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The Elastic Rod Approach toward System Theory for Soft Robotics ⋆

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Abstract: In this paper, an approach toward system theory for soft robotics is considered. An overview of a theoretical scenario is presented by focusing on an elastic rod which is regarded as one of the most essential objects for soft mechanical elements of soft robots. The presented topics include geometry of its backbone curve, kinematics, shape, mechanics (mainly its statics), and discretization, with emphasizing on some important system properties of an elastic rod which will be useful for shape computation and stiffness identification.

Keywords: Robotics, Robots, Robot kinematics, Robot dynamics, Robot control, Robot calibration, Flexible arms, Distributed parameter systems, Discretization.

1. INTRODUCTION

1.1 Background and Purpose

Soft Robotics deals with robots including highly deformable continuum mechanical parts and having some actuators and sensors for being controlled. Soft Robotics is one of the branches in robotics, but a strongly interdisciplinary research field which gathers much attention from the outside of robotics such as chemistry, material science and biology. Soft robotics is not a new research field. Actually, we can find many soft robots which were invented by a quarter of a century ago (1; 2; 3). It is a fact that the number of research papers on soft robotics is rapidly increasing and soft robotics is one of main topics in robotics recently. Now it is the time to consider to control soft robots successfully so as to be useful for some practical applications.

As a controlled system, a soft robot has the following properties in general:

(1) The system is infinite dimensional, because it includes a largely deformable continuum part which has infinite kinematic degrees of freedom.
(2) The system is essentially under-actuated, because the number of actuators to be attached is finite for this infinite dimensional system.
(3) The system is essentially under-observable, because the number of sensors to be attached is finite for this infinite dimensional system.
(4) The system has non-trivial equilibrium points, because the static shape of a soft robots is determined by statically balancing of an infinite dimensional mechanical system. [1]

Looking these general system properties directly, it seems quite challenging to consider a system theory for soft robots. The purpose of this research is to establish a useful system theory for soft robotics so as to make full use of the functions of soft robots by controlling them appropriately.

1.2 Related Work

Control aspects of soft robots were discussed in some comprehensive review papers (4; 5; 6) [2]. Historically, the theory of robot control was built up based on some important system properties (8). One famous good example is that the task space PD feedback control law for a robot manipulator successfully can be found by an energy-based Lyanpunov function based on positive definiteness of the inertia matrix and skew symmetry of the matrix related to the Coriolis and centrifugal forces (9). However, system properties of soft robots important for controlling them have not been fully discussed yet.

1.3 Proposed Direction

It is reasonable to take a strategy to build up a theory by focusing on a certain important class of soft object which is a main part of typical soft robots, and then to extend it to other classes. In spite of the hopeless general system properties of soft robots shown above, there is a possibility to control a soft robot system whose soft body is characterized by an elastic rod, one of the most typical objects frequently appeared in soft robotics, due to recent advanced application of the elastic rod theory (10) to soft robotics with utilize the system properties of an elastic rod.

In this paper, a system theory for soft robotics based on an elastic rod, a typical soft element and one of the simplest continuum objects studied in the rod theory, is discussed

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[1] This property was pointed out by Prof. Masato Ishikawa of Osaka University.

[2] Recently, an excellent special issue on soft robot mechanisms including many insightful articles was published, but it is available only in Japanese so far (7).

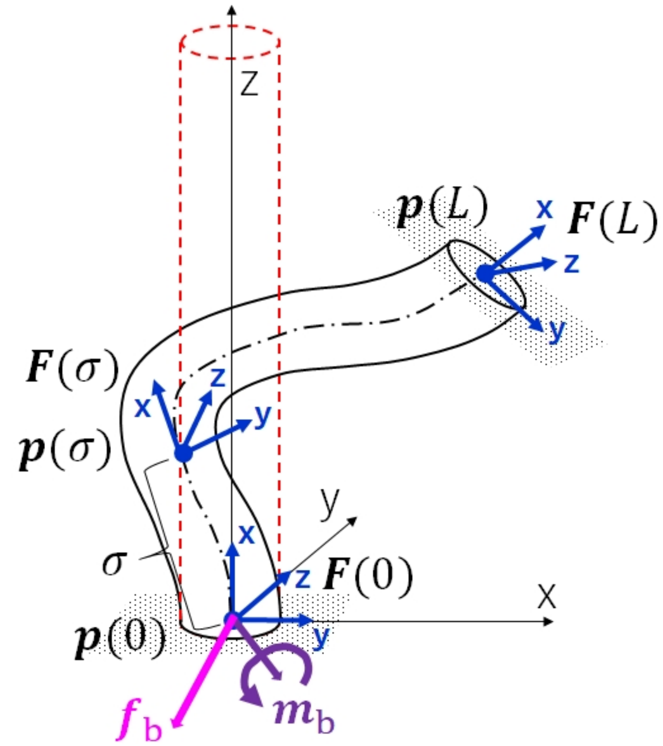

Fig. 1. A deformed elastic rod.

as an important step for the research purpose. An elastic rod can be utilized to investigate flexibility of biological trunk which is considered as one of important factors in interacting with the environment successfully. The soft robot system theory based on an elastic rod is expected to become like a Linear Time-Invariant (LTI) system from which a variety of branches of control theory stem.

2. ROD MODEL AND RELATED NOTIONS

In this section, a Kirchhoff elastic model is reviewed briefly and related notions on an elastic rod are introduced with geometric considerations.

2.1 Review of Elastic Rod Model

There are many methods to model an elastic rod. Here a Kirchhoff elastic model [3] is utilized because the entire shape of an elastic rod can be captured efficiently, and important system properties can be extracted (10).

Consider an elastic rod with length L (Fig. 1). An elastic rod deforms by applying forces to its both ends. Here this situation can be represented by fixing the both ends to certain places. One end of the rod is called the base while the other end is called the tip. The longitudinal direction of a rod is defined as the direction from the base to the tip.

The shape of the rod can be captured by the spatial curve obtained by continuously connecting the geometric centers of the cross section perpendicular to the longitudinal direction of a rod. This curve is called the backbone curve (13). Without loss of generality, it is assumed that the backbone curve of an elastic rod is straight if no external forces/torques is applied on it.

[3] In a Kirchhoff elastic rod, we assume an inextensible and unshearable rod with a linear constitutive equation.

A typical way to understand geometric aspects of an elastic rod is to attach frames continuously to all the point on the backbone curve (11). More details of the frame setting can be seen in (12).

Let $\sigma \in [0\ L]$ be the arc length parameter of the backbone curve. Let $\boldsymbol{p}(\sigma) \in \Re^3$ be the position vector of the point on the backbone curve at σ. Let $\boldsymbol{F}(\sigma) \in \mathrm{SO}(3)$ be the orientation matrix of the frame attached to the point on the backbone curve at σ. Then, the geometry of the backbone curve, i.e., the kinematics of an elastic rod, can be expressed by the following differential equations w.r.t. the arc length parameter σ:

$$\frac{d\boldsymbol{F}}{d\sigma}(\sigma) = \boldsymbol{F}(\sigma)\,[\boldsymbol{\theta}(\sigma)\times] \tag{1}$$

$$\frac{d\boldsymbol{p}}{d\sigma}(\sigma) = \boldsymbol{F}(\sigma)\boldsymbol{e}_{\mathrm{x}} \tag{2}$$

where $\boldsymbol{\theta}(\sigma) := [\theta_{\mathrm{t}}(\sigma)\ \theta_{\mathrm{n}}(\sigma)\ \theta_{\mathrm{b}}(\sigma)]^T \in \Re^3$, and variables $\theta_{\mathrm{t}}(\sigma)$, $\theta_{\mathrm{n}}(\sigma)$, $\theta_{\mathrm{b}}(\sigma) \in \Re$ are the infinitesimal rotational amount around three axes of the frame at σ, respectively. Vector $\boldsymbol{e}_{\mathrm{x}} = [1\ 0\ 0]^T$ is the unit vector in the x-direction. Expression $\boldsymbol{a} \times \boldsymbol{b}$ denotes the outer product of vectors $\boldsymbol{a}$ and $\boldsymbol{b} \in \Re^3$, and notation $[\cdot\times]$ stands for the operator from a three dimensional vector to a three dimensional skew symmetric matrix such that $\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}\times]\,\boldsymbol{b}$.

Assume that the base of an elastic rod is fixed to the position $\boldsymbol{p}_{\mathrm{b}} \in \Re^3$ with the orientation $\boldsymbol{F}_{\mathrm{b}} \in SO(3)$. Moreover, assume that the tip of an elastic rod is fixed to the position $\boldsymbol{p}^* \in \Re^3$ with the orientation $\boldsymbol{F}^* \in SO(3)$. Then, the boundary conditions for the rod ends can be expressed by

$$\boldsymbol{p}(0) = \boldsymbol{p}_{\mathrm{b}} \tag{3}$$

$$\boldsymbol{F}(0) = \boldsymbol{F}_{\mathrm{b}} \tag{4}$$

$$\boldsymbol{p}(L) = \boldsymbol{p}^* \tag{5}$$

$$\boldsymbol{F}(L) = \boldsymbol{F}^* \tag{6}$$

2.2 Shape

A shape of an elastic rod, which is one of the most important notions in order to capture motion of soft robots, is introduced here.

It is appropriate to represent a shape of an elastic rod as a set (or an order set) of the pairs each of which consists of the position of a featured point and the orientation of the associated frame in a three dimensional space because humans can recognize its shape from the whole made by placing each oriented frame at each position of the associated point if a sufficient number of featured positions with orientations are selected. It is geometrically desirable to define the shape so as to be independent of the position and orientation of an elastic rod. If humans recognize two shapes of elastic rods are identical intuitively, those values of the shapes must be equal, and vice versa. The backbone curve itself seems one of the candidate as the shape of an elastic rod, but it does not include the rotations around the longitudinal direction along the backbone, and its representation depends on the rod position and orientation. One way to represent the shape of an elastic rod s is to define it as the following mapping:

$$s : [0\ L] \to \Re^3 \times SO(3) \tag{7}$$

$$\sigma \mapsto \left({\boldsymbol{F}_{\rm b}}^T \left\{ \boldsymbol{p}(\sigma) - \boldsymbol{p}_{\rm b} \right\}, {\boldsymbol{F}_{\rm b}}^T \boldsymbol{F}(\sigma) \right) \tag{8}$$

where note that $\boldsymbol{p}(\sigma)$ and $\boldsymbol{F}(\sigma)$ have to satisfy the rod kinematics (1) and (2). Therefore, the range of this mapping becomes a subset of $\Re^3 \times SO(3)$. Hereinafter, the space of all the possible shapes is denoted by $\mathcal{S}$. Here we define the straight shape $s_0 \in \mathcal{S}$ as

$$s_0(\sigma) := (\sigma \boldsymbol{e}_{\rm x}, \boldsymbol{I}_3) \tag{9}$$

Another way to represent a shape of an elastic rod $\bar{s}$ is to directly employ $\boldsymbol{\theta}(\sigma)$ which expresses the amounts of bendings and twisting of the rod:

$$\bar{s} : [0\ L] \to \Re^3 \tag{10}$$

$$\sigma \mapsto \boldsymbol{\theta}(\sigma) \tag{11}$$

$\boldsymbol{\theta}(\sigma)$ is a value corresponding to the curvature and torsion of a curve, and thus it is not what we call a shape intuitively, but this shape representation is easy to treat because it does not confined to the rod kinematics. The space of all the possible shape by this representation is denoted by $\bar{\mathcal{S}}$. The straight shape of this shape representation $\hat{s}_0 \in \bar{\mathcal{S}}$ corresponding to the straight shape in the previous shape representation s_0 is as follows:

$$\bar{s}_0(\sigma) := \boldsymbol{0} \tag{12}$$

3. MECHANICS

In this section, mechanics of an elastic rod is discussed. The central topic is statics of an elastic rod.

3.1 Review of Rod Statics

Suppose that the both ends of an elastic rod is fixed certain positions with orientations. In this situation, the elastic rod is deformed forcibly, which means that some external forces and torques are applied to the both ends of an elastic rod. Let $\boldsymbol{f}_{\rm t}, \boldsymbol{m}_{\rm t} \in \Re^3$ be the external force and torque applied to the tip of an elastic rod. If an elastic rod is in statically balancing configurations, the following equations hold:

$$\boldsymbol{f}_{\rm b} = \boldsymbol{f}_{\rm t} \tag{13}$$

$$\boldsymbol{m}_{\rm b} = \boldsymbol{m}_{\rm t} + (\boldsymbol{p}^* - \boldsymbol{p}(0)) \times \boldsymbol{f}_{\rm t} \tag{14}$$

where vectors $\boldsymbol{f}_{\rm b}$ and $\boldsymbol{m}_{\rm b} \in \Re^3$ are the force and torque applied from an elastic rod to the base ground, respectively. The pair of these vectors is called *the base wrench.*

For an elastic rod in statically balancing configurations, the following equations corresponding to the Euler equations for calculus of variation are satisfied with the base wrench:

$$\boldsymbol{F}(\sigma) \mathrm{diag}\{\boldsymbol{k}(\sigma)\}\, \boldsymbol{\theta}(\sigma) = \{\boldsymbol{p}(0) - \boldsymbol{p}(\sigma)\} \times \boldsymbol{f}_{\rm b} + \boldsymbol{m}_{\rm b} \tag{15}$$

where $\boldsymbol{k}(\sigma) = [k_{\rm t}(\sigma)\ k_{\rm n}(\sigma)\ k_{\rm b}(\sigma)]^T$ is the rod stiffness vector at σ, and its non-negative real elements are the twisting stiffness and the bending stiffnesses around the axes of the frame at σ, respectively. This equations express that the torque balance between the reaction of the joint stiffness and the base wrench must hold at each point on the backbone curve.

Here we only consider statically balancing situations, but if we consider rod inertial and viscosity as well as elasticity, we can obtain the differential equations w.r.t. time. The obtained equations will includes parameter σ which takes any real value in continuous interval $[0\ L]$, which shows that the dynamical system of an elastic rod is infinite dimensional.

Equations (1)-(6) and (15) are the set of equations which must be satisfied by statically balancing shapes, and the solution of the equations is not trivial. Therefore, the dynamical system of an elastic rod has non-trivial equilibrium states. Hereinafter, the space of the statically balancing shapes is denoted by $\mathcal{E} \subset \mathcal{S}$ or $\bar{\mathcal{E}} \subset \bar{\mathcal{S}}$.

3.2 Rod Stiffness

Rod stiffness is as important as rod shape. Here rod stiffness is introduced in order to express the stiffness along the rod trunk. Rod stiffness will be utilized for representing flexibility of a biological trunk.

The stiffness of an elastic rod r is defined by the following mapping:

$$r : [0\ L] \to {\Re_+}^3 \tag{16}$$

$$\sigma \mapsto \boldsymbol{k}(\sigma) \tag{17}$$

The space of the rod stiffness is denoted by $\mathcal{R}$. The zero rod stiffness $r_0 \in \mathcal{R}$ is defined by the following expression:

$$r_0 := \boldsymbol{0} \tag{18}$$

4. IMPORTANT SYSTEM PROPERTIES

In this section, we explain the two very important system properties of an elastic rod in order to understand an elastic rod as a system as a whole.

4.1 Wrench-Shape Bijectiveness

Suppose that the stiffness of an elastic rod is given. Except the straight shape, the map from the base wrench to the equilibrium shape is bijective (14). This is a very useful system property of an elastic rod because any statically balancing shape is characterized by some base wrench which is a six dimensional vector.

The space of the base wrench is $\Re^6$. The set of the base wrenches corresponding to the straight shapes W_0 can be expressed by

$$W_0 := \left\{ (\boldsymbol{f}_{\rm b}, \boldsymbol{m}_{\rm b}) \in \Re^6 \,\middle|\, {\boldsymbol{F}_{\rm b}}^T \boldsymbol{f}_{\rm b} = c\boldsymbol{e}_{\rm x}, \boldsymbol{m}_{\rm b} = 0, c \in \Re \right\} \tag{19}$$

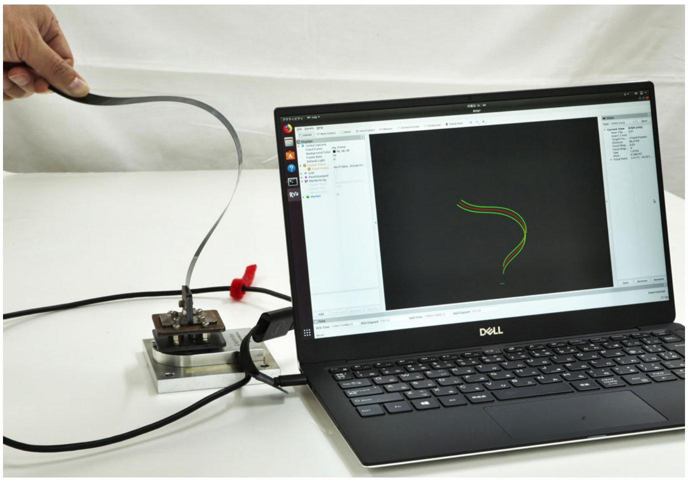

Fig. 2. Demonstrating real-time shape estimation of an elastic rod.

Therefore, this rod system property means that the mapping $f : \Re^6 - W_0 \to \mathcal{E} - s_0$ is bijective.

4.2 Rod Integrability

Suppose that the stiffness of an elastic rod is given again. Given a base wrench, the differential equations w.r.t. σ (1) and (2) can be integrated from the rod base to the rod tip with using (15). This system property is called the rod integrability here. This property can be understood from the Antman's famous book on the elasticity (10) or the formulation by Rucker and Webster (15), but was not shown explicitly. The word "Rod Integration" can be seen in the paper by Till et al. (16). One of the important remarks on this system property is that it is possible to compute the rod integration fast even in real time by taking a proper discretization method and some approximation, which will be explain in the next section. The author and his colleagues succeeded to implement a real-time shape estimation algorithm for reconstructing a rod shape from a six-axes force/torque sensor attached at the base end of the rod without employing any convergence calculation (17; 18).

Fig. 2 is a snapshot of demonstrating real-time shape estimation of an elastic rod by using a six-axes force/torque sensor. The graphic rod shape drawn on the laptop PC screen (right) is similar to the real rod shape deformed by a human hand (left). This demonstration is an illustrative example of understanding the rod integrability property as well as the wrench-shape bijectiveness property intuitively.

5. DISCRETIZATION

Discretization of an elastic rod model is important not only for computing some practical values on a rod system, but also for understanding the system properties more deeply.

5.1 Review of Discretized Elastic Rod Model

It has been known that it is reasonable to approximate a continuum elastic rod with a seral chain of n rigid bodies connected with n there-degrees-of-freedom elastic joints. This discretized version of an elastic rod model can be expressed as follows (12). First, the difference equations corresponding to the rod kinematics can be represented by

$$\boldsymbol{F}_i = \boldsymbol{F}_{i-1}\boldsymbol{R}_{\mathrm{J}}(\boldsymbol{\theta}_i) \tag{20}$$

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + l\boldsymbol{F}_i\boldsymbol{e}_{\mathrm{x}} \tag{21}$$

Second, the discretized Euler equations which express torque balance at any position of the elastic joints can be expressed by

$$\boldsymbol{A}_i \mathrm{diag}\{\boldsymbol{k}_{\mathrm{d},i}\}\,\boldsymbol{\theta}_i = (\boldsymbol{p}_0 - \boldsymbol{p}_{i-1}) \times \boldsymbol{f}_{\mathrm{b}} + \boldsymbol{m}_{\mathrm{b}}, \tag{22}$$

where $i \in \{1, \cdots, n\}$ is the index for rigid bodies or elastic joints numbered from the base to the tip in turn. Vector $\boldsymbol{p}_i \in \Re^3$ is the position vector of the $(i-1)$-th join while matrix $\boldsymbol{F}_i \in SO(3)$ is the orientation matrix of the i-th rigid body. Variables $\theta_{\mathrm{t},i}, \theta_{\mathrm{n},i}, \theta_{\mathrm{b},i} \in \Re$ are the relative angles from the $(i-1)$-th frame to the i-th frame around the axes of the $i-1$ frame, respectively. Vector $\boldsymbol{\theta}_i \in \Re^3$ is made by arranging those variables in a column, i.e., $\boldsymbol{\theta}_i := [\theta_{\mathrm{t},i}\ \theta_{\mathrm{n},i}\ \theta_{\mathrm{b},i}]^T$. Constant l is the length of the rigid body defined by $l = L/n$. Vector $\boldsymbol{k}_{\mathrm{d},i} = [k_{\mathrm{dt},i}\ k_{\mathrm{dn},i}\ k_{\mathrm{db},i}]^T$ is the discretized version of the stiffness vector which consists of the three rotational spring constants around the frame $k_{\mathrm{dt},i}, k_{\mathrm{dn},i}, k_{\mathrm{db},i}$. Matrix $\boldsymbol{R}_{\mathrm{J}} \in SO(3)$ is the matrix expressing the rotational action of the three-degrees-of-freedom spring joint which can be defined by

$$\boldsymbol{R}_{\mathrm{J}}(\boldsymbol{\theta}_i) = \boldsymbol{R}(\boldsymbol{e}_{\mathrm{x}}, \theta_{\mathrm{t},i})\boldsymbol{R}(\boldsymbol{e}_{\mathrm{y}}, \theta_{\mathrm{n},i})\boldsymbol{R}(\boldsymbol{e}_{\mathrm{z}}, \theta_{\mathrm{b},i}) \tag{23}$$

Matrix $\boldsymbol{A}_i := [\boldsymbol{a}_{\mathrm{t},i}\boldsymbol{a}_{\mathrm{n},i}\boldsymbol{a}_{\mathrm{b},i}]$ is the matrix obtained by arranging the unit-length axis vectors of the i-th joint in a row. Each axis vector can be defined by

$$\boldsymbol{a}_{\mathrm{t},i} = \boldsymbol{F}_{i-1}\boldsymbol{e}_{\mathrm{x}} \tag{24}$$

$$\boldsymbol{a}_{\mathrm{n},i} = \boldsymbol{F}_{i-1}\boldsymbol{R}(\boldsymbol{e}_{\mathrm{x}}, \theta_{\mathrm{t},i})\boldsymbol{e}_{\mathrm{y}} \tag{25}$$

$$\boldsymbol{a}_{\mathrm{b},i} = \boldsymbol{F}_{i-1}\boldsymbol{R}(\boldsymbol{e}_{\mathrm{x}}, \theta_{\mathrm{t},i})\boldsymbol{R}(\boldsymbol{e}_{\mathrm{y}}, \theta_{\mathrm{n},i})\boldsymbol{e}_{\mathrm{z}} \tag{26}$$

where $\boldsymbol{R}(\boldsymbol{a}, \theta) \in SO(3)$ is the matrix expressing the rotational action around the unit-length directional vector with the amount of angle θ.

Third, the corresponding boundary conditions can be written by

$$\boldsymbol{p}_0 = \boldsymbol{p}_{\mathrm{b}} \tag{27}$$

$$\boldsymbol{F}_0 = \boldsymbol{F}_{\mathrm{b}}. \tag{28}$$

$$\boldsymbol{p}_n = \boldsymbol{p}^* \tag{29}$$

$$\boldsymbol{F}_n = \boldsymbol{F}^*. \tag{30}$$

In this discretization, when the number of partition n becomes large, $\boldsymbol{F}_i$ approaches to $\boldsymbol{F}(L \cdot i/n)$, $\boldsymbol{F}_{i-1}$ next to $\boldsymbol{F}_i$ approaches to $\boldsymbol{F}_i$, and then, $\boldsymbol{R}(\boldsymbol{e}_{\mathrm{x}}, \theta_{\mathrm{t},i})$ approaches to the identity matrix. Therefore, note that, at the limit of$n \to \infty$, $\boldsymbol{A}_i$ converges to $\boldsymbol{F}(\sigma)$, and $\boldsymbol{k}_{\mathrm{d},i}$ goes to $\boldsymbol{k}(L \cdot i/n)/l$.

5.2 Discretized shape and rod stiffness

Due to the discretization, the rod system becomes finite dimensional, and it is possible to express the rod shape using vectors and matrices.

The discretized version of the rod shape $\boldsymbol{s}_{\rm d} \in \left(\Re^3 \times SO(3)\right)^n$ can be defined by

$$\begin{aligned}\boldsymbol{s}_{\rm d} := \Big\{ & \Big(\boldsymbol{F}_{\rm b}{}^T \{\boldsymbol{p}_1 - \boldsymbol{p}_{\rm b}\}, \boldsymbol{F}_{\rm b}{}^T \boldsymbol{F}_1\Big), \\ & \cdots, \Big(\boldsymbol{F}_{\rm b}{}^T \{\boldsymbol{p}_i - \boldsymbol{p}_{\rm b}\}, \boldsymbol{F}_{\rm b}{}^T \boldsymbol{F}_i\Big), \\ & \cdots, \Big(\boldsymbol{F}_{\rm b}{}^T \{\boldsymbol{p}_n - \boldsymbol{p}_{\rm b}\}, \boldsymbol{F}_{\rm b}{}^T \boldsymbol{F}_n\Big)\Big\} \end{aligned} \tag{31}$$

Alternatively, the rod shape $\bar{\boldsymbol{s}}_{\rm d} \in \Re^{3n}$ can be defined by another way as follows:

$$\bar{\boldsymbol{s}}_{\rm d} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ \vdots \\ \boldsymbol{\theta}_i \\ \vdots \\ \boldsymbol{\theta}_n \end{bmatrix} \tag{32}$$

Moreover, the discretized version of the rod stiffness $\bar{\boldsymbol{r}}_{\rm d} \in \Re_+{}^{3n}$ can be defined by

$$\boldsymbol{k}_{\rm d} = \begin{bmatrix} \boldsymbol{k}_{{\rm d},1} \\ \vdots \\ \boldsymbol{k}_{{\rm d},i} \\ \vdots \\ \boldsymbol{k}_{{\rm d},n} \end{bmatrix} \tag{33}$$

5.3 Understanding System Properties from Discretization

It is possible to understand the system properties shown in section 4 better from the discretized rod model.

The discretized Euler equations which represent the torque balance at each joint can be rewritten as follows because $\boldsymbol{A}_i \approx \boldsymbol{F}_{i-1}$ when the number of partition n is sufficiently large:

$$\boldsymbol{\theta}_i \approx {\rm diag}\{\boldsymbol{k}_{{\rm d},i}\}^{-1} \boldsymbol{F}_{i-1}{}^T \{(\boldsymbol{p}_0 - \boldsymbol{p}_{i-1}) \times \boldsymbol{f}_{\rm b} + \boldsymbol{m}_{\rm b}\} \tag{34}$$

From these equations, it is easy to understand that, given the rod stiffness and the base wrench, it is possible to calculate $\boldsymbol{\theta}_i$ from $\boldsymbol{p}_{i-1}$ and $\boldsymbol{F}_{i-1}$, and furthermore, to calculate $\boldsymbol{p}_i$ and $\boldsymbol{F}_i$ due to the discretized rod kinematics. This corresponds to the rod integrability. On the other hand, from the structure of equation (34), it is not difficult to show that, given the rod stiffness, as far as the shape is not straight, a different base wrench yields a different equilibrium shape, which proves that the mapping form the base wrench to the equilibrium shape is injective. Moreover, since $\boldsymbol{\theta}_i$ which satisfies equation (22) means the equilibrium shape, the mapping is proven to be surjective, too. Therefore, from the discretized elastic rod model, we can understand the bijective property of the mapping from the base wrench to the equilibrium shape very easily.

5.4 System Properties on Rod Stiffness

It should be noted that the shape and the rod stiffness are commutative, i.e., the following equation holds

$${\rm diag}\{\boldsymbol{k}_{{\rm d},i}\}\, \boldsymbol{\theta}_i = {\rm diag}\{\boldsymbol{\theta}_i\}\, \boldsymbol{k}_{{\rm d},i} \tag{35}$$

Thus, in the case of sufficiently large n, the discretize Euler equations (22) can be rewritten by

$$\boldsymbol{k}_{{\rm d},i} \approx {\rm diag}\{\boldsymbol{\theta}_i\}^{-1} \boldsymbol{F}_{i-1}{}^T \{(\boldsymbol{p}_0 - \boldsymbol{p}_{i-1}) \times \boldsymbol{f}_{\rm b} + \boldsymbol{m}_{\rm b}\} \tag{36}$$

which shows that, given the equilibrium shape, in a similar manner as the previous proof of the wrench-shape bijectiveness, the mapping from the base wrench to the rod stiffness which realizes the equilibrium shape is also bijective.

Based on the rod system properties, we can consider the analogy between an elastic rod and one-degree-of-freedom linear spring $f = kx$ (f: the force applied to the linear spring, x: the extension of the spring, k: the spring constant). The detail discussion of this analogy can be found in (19). This analogy will be useful for considering the problems of the shape computation and stiffness identification properly.

6. CONCLUSION

In this paper, an approach for system theory for soft robotics including kinematics, statics, shape computation, stiffness identification was presented with emphasizing on an elastic rod, one of the most typical largely deformable object in soft robotics.

Essential future directions includes extension of the discussion here along more general Cosserat rod theory, consideration of system theory for other typical soft robots such as soft pneumatic actuators, and treatment of dynamics which is essential for successful physical interaction to the environment.

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