R}} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that for any point $(x,z,v) \in \mathbb{R}^n \times \mathcal{Z}_{\mathrm{R}}$ with $V_\delta(x,z,v) \leq c^2$, and any $c \in [0, \delta_{\mathrm{loc}}]$, we have

$$\hat{w}^\varepsilon(x, \kappa(x,z,v)) \leq w_\delta^\varepsilon(z,v,c)\,. \tag{19}$$

Furthermore, w_δ satisfies the following monotonicity properties: Firstly, for any point $(x,z,v) \in \mathbb{R}^n \times \mathcal{Z}_{\mathrm{R}}$ such that $V_\delta(x,z,v) \leq (c_1 - c_2)^2$ with constants $0 \leq c_2 \leq c_1 \leq \delta_{\mathrm{loc}}$, we have

$$w_\delta^\varepsilon(x, \kappa(x,z,v), c_2) \leq w_\delta^\varepsilon(z,v,c_1)\,. \tag{20}$$

Secondly, for any constant $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$, we have

$$w_\delta^{\varepsilon_1}(x, \kappa(x,z,v), c) \leq w_\delta^{\varepsilon_2}(z,v,c)\,. \tag{21}$$

This assumption establishes $\omega_\delta^\varepsilon$ as an ε-likely upper bound on the uncertainty that can occur at a state x of an incrementally stabilized trajectory in a neighborhood of a point $(z,v) \in \mathcal{Z}_{\mathrm{R}}$, where the neighborhood is given by $V_\delta(x,z,v) \leq c^2$. Based thereupon, we can bound the increase of the ILF due to the disturbance in the next time step with probability ε.

Proposition 7. Let Ass. 1, 2, and 6 hold. Then, there exists a function $\tilde{w}_\delta^\varepsilon : \mathcal{Z}_{\mathrm{R}} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that for any point $(x,z,v) \in \mathbb{R}^n \times \mathcal{Z}_{\mathrm{R}}$ with $V_\delta(x,z,v) \leq c^2$, any $c \in [0, \delta_{\mathrm{loc}}]$, any $(z^+, v^+) \in \mathcal{Z}_{\mathrm{R}}$ with $z^+ = f(z,v)$, and disturbance d_w as random variable, we have

$$\mathbb{P}\big[V_\delta(z^+ + d_w(x, \kappa(x,z,v)), z^+, v^+) \leq (\tilde{w}_\delta^\varepsilon(z,v,c))^2\big] \geq \varepsilon\,. \tag{22}$$

Furthermore, $\tilde{w}_\delta^\varepsilon$ satisfies the same monotonicity properties as w_δ^ε, i.e., (20) and (21) hold for $\tilde{w}_\delta^\varepsilon$.

Proof. The proof follows trivially from the assumptions, by setting $\tilde{w}_\delta^\varepsilon(z,v,c) = \sqrt{c_{\delta,u}}\, w_\delta^\varepsilon(z,v,c)$. □

The function $\tilde{w}_\delta^\varepsilon$ can be constructed similarly as in Köhler et al. (2019), an example thereof is given in Sec. 4.

In the absence of chance constraint, we could now construct the tube as in Köhler et al. (2019). For the chance constraints (4), however, additional considerations are required, in order to ensure closed-loop constraint satisfaction, which is discussed later in Sec. 3.2.

3. STOCHASTIC MODEL PREDICTIVE CONTROL FRAMEWORK

This section presents the proposed stochastic MPC framework for nonlinear uncertain systems. The overall scheme is introduced in Sec. 3.1. In Sec. 3.2 the constraint backoff for the chance constraints are discussed. The theoretical analysis in Sec. 3.4 uses the terminal ingredients described in Sec. 3.3.

3.1 Proposed nonlinear MPC scheme

Our scheme indirectly characterizes the tube as the sublevel sets of the ILF V_δ (Ass. 2) using online predicted tube sizes s^p. Then, these tube sizes are used to tighten the state and input constraints ensuring constraint satisfaction, cf. (23f–g). The main contribution allows for ICCs and stochastic uncertainties by incorporating additional larger tubes with size s^p for each likelihood p required by a constraint. These are derived from the robust tube size s^1 as described in the following section.

This lead to the deterministic optimization problem

$$V_N(x(t)) = \min_{u_{\cdot|t}, w^{\mathrm{R}}_{\cdot|t}, w^p_{\cdot|t}} J_N(x_{\cdot|t}, u_{\cdot|t}) \tag{23a}$$

$$\text{s.t. } x_{0|t} = x(t), \quad s^p_{0|t} = 0, \tag{23b}$$

$$x_{k+1|t} = f(x_{k|t}, u_{k|t}), \tag{23c}$$

$$s^p_{k+1|t} = \rho s^1_{k|t} + w^p_{k|t}, \tag{23d}$$

$$w^p_{k|t} \geq \tilde{w}^p_\delta(x_{k|t}, u_{k|t}, s^p_{k|t}), \tag{23e}$$

$$h_j(x_{k+1|t}, u_{k+1|t}) + c_j^{\mathrm{P}} s^p_{k+1|t} \leq 0, \tag{23f}$$

$$g_i(x_{k|t}, u_{k|t}) + c_i^{\mathrm{R}} s^1_{k|t} \leq 0, \tag{23g}$$

$$s^1_{k|t} \leq \bar{s}, \quad w^p_{k|t} \leq w^1_{k|t} \leq \bar{w}, \tag{23h}$$

$$(x_{N|t}, s^1_{N|t}) \in \mathcal{X}_f, \tag{23i}$$

$$i = 1, \ldots, q_{\mathrm{R}}, \quad j = 1, \ldots, q_{\mathrm{P}},$$
$$k = 0, \ldots, N-1, \quad p \in \mathcal{P} \cup \{1\},$$

which is to be solved at each time instant. The solution of (23) are optimal trajectories for the state $x^*_{\cdot|t}$, the input $u^*_{\cdot|t}$, the tube sizes $s^{p,*}_{\cdot|t}$, the disturbance bounds $w^{p,*}_{\cdot|t}$, and the value function V_N. The terminal ingredients V_f, $\mathcal{X}_f$, $\bar{s}$, and $\bar{w}$ are introduced in Sec. 3.3.

The first portion of the resulting optimal input sequence is applied to the system, resulting in the closed-loop system is given by

$$x(t+1) = f(x(t), u(t), d(t))\,, \qquad u(t) := u^*_{0|t}\,. \tag{24}$$

3.2 Chance Constraints

For the sake of simplicity, we will consider in this section without loss of generality only single ICCs

$$\mathbb{P}_t[h(x(t+1), u(t+1)) \leq 0] \geq p\,. \tag{25}$$

In the literature, the chance constraints are commonly handled by so-called constraint backoffs. This idea originates in linear MPC with additive stochastic disturbances (van Hessem and Bosgra, 2002). There, one can simply *backoff* the constraint, by enforcing at least a precomputed constant distance from the constraint boundary. In this work, however, we consider nonlinear systems with general disturbances, where the required backoff not only becomes state-dependent, but also intractable to compute.

Using the sublevel sets of the ILF, we can construct a tube around the prediction, which contains the disturbed

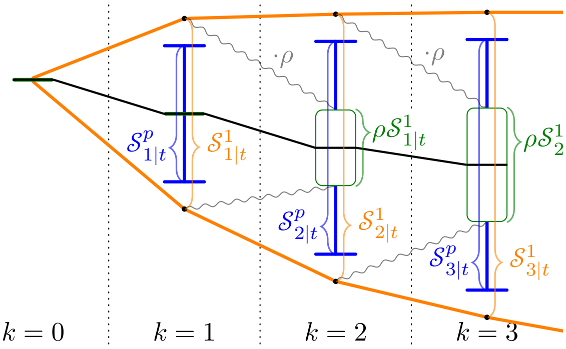

Fig. 1. Illustration of the idea behind the proposed incremental backoff. The robust and stochastic tubes are shown in orange and blue, respectively.

closed-loop trajectory with at least probability p. The size of this tube will be used as our backoff.

The idea of the tube construction is illustrated in Fig. 1. Starting off, we begin with the prediction (black). Using Prop. 7 for $\varepsilon = 1$, a robust tube (orange) can be constructed around this prediction, inside which the true state will certainly lie. This tube is constituted by the sublevel set

$$\mathcal{S}^1_{k|t} := \left\{ x \in \mathbb{R}^n \middle| V_\delta(x, x_{k|t}, u_{k|t}) \leq s^1_{k|t} \right\} . \tag{26}$$

If one assumes that the previous time step was without disturbance, i.e., $d_w = 0$, then a contraction ρ of the robust set (Ass. 2) is reached by the incremental stabilization κ. Thus, we obtain an inner tube (green) lacking the influence of the last disturbance with the sets $\mathcal{S}^0_{k+1|t} := \rho\mathcal{S}^1_{k|t}$.

Using Prop. 7 for $\varepsilon = p \in (0, 1)$, the disturbance is added to the tube. Thereby, we obtain an ε-likely tube, indicate by the blue error bars. This tube confines the state with a probability greater than ε at each time step in the sets

$$\begin{aligned} \mathcal{S}^p_{k+1|t} &:= \mathcal{S}^1_{k|t} + \left\{ x \in \mathbb{R}^n \middle| V_\delta(x, x_{k|t}, u_{k|t}) \leq w^p_{k|t} \right\} \\ &= \left\{ x \in \mathbb{R}^n \middle| V_\delta(x, x_{k|t}, u_{k|t}) \leq \rho s^1_{k|t} + w^p_{k|t} =: s^p_{k+1|t} \right\} . \end{aligned} \tag{27}$$

By this construction, we can employ the size s^p of the sublevel sets of V_δ, i.e., the size of our tube, as our backoff to ensure ICC satisfaction.

3.3 Terminal ingredients

By using the minimal bound on the uncertainty $\bar{w}_{\min}$ and the maximal tube size $\bar{s}$

$$\bar{w}_{\min} = \inf_{(x,u)\in\mathcal{Z}_{\mathrm{R}}} \tilde{\omega}_\delta(x, u, 0) , \qquad \bar{s} = \sqrt{\delta_{\mathrm{loc}}} , \tag{28}$$

we capture the desired properties of the terminal ingredients in the following assumption.

Assumption 8. There exist a terminal controller $k_f : \mathbb{R}^n \to \mathbb{R}^m$, a terminal cost function $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a terminal set $\mathcal{X}_f \subset \mathbb{R}^{n+1}$, and a constant $\bar{w} \in \mathbb{R}_{\geq 0}$ such that the following holds for all $(x, s) \in \mathcal{X}_f$, all $d_w \in \mathbb{R}^n$, all $w \in [\bar{w}_{\min}, \bar{w}]$, and all $s^+ \in [0, \rho s - \rho^N w + \tilde{w}_\delta(x, k_f(x), s)]$, such that $V_\delta(x^+ + d_w, x^+, k_f(x^+)) \leq \rho^{2N} w^2$ with $x^+ = f(x, k_f(x))$:

$$\begin{aligned} V_f(x) - \ell(x, k_f(x)) &\geq V_f(x^+) , && (29a) \\ (x^+ + d_w, s^+) &\in \mathcal{X}_f , && (29b) \\ \tilde{w}^1_\delta(x, k_f(x), s) &\leq \bar{w} , && (29c) \\ g_i(x, k_f(x)) + c^{\mathrm{R}}_i s &\leq 0, && (29d) \\ \rho s - \rho^N w + \tilde{w}^{p_j}_\delta(x, k_f(x), s) &=: \beta^{p_j} && (29e) \\ h_j(x^+, k_f(x^+)) + c^{\mathrm{P}}_j \beta^{p_j} &\leq 0, && (29f) \\ s &\leq \bar{s} , && (29g) \end{aligned}$$

with $i = 1, \ldots, q_{\mathrm{R}}$, and $j = 1, \ldots, q_{\mathrm{P}}$. Furthermore, the terminal cost V_f is continuous on the compact set $\mathcal{X}_{f,x} := \{x \,|\, \exists s \in [0, s], (x, s) \in \mathcal{X}_f\}$, i.e., there exists a function $\alpha_f \in \mathcal{K}_\infty$ such that

$$V_f(z) \leq V_f(x) + \alpha_f(\|x - z\|), \forall x, z \in \mathcal{X}_{f,x} . \tag{30}$$

These technical conditions are similar to the standard conditions in nominal MPC for the augmented state (x, s) and input (u, w, w^p). Details on constructive satisfaction can be found in Köhler et al. (2019).

The only extension to the robust case is the inclusion of the ICC in the construction of the terminal set, i.e., (29f). This ensures that also the ICCs are satisfied by the terminal controller, using the same backoff technique as just described in Sec. 3.2. For the construction of the terminal set, these constraints are treated similarly to the hard constraint (29d).

3.4 Theoretical analysis

In the following theorem, we provide guarantees on the closed-loop properties of the proposed MPC scheme.

Theorem 9. Let Ass. 1 − 4, 6, and 8 hold, and suppose that (23) is feasible at t = 0. Then (23) is recursively feasible, the constraints (3), (4) are satisfied and the origin is practically asymptotically stable for the resulting closed-loop system.

Proof. The proof is based on an extension of the main idea behind Köhler et al. (2019, Thm. 1), as such we will refer to their results, whenever it is possible. This will enable us to focus on handling the chance constraints, as the impact of the hard constraints is equivalent.

The core idea is to use the control law κ from Ass. 2 to construct a candidate solution, ensuring recursive feasibility, and bounding the cost increase.

I. *Candidate Solution*: For convenience, define

$$\begin{aligned} u^*_{N|t} &= k_f(x^*_{N|t}) , \quad u^*_{N+1|t} = k_f(x^*_{N+1|t}), && (31a) \\ x^*_{N+1|t} &= f(x^*_{N|t}, u^*_{N|t}) , && (31b) \\ w^{p,*}_{N|t} &= \tilde{w}^p_\delta(x^*_{N|t}, u^*_{N|t}, s^*_{N|t}) . && (31c) \end{aligned}$$

Consider the adapted candidate solution, i.e.,

$$\begin{aligned} x_{0|t+1} &= x(t + 1) = f(x_{0|t}, u_{0|t}) + d_w(x_t, u_t) , && (32a) \\ u_{k|t+1} &= \kappa(x_{k|t+1}, x^*_{k+1|t}, u^*_{k+1|t}) , && (32b) \\ x_{k+1|t+1} &= f(x_{k|t+1}, u_{k|t+1}) , && (32c) \\ s^p_{k+1|t+1} &= \rho s^1_{k|t+1} + w^p_{k|t+1} , \quad s^p_{0|t+1} = 0 , && (32d) \\ w^p_{k|t+1} &= \tilde{w}^p_\delta(x_{k|t+1}, u_{k|t+1}, s_{k|t+1}) , && (32e) \end{aligned}$$

with $k = 0, \ldots, N - 1$ and $p \in \mathcal{P}$. As in Köhler et al. (2019, eq. 17), we obtain using Prop. 7 (22) with $\varepsilon = 1$ and repeatedly applying Ass. 2 (11) that for $k = 0, \ldots, N$

$$V_\delta(x_{k|t+1}, x^*_{k+1|t}, u^*_{k+1|t}) \leq \rho^{2k}[w^*_{0|t}]^2 \leq \delta_{\text{loc}} \,. \tag{33}$$

Thus, the candidate and previous optimal solution stay in the region $V_\delta(z,x,v) \leq \delta_{\text{loc}}$, for which we have a local incremental Lyapunov function V_δ by Ass. 2.

II. *Tube Dynamics*: From Köhler et al. (2019, Proof of Thm. 1, Part II, eq. 18-19), we have the inequalities

$$s^1_{k|t+1} \leq s^{1,*}_{k+1|t} - \rho^k w^{1,*}_{0|t} \,, \tag{34}$$

$$w^1_{k|t+1} \leq w^{1,*}_{k+1|t} \,, \tag{35}$$

for $k = 0, \dots, N-1$. Analogously to the derivation of (35), we can show

$$w^p_{k|t+1} \leq w^{p,*}_{k+1|t} \,. \tag{36}$$

This enables us to consider the general case of $s^p_{k|t+1}$, yielding that for all $p \in \mathcal{P} \cup \{1\}$ the inequality

$$s^p_{k|t+1} \leq s^{p,*}_{k+1|t} - \rho^k w^{1,*}_{0|t} \tag{37}$$

holds for all $k = 0, \dots, N-1$ and $j = 1, \dots, q_{\text{P}}$, since

$$\begin{aligned} s^p_{0|t+1} &\overset{(23b)}{=} 0 \overset{(23d)}{=} s^{1,*}_{1|t} - \rho^0 w^{1,*}_{0|t} \\ s^p_{k+1|t+1} &\overset{(23d)}{=} \rho s^1_{k|t+1} + w^p_{k|t+1} \\ &\overset{(34)}{\leq} \rho s^{1,*}_{k+1|t} - \rho^{k+1} w^{1,*}_{0|t} + w^p_{k|t+1} \\ &\overset{(36)}{\leq} \rho s^{1,*}_{k+1|t} - \rho^{k+1} w^{1,*}_{0|t} + w^{p,*}_{k+1|t} \\ &\overset{(23d)}{=} s^{p,*}_{k+2|t} - c^{\text{P}}_j \rho^{k+1} w^{1,*}_{0|t} \,. \end{aligned}$$

III. *Satisfaction of Hard Constraints, Terminal Constraints, and Tube Bounds*: The constraints (23g–i) are satisfied by the candidate solution (32) as shown in Köhler et al. (2019, Proof for Thm. 1, Part III−V)

IV. *Satisfaction of Deterministic ICC Replacement*: In the following, we show that the deterministic constraints (23f) used in place of the ICCs (4) hold for $k = 0, \dots, N-1$.

For $k = 0, \dots, N-2$, we have

$$\begin{aligned} & h_j(x_{k+1|t+1}, u_{k+1|t+1}) + s^p_{k+1|t+1} \\ \overset{(18),(33)}{\leq} \; & h_j(x^*_{k+2|t}, u^*_{k+2|t}) + c^{\text{P}}_j \rho^{k+1} w^*_{0|t} + s^p_{k+2|t+1} \\ \overset{(37)}{\leq} \; & h_j(x^*_{k+2|t}, u^*_{k+2|t}) + s^{p,*}_{k+2|t} \overset{(23f)}{\leq} 0 \end{aligned}$$

The terminal condition (23i) ensures constraint satisfaction for $k = N-1$ with

$$\begin{aligned} & h_j(x_{N|t+1}, u_{N|t+1}) + c^p_j s^p_{N|t+1} \\ \overset{(33),(18),(37)}{\leq} \; & h_j(x^*_{N+1|t}, u^*_{N+1|t}) + c^p_j s^{p,*}_{N+1|t} \overset{(29f)}{\leq} 0 \end{aligned}$$

V. *Practical Stability*: As shown in Köhler et al. (2019, Proof of Thm. 1,Part VI), there exist α^-, α^+, $\alpha_w \in \mathcal{K}_\infty$ such that

$$\alpha^-(\|x_t\|) \leq V_N(x_t) \leq \alpha^+(\|x_t\|) \,, \tag{38}$$

$$V_N(x_{t+1}) - V_N(x_t) \leq -\alpha^-(\|x_t\|) + \alpha_w(\bar{w}) \,. \tag{39}$$

Thus, the closed-loop is practically asymptotically stable.

VI. *Closed-looped Chance Constraint Satisfaction*: In the following, we show if (23f) holds that the ICCs (4) are satisfied at time t. This follows the line of thought outline in Sec. 3.2. This will also imply that the ICCs hold in closed-loop, as we have shown in Part iv of this proof that (23f) will be satisfied at every time-step.

Again, we consider just a single ICC (25). By Prop. 5, $V_\delta(x_{k|t+1}, x_{k+1|t}, u_{k+1|t}) \leq c^2$ implies

$$h(x_{k|t+1}, u_{k|t+1}) - h(x_{k+1|t}, u_{k+1|t}) \leq c^{\text{P}} \cdot c \tag{40}$$

By Ass. 2 (11), we have $V_\delta(x_{k|t+1}, x_{k+1|t}, u_{k+1|t}) \leq (\rho s^1_{k|t})^2$ for $d_w = 0$. Using Prop. 7, we can bound the additional increase of V_δ due to the $d_w \neq 0$. Together, this yields

$$\mathbb{P}\Big[c = \sqrt{V_\delta(x_{k|t+1}, x_{k+1|t}, u_{k+1|t})} \leq \rho s^1_{k|t} + w^p_{k|t}\Big] \geq p \,. \tag{41}$$

Then, given that $V_\delta(x_{k-1|t+1}, v_{k|t}, u_{k|t}) \leq (s^1_{k|t})^2$, substituting (41) into (40) yields

$$\mathbb{P}\Big[h(x_{k|t+1}, u_{k|t+1}) \leq h(x_{k+1|t}, u_{k+1|t}) + c^{\text{P}} s^p_{k+1|t}\Big] \geq p \,, \tag{42}$$

hence, we obtain that

$$h(x_{k+1|t}, u_{k+1|t}) + c^{\text{P}} \cdot s^p_{k+1|t} \leq 0 \tag{43}$$

$$\implies \mathbb{P}[h(x_{k|t+1}, u_{k|t+1}) \leq 0] \geq p \iff (25) \,. \qquad \square$$

3.5 Discussion

In the following, we discuss various properties of the proposed SMPC scheme, as well as relations to other existing MPC schemes for uncertain systems.

Remark 10. Compared to a nominal MPC scheme, s^p and w^p augment the state and the input, resp., for each $p \in \mathcal{P} \cup \{1\}$. Thus, the online computational demand for solving (23) is comparable to a nominal MPC scheme with $n + 1 + |\mathcal{P}|$ states, $m + 1 + |\mathcal{P}|$ inputs, and additional $1 + |\mathcal{P}|$ nonlinear constraints for each time step. Correspondingly, while it is possible to have multiple probability levels p_j for the ICCs, using the same p for all ICCs will be computationally significantly cheaper.

Remark 11. The SMPC method in Santos et al. (2019) for additive uncertainty is based on a $\mathcal{K}$-function or as a special case on a Lipschitz constant L. Therein, the authors employed the inverse of empirical cumulative distributions $\hat{F}_W$ as uncertainty description, which can be equivalently used in our method by setting $\hat{w}^p = \hat{F}^{-1}_W(p)$. In particular, the result on Lipschitz constants L are contained as a special case in our framework with $V_\delta(x,z,v) = \|x - z\|^2$, $\kappa(x,z,v) = v$, and $\rho = L$. Similarly, the use of a $\mathcal{K}$-function σ is also a special of our scheme using the same V_δ and κ by rewriting ρ and $\tilde{w}_\delta$ in terms of σ. The main difference is that we use a backoff for the ICCs depending on the prediction in order to consider state- and input-dependent disturbances, whereas Santos et al. (2019) can employ constant backoff as the current state has no impact on the considered additive stationary stochastic uncertainty. Overall, the proposed framework using ILF is typically less conservative than either one of these choices (Köhler et al., 2018).

Remark 12. While other methods (e.g., Santos et al. (2019)), often require prestabilization in order to limit the tube growth for unstable systems. In our method, this is not necessary as the tube size depends on the controller κ in ILF, which is efficiently able to handle instability.

4. NUMERICAL SIMULATION

A widely used benchmark case study in the SMPC literature is the DC-DC-converter regulation problem. The discrete-time dynamics translated to the origin are described by Lazar et al. (2008), including their parameters, as

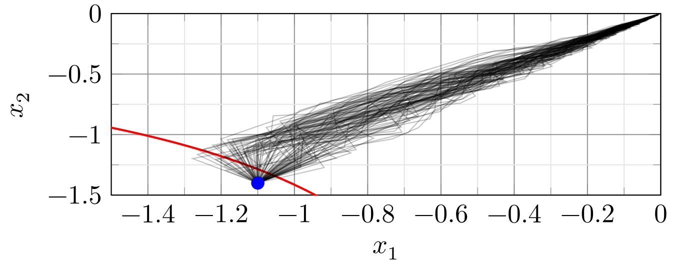

Fig. 2. Evolution of the closed-loop states under 100 disturbance realizations with chance constraints (red) and initial condition $x(0) = [-1.1, -1.4]^\top$.

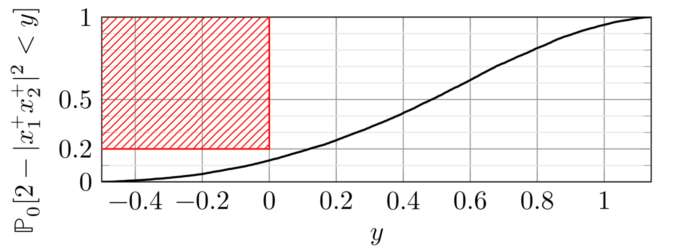

Fig. 3. Empirical cumulative distribution of the chance constraint (red) under 10 000 disturbance realizations.

$$x^+ = \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 + \alpha x_2 + \left(\beta - \frac{T}{L} x_2\right) u \\ \left(\frac{T}{C} x_1 + \gamma\right) u + \left(1 - \frac{T}{RC}\right) x_2 + \delta x_1 \end{bmatrix}. \quad (44)$$

We consider a (possibly time-varying) parameter uncertainty in $\theta = [\alpha, \delta]$ with a Gaussian distribution with $\Sigma_\theta = 0.1 \cdot \mathrm{I}_{2\times 2}$ variance truncated to a maximal deviation of 1.6σ. The objective is to minimize the quadratic cost $\ell(x, u) = x^\top x + u^2$ over the finite horizon $N = 7$. The system is subject to hard input constraint $|u| \leq 0.5$, and the electric power is chance-constrained by $\mathbb{P}_t[|x_1^+ x_2^+|^2 \leq 2] \geq 0.8$. Using an ILF $V_\delta(x, z, v) = \|x - z\|_P^2$ and the controller $\kappa(x, z, v) = K(x - z) + v$, a contraction rate of $\rho \approx 0.82$ can be achieved. The disturbance bounds $\tilde{w}_\delta^p(x, u, c) = \left\| P^{\frac{1}{2}} \frac{\partial f(x,u)}{\partial[\alpha,\delta]} \Sigma_\theta^{\frac{1}{2}} \right\| \varepsilon(p) + L_w c$ is derived analogously to (Köhler et al., 2019, Prop. 3) with Lipschitz constant $L_w^1 \approx 0.15$, $L_w^{0.6} \approx 0.06$ and $\mathbb{P}[\|\theta\|^2_{\Sigma_\theta^{-1}} \leq \varepsilon(p)] \geq p$.

In Fig. 2, we can see 100 realizations of the closed-loop under the proposed SMPC. The initial condition was chosen such that unconstrained operation would violate the chance constraint. In 87% of the 10 000 simulated realizations the power constraint is satisfied, this can also be seen in the empirical cumulative distribution in Fig. 3.

To achieve guarantees despite nonlinearity of the system and the constraints while maintaining low computational complexity, some relaxations were made that necessarily lead to conservatism. Yet, this is still less conservative than other methods, e.g., using Lipschitz constants (cf. Rem. 11) or approximating the disturbance as constantly bounded. Additionally, with our approach conservatism could be further reduced by using a less conservative disturbance bound $\tilde{w}_\delta$ at the price of additional computational complexity. Therefore, our method can be tuned to the desired compromise between conservatism and complexity. At the same time with the fairly conservative, but simple, bound presented in this example, we achieved tighter satisfaction than existing methods.

5. CONCLUSION

We proposed a nonlinear SMPC framework based on incremental stabilizability for nonlinear systems incorporating general state- and input- dependent uncertainty descriptions. The scheme can ensure the satisfaction of individual chance constraints and hard constraints, as well as recursive feasibility. By using a specially designed growing tube, we achieve this with only a small increase in the computation cost over nominal MPC.

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