Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Stability Analysis of Real-Time Methods for Equality Constrained NMPC

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Abstract: In this paper, a proof of asymptotic stability for the combined system-optimizer dynamics associated with a class of real-time methods for equality constrained nonlinear model predictive control is presented. General Q-linearly convergent online optimization methods are considered and asymptotic stability results are derived for the case where a single iteration of the optimizer is carried out per sampling time. In particular, it is shown that, if the underlying sampling time is sufficiently short, asymptotic stability can be guaranteed. The results constitute an extension to existing attractivity results for the well-known real-time iteration strategy.

Keywords: predictive control, convergence of numerical methods, stability analysis

1. INTRODUCTION

Nonlinear model predictive control (NMPC) is an optimization based control strategy that relies on the solution of parametric nonlinear noncovex programs (NLP) in order to compute an implicit feedback policy. Due to the considerable computational burden associated with the solution of such NLPs, NMPC has first found application in fields where the sampling times are generally slow enough to carry out the required computations. In particular, since the 1970s, successful applications of NMPC have been reported in the process control industry (Rawlings et al., 2017).

In more recent years, due to the significant progress in the development of efficient algorithms and software implementations and due to the increasing computational power available on embedded control units, NMPC has gradually become a viable strategy for applications with much shorter sampling times. Among others, we report on recent applications such as (Zanelli et al., 2019b), (Albin et al., 2017) and (Besselmann et al., 2015), where sampling times in the milli- and microsecond range are met.

In order to alleviate the computational burden associated with NMPC, inexact approaches are often exploited that rely on the computation of approximate solutions to the underlying NLPs. The so-called real-time iteration (RTI) method proposed in (Diehl, 2002) exploits a single iteration of a sequential quadratic programming (SQP) algorithm in order to compute an approximate solution of the current instance of the nonlinear program. By using this solution to warmstart the SQP algorithm at the next sampling time, it is possible to track an optimal solution and eventually converge to it, as the system's state is steered to a steady state. An attractivity proof for such an algorithm is derived in (Diehl et al., 2007) for a simplified setting where inequalities are not present or inactive in the entire region of attraction of the closed-loop system. Other real-time algorithms with stability guarantees are the relaxed-barrier anytime MPC for linear-quadratic problems (Feller and Ebenbauer, 2017), and the approach for general nonlinear systems in (Graichen and Kugi, 2010) that assumes a decrease over time of the cost function. Finally, in the recent paper by Liao-McPherson et al. (2019), under rather general settings, stability is established with the requirement that a sufficiently large number of iterations are carried out per sampling time.

In the present paper, the results in (Diehl et al., 2007) are extended such that not only attractivity, but also stability of the combined system-optimizer dynamics can be guaranteed.

1.1 Notation

Throughout the paper we will denote the Euclidean norm by $\|\cdot\|$, when referring to vectors, and, with the same notation, to the spectral norm

$$\|A\| := \sqrt{\lambda_{\max}(A^\top A)}, \tag{1}$$

when referring to a (real) matrix A. All vectors are column vectors and we denote the concatenation of two vectors by

$$(x, y) := \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2}$$

We denote the derivative (gradient) of any function by $\nabla f(x) = \frac{\partial f}{\partial x}(x)^\top$ and the Euclidean ball of radius r centered at x as

$$\mathcal{B}(x, r) := \{y : \|x - y\| \leq r\}. \tag{3}$$

Finally, we denote the identity matrix by $\mathbb{I}$.

⋆ This research was supported by the German Federal Ministry for Economic Affairs and Energy (BMWi) via eco4wind (0324125B) and DyConPV (0324166B), by DFG via Research Unit FOR 2401 and by the EU via ITN-AWESCO (642 682)

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2. INDEPENDENT SYSTEM AND OPTIMIZER DYNAMICS

Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{s(\cdot),u(\cdot)} \quad & \int_0^{T_f} l(s(t),u(t))\mathrm{d}t + m(s(T_f)) \\ \text{s.t.} \quad & s(0) - x = 0, \\ & \dot{s}(t) = \phi(s(t),u(t)), \quad t \in [0,T_f], \end{aligned} \tag{4}$$

where $s : \mathbb{R} \to \mathbb{R}^{n_x}$ and $u : \mathbb{R} \to \mathbb{R}^{n_u}$ represent the state and input of a system, respectively, whose dynamics are described by $\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. The functions $l : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ and $m : \mathbb{R}^{n_x} \to \mathbb{R}$ represent the Lagrange and Mayer cost terms, respectively. Finally, x is a parameter describing the current state of the system and we assume, without loss of generality, that $\phi(0,0) = 0$.

We will regard a discretized version of (4) obtained with *some* discretization method (e.g. multiple shooting):

$$P(x) : \quad \begin{aligned} \min_{y} \quad & f(y) \\ \text{s.t.} \quad & g(y) + Bx = 0, \end{aligned} \tag{5}$$

where $y \in \mathbb{R}^n$ describes the primal variables of the discretized problem, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^{n_g}$. The parameter x enters the equality constraints through the linear map defined by the constant matrix $B \in \mathbb{R}^{n_g \times n_x}$.

Assumption 1. The functions f and g are twice continuously differentiable and have bounded first and second order derivatives.

Let $X \subseteq \mathbb{R}^{n_x}$ denote the set that contains all the possible values of x such that $P(x)$ has at least one solution. We will restrict our attention to the set $X_{\bar{V}} := \{x : V(x) \leq \bar{V}\}$, with

$$V(x) := f(\bar{y}(x)), \tag{6}$$

and where $\bar{y}(x)$ solves $P(x)$. Let $\bar{u}(x)$ denote the feedback policy

$$\bar{u}(x) := M_{u,y}\bar{y}(x) \tag{7}$$

implicitly defined by $P(x)$, for some constant projection matrix $M_{u,y}$, where $\|M_{u,y}\| = 1$ is assumed for simplicity.

2.1 System Dynamics

The system under control obeys the following sampled-feedback closed-loop dynamics:

Definition 2. (System Dynamics). Let the following differential equation describe the dynamics of the system controlled using a constant input u_0:

$$\begin{aligned} \frac{d\psi}{dt}(t;x_0,u_0) &= \phi(\psi(t;x_0,u_0),u_0), \\ \psi(0;x_0,u_0) &= x_0. \end{aligned} \tag{8}$$

Here $\psi : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ describes the trajectories of the system, x_0 denotes the state of the system at a given sampling instant and u_0 the corresponding constant input. We will refer to the strictly positive parameter $T > 0$ as the *sampling time* associated with the corresponding discrete-time system

$$x_{\text{next}} = \psi(T;x,u). \tag{9}$$

In the following, we summarize an adapted version of standard assumptions used to ensure the stability properties of the nominal NMPC scheme.

Assumption 3. (Lyapunov Stability). Assume that there exists positive constants a_1, a_2, a_3 and T_0 such that the following holds for any $x \in X_{\bar{V}}$ and any $T \leq T_0$:

$$a_1\|x\|^2 \leq V(x) \leq a_2\|x\|^2, \tag{10a}$$

$$V(\psi(T;x,\bar{u}(x))) - V(x) \leq -T \cdot a_3\|x\|^2. \tag{10b}$$

Remark 4. Notice that Assumption 3, for a fixed T boils down to the standard assumption for exponential asymptotic stability (see e.g. Theorem 2.21 in (Rawlings et al., 2017)). Moreover, the dependency on T in (14) can be justified, for example, by assuming that a continuous-time Lyapunov function $V_c(x(t))$ exists such that $\frac{\mathrm{d}}{\mathrm{d}t}V_c(x(t)) \leq -\underline{a}\|x\|^2$, for some positive constant $\underline{a}$ and that $V(x)$ is a sufficiently good approximation of $V_c(x)$ in the following sense.

In particular, regard the simpler case in which the system under consideration is linear time-invariant, i.e. $\dot{x}(t) = \phi(x(t),u(t)) = A_cx(t) + B_cu(t)$. Its discretized counterpart reads $x_{\text{next}} = A_dx + B_du$, where

$$A_d := \exp(A_cT_d),\ B_d := \left(\int_0^{T_d} \exp(A_c\tau)\mathrm{d}\tau\right) B_c.$$

When controlling a discrete-time linear time-invariant system with the linear feedback policy $u = K_dx$, we know that, if $x^\top P_dx$ is a Lyapunov function for the resulting closed-loop system $x_{\text{next}} = (A_d + B_dK_d)x$, then it must satisfy the following discrete-time Lyapunov equation:

$$(A_d + B_dK_d)^\top P_d(A_d + B_dK_d) - P_d + Q_d = 0, \tag{11}$$

for some positive-definite Q_d. It is easy to show that, if the discretization time T_d is sufficiently small, then $x^\top P_dx$ is a Lyapunov function for the continuous-time closed-loop system $\dot{x}(t) = (A_c + B_cK_d)x(t)$, where we use the discrete-time gain K_d. In particular, it suffices to show that a positive-definite matrix Q_c exists such that the following continuous-time Lyapunov equation is satisfied:

$$(A_c + B_cK_d)^\top P_d + P_d(A_c + B_cK_d) + Q_c = 0. \tag{12}$$

To this end, we note that $A_d = \mathbb{I} + T_dA_c + O(T_d^2)$ and $B_d = T_dB_c + O(T_d^2)$, such that we obtain

$$\begin{aligned} &\left(\frac{A_d - \mathbb{I}}{T_d} + O(T_d) + \left(\frac{B_d}{T_d} + O(T_d)\right)K_d\right)^\top P_d \\ &\quad + P_d\left(\frac{A_d - \mathbb{I}}{T_d} + O(T_d) + \left(\frac{B_d}{T_d} + O(T_d)\right)K_d\right) \\ &\quad + Q_c = 0 \end{aligned}$$

and, multiplying by T_d,

$$\begin{aligned} &(A_d - \mathbb{I} + B_dK_d)^\top P_d + P_d(A_d - \mathbb{I} + B_dK_d) \\ &\quad + E^\top E = -T_dQ_c, \end{aligned} \tag{13}$$

where $E = O(T_d)$. Let $\tilde{A}_d := A_d + B_dK_d$. Simplifying, we obtain

$$\begin{aligned} &(A_d - \mathbb{I} + B_dK_d)^\top P_d + P_d(A_d - \mathbb{I} + B_dK_d) + E^\top E \\ &= \tilde{A}_d^\top P_d + P_d\tilde{A}_d - 2P_d + E^\top E \\ &\preceq \tilde{A}_d^\top P_d + P_d\tilde{A}_d - 2P_d + E^\top E + \\ &\quad (\tilde{A}_d - \mathbb{I})^\top P_d(\tilde{A}_d - \mathbb{I}) \\ &= \tilde{A}_d^\top P_d\tilde{A}_d - P_d + E^\top E = -Q_d + E^\top E, \end{aligned}$$

where we have exploited the fact that $(\tilde{A}_d - \mathbb{I})^\top P_d(\tilde{A}_d - \mathbb{I}) \succeq 0$. Due to the fact that $E^\top E = O(T_d^2)$ and $Q_d \succ 0$, we obtain that $-T_dQ_c \preceq -Q_d + O(T_d^2)$, and, for

any sufficiently small discretization time T_d, there must exist a positive-definite Q_c such that the continuous-time Lyapunov equation (12) is satisfied. Finally, with similar arguments it is possible to show that, if T_d is small enough, for any sufficiently small sampling time T, $x^\top P_d x$ is a valid Lyapunov function for the closed-loop system $x_{\text{next}} = (A_s + B_s K_d)x$, where

$$A_s := \exp(A_c T),\ B_s := \left(\int_0^T \exp(A_c\tau)\mathrm{d}\tau\right) B_c.$$

Assumption 5. (Second Order Growth). Assume that, for any $u \in \mathbb{R}^{n_u}$, for any $x \in X_{\bar{V}}$ and any $T \leq T_0$, the following holds:

$$\begin{aligned} V(\psi(T;x,u)) - V(x) \leq \\ - T \cdot a_3\|x\|^2 + T \cdot O(\|u - \bar{u}(x)\|^2). \end{aligned} \tag{14}$$

Remark 6. Assumption 5 can be informally justified by analyzing the properties of an underlying continuous-time Lyapunov function $V_c(x)$ and using an argument similar to the one used in Assumption 2.18 in (Diehl et al., 2007) in a discrete-time setting. In particular, under suitable differentiability assumptions, for any $\delta > 0$, we can write

$$\begin{aligned} V_c(x) &= \min_{u_\delta(\cdot)} \left\{ \int_0^\delta l(\psi(\tau,x,u_\delta(\tau)),u_\delta(\tau))\mathrm{d}\tau \right. \\ &\quad \left. + \tilde{V}_c(\psi(\delta,x,u_\delta(\cdot))) \right\} \\ &= \tilde{V}_c(\psi(\delta,x,u_\delta(\cdot))) + \int_0^\delta l(\psi(\tau,x,u_\delta(\tau)),u_\delta(\tau))\mathrm{d}\tau \\ &\quad + \delta \cdot O(\|u_\delta(\cdot) - \bar{u}_\delta(\cdot)\|^2), \end{aligned}$$

where $\tilde{V}(x)$ is the optimal value function for a problem with shrunk horizon $T_f - \delta$. Using the fact that $V_c(x) \leq \tilde{V}_c(x)$ and a quadratic lower bound on l we can conclude

$$\begin{aligned} V_c(\psi(\delta,x,u_\delta(\cdot))) &\leq \tilde{V}_c(\psi(\delta,x,u_\delta(\cdot))) \\ &\leq V_c(x) - \int_0^\delta l(\psi(\tau,x,u_\delta(\tau)),u_\delta(\tau))\mathrm{d}\tau \\ &\quad + \delta \cdot O(\|u_\delta(\cdot) - \bar{u}_\delta(\cdot)\|^2) \\ &\leq V_c(x) - \delta \cdot \tilde{a}_3\|x\|^2 + \delta \cdot O(\|u_\delta(\cdot) - \bar{u}_\delta(\cdot)\|^2), \end{aligned}$$

which justifies Assumption 5, if $V(x)$ is a "sufficiently" good approximation of $V_c(x)$ in the sense of Remark 4.

The following slightly tailored version of (Rawlings et al., 2017, Theorem 2.21) provides asymptotic stability of the closed-loop dynamics.

Theorem 7. Let Assumption 3 hold. Then, the origin is an exponentially asymptotically stable equilibrium for the closed-loop system $x_{\text{next}} = \psi(T;x,M_{u,y}\bar{y}(x))$ for any $T \leq T_0$.

Proof. See (Rawlings et al., 2017).

2.2 Optimizer Dynamics

The first-order necessary optimality conditions associated with (5) read as follows:

$$\begin{aligned} 0 &= \nabla f(y) + \nabla g(y)\lambda, \\ 0 &= g(y) + Bx \end{aligned} \tag{15}$$

where $\lambda \in \mathbb{R}^{n_g}$ is the Lagrange multiplier associated with the equality constraints. Introducing

$$F(z) := \begin{pmatrix} \nabla f(y) + \nabla g(y)\lambda \\ g(y) \end{pmatrix}, \tag{16}$$

where $z = (y,\lambda)$, Equations (15) can be expressed as

$$0 = F(z) + Cx, \tag{17}$$

where $C := [\ 0\ B^\top]^\top$. Let $\bar{Z}(x)$ be the set of all stationary points satisfying (17) for a given x. The following assumptions are made.

Assumption 8. (Regularity). Assume that, for any $x \in X_{\bar{V}}$ there exists a unique solution $\bar{z}(x)$ and that second order sufficient conditions hold at $\bar{z}(x)$. Moreover, assume that the steady-state solution is $\bar{z}(0) = 0$, i.e. $\bar{Z}(0) = \{0\}$.

Remark 9. Although most classical NMPC stability results rely on the fact that the optimizer finds the global solution, the assumption that a unique solution exists in $X_{\bar{V}}$ is somewhat restrictive and deserves further discussion. On the one hand, this is similar to (Diehl et al., 2007, Assumption 2.3) and to some extent a strong assumption. On the other hand, notice that, loosely speaking, it would suffice to assume that there is a unique "branch" of the solution manifold in a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^{n_z} \times \mathbb{R}^{n_z}$. For the sake of simplicity, in this paper, we will however assume that $\bar{z}(x)$ is the unique solution in $X_{\bar{V}}$. One further additional implication is the fact that the value function $V(x)$ is continuous. This is again similar to what assumed in (Diehl et al., 2007) and in some of the work on inherent robustness of NMPC (see e.g. (Pannocchia et al., 2011)).

Proposition 10. Let Assumptions 1 and 8 hold. Then there exist strictly positive constants σ, $\bar{r}_z$ and $\bar{r}_x$, such that, for any $x \in X_{\bar{V}}$, the solution $\bar{z}(x)$ is uniquely defined over $\mathcal{B}(\bar{z}(x),\bar{r}_z)$ and the following holds:

$$\|\bar{z}(x'') - \bar{z}(x')\| \leq \sigma \|x'' - x'\|, \tag{18}$$

for any $x'', x' \in \mathcal{B}(x,\bar{r}_x)$.

Proof. The result is a direct consequence of the implicit function theorem (also known as Dini's theorem).

Definition 11. (Optimizer Dynamics). Let the following discrete-time system describe the dynamics of the optimizer used to solve the parametric problem (5)

$$z_+ = \varphi(\psi(T;x,M_{u,z}z),z), \tag{19}$$

where $\varphi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ and where $M_{u,z}$ is a properly defined projection matrix. For simplicity, we will assume that $\|M_{u,z}\| = 1$.

Assumption 12. (Contraction). There exists a radius $\hat{r}_z > 0$ and a positive constant $\hat{\kappa} < 1$ such that, for any given stationary point $\bar{z}(x)$ at $x \in X_{\bar{V}}$, and any z in $\mathcal{B}(\bar{z}(x),\hat{r}_z)$, the optimization routine produces z_+ such that

$$\|z_+ - \bar{z}(x)\| \leq \hat{\kappa}\|z - \bar{z}(x)\|. \tag{20}$$

Since we are interested in a real-time strategy that seeks an approximate solution to (17) as the parameter x changes over time, we will exploit the following result on general real-time methods.

Lemma 13. Let Assumptions 1, 8 and 12 hold. Then there exist strictly positive constants r_z and r_x, and finite positive constants $\sigma, \hat{\kappa} > 0$, with $\hat{\kappa} < 1$, such that, for any x in $X_{\bar{V}}$, any z in $\mathcal{B}(\bar{z},r_z)$, and any x_+ in $\mathcal{B}(x,r_x)$, it holds that

$$\|z_+ - \bar{z}(x_+)\| \leq \hat{\kappa}\|z - \bar{z}(x)\| + \sigma\hat{\kappa}\|x_+ - x\|. \tag{21}$$

Proof. See e.g. (Zanelli et al., 2019a).

3. SYSTEM-OPTIMIZER DYNAMICS

Theorem 7 and Lemma 13 provide key properties of the system and optimizer dynamics, respectively. In this section, we analyze the interaction between these two dynamical systems. In particular, we interested in analyzing the behavior of the following coupled system-optimizer dynamics.

Definition 14. (System-Optimizer Dynamics). Let the following equations describe the evolution of the state of the system and of the optimizer's iterates:

$$\begin{aligned} x_+ &= \psi(T; x, M_{u,z}z), \\ z_+ &= \varphi(\psi(T; x, M_{u,z}z), z). \end{aligned} \tag{22}$$

In particular, after a simple coordinate change, we can refer to the following *error* dynamics in compact form:

$$\xi_+ = \Phi(T; \xi), \tag{23}$$

where $\xi := (x, z - \bar{z}(x))$ and $\Phi : \mathbb{R} \times \mathbb{R}^{n_x+n_z} \to \mathbb{R}^{n_x+n_z}$. We will refer to (23) as *system-optimizer dynamics.*

An attractivity proof for real-time iterations based on the assumption that no inequalities are present in the problem formulation or that, equivalently, no active constraints are present in the region of interest, has been proposed in (Diehl et al., 2007). Similarly, an attractivity proof where shifted iterations and a zero terminal constraint are used is derived in (Diehl et al., 2005).

In this paper, we prove instead asymptotic stability for the system-optimizer dynamics, which is in general not implied by attractivity. Moreover, the general contraction considered in Assumption 12 covers a broader class of algorithms where the iterations need not be iterations of a sequential quadratic programming method.

3.1 Perturbed Error Contraction

In order to be able to use the contraction from Lemma 13, we will make a general assumption on the behavior of the closed-loop system in a neighborhood of the equilibrium and for a bounded value of the numerical error.

Assumption 15. There exist positive finite constants $L_{\psi,x}$ and $L_{\psi,u}$ such that, for all $x \in X_{\bar{V}}$ and all z such that $\|z - \bar{z}(x)\| \le r_z$, the following inequality holds:

$$\|\psi(x, u) - x\| \le T(L_{\psi,x}\|x\| + L_{\psi,u}\|M_{u,z}z\|). \tag{24}$$

Proposition 16. Let Assumptions 1, 8 and 15 hold and define the following constants:

$$\eta := L_{\psi,u} + L_{\psi,x}\sigma \tag{25}$$

and

$$\theta := L_{\psi,u}. \tag{26}$$

Then, the following holds:

$$\|\psi(x, u) - x\| \le T(\eta\|x\| + \theta\|z - \bar{z}(x)\|), \tag{27}$$

for any $x \in X_{\bar{V}}$ and any z such that $\|z - \bar{z}(x)\| \le r_z$.

Proof. In the following let $u = M_{u,z}z$ for ease of notation. Due to Assumption 15 we have that

$$\|\psi(x, u) - x\| \le T(L_{\psi,x}\|x\| + L_{\psi,u}\|u\|) \tag{28}$$

and, due to the assumption of regularity at the solution $\bar{z}(x)$ and the fact that $\bar{z}(0) = 0$, we can write

$$\|u\| \le \|\bar{z}(x)\| + \|z - \bar{z}\| \le \sigma\,\|x\| + \|z - \bar{z}\|\,,$$

and the following holds:

$$\begin{aligned} \|\psi(x, u) - x\| \le\ & T(L_{\psi,x} + L_{\psi,u}\sigma)\,\|x\| \\ & + TL_{\psi,u}\,\|z - \bar{z}\|\,. \end{aligned} \qquad \square$$

Proposition 17. Let Assumptions 1, 8, 12 and 15 hold. Moreover, define

$$T_1' := \min\left\{\frac{r_x}{\eta r_{\bar{V}} + \theta r_z}, \frac{r_z(1-\hat{\kappa})}{\sigma\hat{\kappa}(\theta r_z + \eta r_{\bar{V}})}\right\}, \tag{29}$$

where

$$r_{\bar{V}} := \sqrt{\frac{\bar{V}}{a_1}}. \tag{30}$$

Then, for any $x \in X_{\bar{V}}$, any $\|z - \bar{z}(x)\| \le r_z$ and any $T \le T_1 := \{T_1', T_0\}$, the following holds:

$$\|z_+ - \bar{z}(x_+)\| \le \kappa\|z - \bar{z}\| + T \cdot \gamma\|x\|, \tag{31}$$

where

$$\kappa := \hat{\kappa}(1 + T\sigma\theta) < 1, \quad \gamma := \sigma\hat{\kappa}\eta. \tag{32}$$

Moreover, $\|z_+ - \bar{z}(x_+)\| \le r_z$.

Proof. Given that $\|z - \bar{z}\| \le r_z$ and that, due to Assumption 15 and the definition of T_1', we have $\|x_+ - x\| \le r_x$ for all $x \in X_{\bar{V}}$ and we can apply the contraction from Lemma 13:

$$\|z_+ - \bar{z}(x_+)\| \le \hat{\kappa}\,\|z - \bar{z}\| + \sigma\hat{\kappa}\,\|x_+ - x\|\,. \tag{33}$$

Applying the inequality from Proposition 16, we obtain

$$\|z_+ - \bar{z}(x_+)\| \le \kappa\|z - \bar{z}\| + T\gamma\|x\|, \tag{34}$$

where

$$\kappa := \hat{\kappa}(1 + T\sigma\theta), \quad \gamma := \sigma\hat{\kappa}\eta. \tag{35}$$

Finally, due to the second term in the definition of T_1', we have that $\|z_+ - \bar{z}(x_+)\| \le r_z$ and $\kappa < 1$ since

$$\begin{aligned} T \le T_1' &= \frac{r_z(1-\hat{\kappa})}{\sigma\hat{\kappa}(\theta r_z + \eta r_{\bar{V}})} \\ &= \frac{r_z}{r_z}\frac{(1-\hat{\kappa})}{\sigma\hat{\kappa}(\theta + \eta r_{\bar{V}}/r_z)} < \frac{1-\hat{\kappa}}{\hat{\kappa}\sigma\theta}. \end{aligned} \qquad \square \tag{36}$$

3.2 Perturbed Lyapunov Contraction

In the following, we analyze the impact of the fact that the approximate feedback policy $M_{u,z}z$ is used, instead of the optimal one $M_{u,z}\bar{z}(z)$, on the nominal Lyapunov contraction. Throughout the rest of the paper, we will make use of the following shorthands:

$$V := V(x) \tag{37}$$

and

$$V_+ := V(\psi(T; x, M_{u,z}z)) \tag{38}$$

to denote the values taken by the optimal cost at the "current" state and at the one reached applying the suboptimal control action $M_{u,z}z$ starting from x. Similarly, we introduce

$$E := \|z - \bar{z}(x)\| \tag{39}$$

and

$$E_+ := \|\varphi(\psi(T; x, M_{u,z}z), z) - \bar{z}(\psi(T; x, M_{u,z}z))\| \tag{40}$$

to denote the numerical error attained at the "current" and next iteration of the optimizer, where the error is computed with respect to the exact solution associated with the "current" and next state of the system. Note that, although V, V_+, E, E_+ depend on (x, z), we omit that dependency for a more compact notation.

In Lemma 20, we will show that, under the condition that the sampling time T is sufficiently small, we can guarantee positive invariance of a properly defined set.

Proposition 18. Let Assumptions 1, 3, 5 and 8 hold. Then, there exist finite positive constants $\mu, \bar{V}_q \leq \bar{V}$ and $r_q \leq r_z$, such that, for any $E \leq r_q$ and any x in $X_{\bar{V}_q}$, where $X_{\bar{V}_q} := \{x : V(x) \leq \bar{V}_q\}$, the following holds:

$$V_+ \leq (1 - T\bar{a})V + T\mu E^2, \tag{41}$$

with $\bar{a} := \frac{a_3}{a_2}$.

Proof. Assumption 5 implies that there must exist strictly positive constants $\mu, \bar{V}_q \leq \bar{V}$ and $r_q \leq r_z$ such that the following holds

$$\begin{aligned} V(\psi(x, M_{u,z}z)) &\leq V(x) - Ta_3\|x\|^2 - T\mu E^2 \\ &\leq V(x) - T\frac{a_3}{a_2}V(x) - T\mu E^2 \\ &= (1 - T\bar{a})V(x) - T\mu E^2, \end{aligned} \tag{42}$$

for any $E \leq r_q$, any $x \in X_{\bar{V}_q}$ and any $T \leq T_1$. □

Definition 19. Define the following set:

$$\Sigma := \{(x, z) : V(x) \leq \bar{V}_q, \|z - \bar{z}(x)\| \leq \tilde{r}_q\}. \tag{43}$$

The following theorem shows positive invariance of Σ.

Lemma 20. (Invariance of Σ). Let Assumptions 1, 3, 5, 8, 12 and 15 hold. Define

$$\tilde{r}_q := \min\left\{r_q, \sqrt{\frac{\bar{a}\bar{V}_q}{\mu}}\right\} \text{and} \, T_2' := \frac{(1-\kappa)\tilde{r}_q\sqrt{a_1}}{\sqrt{\bar{V}}\gamma}. \tag{44}$$

Then, for any $(x, z) \in \Sigma$ and any $T \leq T_2 := \min\{T_2', T_1\}$, it holds that $(x_+, z_+) \in \Sigma$. Moreover, the following coupled system-optimizer contractions hold:

$$\begin{aligned} V_+ &\leq (1 - T\bar{a})V + T\mu E^2, \\ E_+ &\leq T\hat{\gamma}V^{\frac{1}{2}} + \kappa E, \end{aligned} \tag{45}$$

where $\hat{\gamma} = \frac{\gamma}{\sqrt{a_1}}$.

Proof. Given that $E \leq \tilde{r}_q \leq r_q$ and $x \in X_{\bar{V}_q}$, we can apply the contraction from Proposition 18, such that

$$V_+ \leq (1 - T\bar{a})V + T\mu E^2, \tag{46}$$

holds. Moreover, due to the definition of $\tilde{r}_q$, $V_+ \leq \bar{V}_q$ holds, which implies that x_+ is in $X_{\bar{V}_q}$. Similarly, due to the fact that $E \leq \tilde{r}_q \leq r_z$ and $x \in X_{\bar{V}_q} \subseteq X_{\bar{V}}$, we can apply the result from Proposition 17, which shows that

$$E_+ \leq \kappa E + T\gamma\|x\| \quad \text{and} \quad E_+ \leq r_z \tag{47}$$

must hold. Using Assumption 3 in Equation (47), we obtain

$$E_+ \leq \kappa E + T\hat{\gamma}V^{\frac{1}{2}}. \tag{48}$$

Moreover, due to (44), $E_+ \leq \tilde{r}_q$ holds. □

Lemma 20 shows that, we can guarantee that the state of the combined system-optimizer dynamics (x, z) will not leave Σ under the assumption that the sampling time T is short enough. Moreover, due to subadditivity of the square root, the following holds:

$$V_+^{\frac{1}{2}} \leq (1 - T\bar{a})^{\frac{1}{2}}V^{\frac{1}{2}} + (T\mu)^{\frac{1}{2}}E \tag{49}$$

such that we can regard the following simpler dynamics:

Definition 21. (Auxiliary Dynamics). We will refer to the following (linear) dynamical system:

$$\begin{aligned} \nu_+ &= (1 - T\bar{a})^{\frac{1}{2}}\nu + (T\mu)^{\frac{1}{2}}\epsilon, \\ \epsilon_+ &= T\hat{\gamma}\nu + \kappa\epsilon \end{aligned} \tag{50}$$

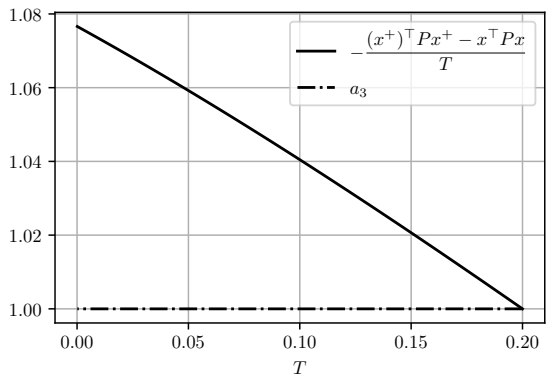

Fig. 1. Lower bound on $V(x)$ and Lyapunov decrease as a function of T. Here we regard $x^+ := (A_T + B_T K_d)x$ and check that $x^\top Px$ is a still a Lyapunov function for such closed-loop system and estimate the decrease rate.

with states $\nu, \epsilon \in \mathbb{R}$ as *auxiliary dynamics.*

Remark 22. Notice that the considerations made by Diehl et al. (2007), in a similar setting, lead to the same type of coupled contraction from Lemma 20. An attractivity proof that implicitly uses auxiliary dynamics that would be obtained directly from (45) is derived in (Diehl et al., 2007). However, due to the fact that such auxiliary system-optimizer dynamics are not Lipschitz at $(0, 0)$, it would not be possible to prove stability with standard linear analysis tools.

4. ASYMPTOTIC STABILITY RESULT

Due to linearity of the auxiliary dynamics (50), we can study asymptotic stability with standard tools from linear systems analysis.

Theorem 23. (Asymptotic Stability). Let Assumptions 1, 3, 5, 8, 12 and 15 hold. Then, the origin $(\nu, \epsilon) = (0, 0)$ is asymptotically stable for the auxiliary dynamics (50).

Proof. A sufficient and necessary condition for the asymptotic stability of (50) is that the eigenvalues of the matrix

$$A = \begin{bmatrix} (1 - T\bar{a})^{\frac{1}{2}} & (T\mu)^{\frac{1}{2}} \\ T\hat{\gamma} & \kappa \end{bmatrix}, \tag{51}$$

are smaller than one in absolute value. In order to compute the eigenvalues, we need to solve

$$\det(\lambda\mathbb{I} - A) = (\lambda - (1 - T\bar{a})^{\frac{1}{2}}) \cdot (\lambda - \kappa) + O\left(T^{\frac{3}{2}}\right) = 0,$$

which entails

$$\lambda_1 = (1 - T\bar{a})^{\frac{1}{2}} + O\left(T^{\frac{3}{2}}\right), \, \lambda_2 = \kappa + O\left(T^{\frac{3}{2}}\right). \tag{52}$$

Hence, for a sufficiently small sampling time T, the origin $(\nu, \epsilon) = (0, 0)$ is asymptotically stable. □

Theorem 23 shows asymptotic stability of the auxiliary dynamics (50), but not of the original system-optimizer dynamics (23). The following corollary shows that we can easily extend the stability result in this sense.

Corollary 24. Let Assumptions 1, 3, 5, 8, 12 and 15 hold. Then, the origin $\xi = (x, z) = (0, 0)$ is locally asymptotically stable for the system-optimizer dynamics (23).

Proof. Let $\chi := (\nu, \epsilon)$ denote the state of the auxiliary dynamics in compact form and regard the sequences $\{\chi_k\}$ and $\{\xi_k\}$ generated by the auxiliary and original coupled dynamics for any compatible initial conditions $\chi_0 = (\nu_0, \epsilon_0)$ and $\xi_0 = (x_0, z_0)$, with $V(x_0)^{\frac{1}{2}} = \nu_0$ and $\|z_0 - \bar{z}(x_0)\| = \epsilon_0$, respectively. Due to the definition of the auxiliary dynamics, we have that, for any compatible initial conditions chosen in such a way, $V(x_k)^{\frac{1}{2}} \leq \nu_k$ and $\|z_k - \bar{z}(x_k)\| \leq \epsilon_k$ holds for any $k \geq 0$. Then, for any $k \geq 0$, we can write

$$\|\xi_k\| = \sqrt{\|x_k\|^2 + \|z_k - \bar{z}(x_k)\|^2} \leq \sqrt{\frac{1}{a_1} V(x_k) + E_k^2}$$
$$\leq \sqrt{\frac{1}{a_1}\nu_k^2 + \epsilon_k^2} \leq \tilde{a}_1^{-\frac{1}{2}} \sqrt{\nu_k^2 + \epsilon_k^2} = \tilde{a}_1^{-\frac{1}{2}} \|\chi_k\|,$$

where $\tilde{a}_1 = \min\{a_1, 1\}$.

Since $(\nu, \epsilon) = (0, 0)$ is locally asymptotically stable for the auxiliary dynamics, we have that, for any $\epsilon' > 0$, there exists a $\delta' > 0$ such that, if $\|\chi_0\| < \epsilon'$, then $\|\chi_k\| < \delta'$, for any $k \geq 0$. Since

$$\|\chi_0\| \leq \sqrt{\tilde{a}_2}\, \|\xi_0\|, \tag{53}$$

where $\tilde{a}_2 = \max\{a_2, 1\}$, we can choose an arbitrarily small $\epsilon = \tilde{a}_2^{-\frac{1}{2}} \epsilon'$, such that $\|\xi_0\| \leq \epsilon = \tilde{a}_2^{-\frac{1}{2}} \epsilon' \implies \|\chi_0\| \leq \epsilon' \implies \|\chi_k\| \leq \delta', \forall k \geq 0$ and $\|\xi_k\| \leq \delta := \tilde{a}_1^{-\frac{1}{2}} \delta', \forall k \geq 0$, which proves stability.

Finally, local attractivity can be trivially shown by observing that $\lim_{k\to\infty} \|\chi_k\| = 0 \implies \lim_{k\to\infty} \|\xi_k\| = 0$. □

5. ILLUSTRATIVE EXAMPLE

In this section, although the results derived apply to a much more general class of problems (twice-continuously nonlinear dynamics and cost), we discuss an illustrative numerical example where we exploit a simplified setting in order to be able to explicitly compute all the constants used in the assumptions of Theorem 23. In particular, we regard the following unconstrained, linear-quadratic optimal control problem:

$$\begin{aligned} \min_{s(\cdot),u(\cdot)} \quad & \int_0^\infty \begin{bmatrix} s(t) \\ u(t) \end{bmatrix}^\top \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} \begin{bmatrix} s(t) \\ u(t) \end{bmatrix} \\ \text{s.t.} \quad & s(0) - x = 0, \\ & \dot{s}(t) = A_c s(t) + B_c u(t), \quad t \in [0, \infty], \end{aligned} \tag{54}$$

where the continuous-time dynamics are defined by

$$A_c := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_c := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{55}$$

and the matrices

$$Q_c := \mathbb{I}_2 \quad \text{and} \quad R_c := 1 \tag{56}$$

define the cost.

Problem (57) is discretized using multiple shooting with a single shooting node and a fixed discretization time $T_d = 0.1s$ as follows:

$$\begin{aligned} \min_{s_0,s_1,u_0} \quad & T_d \begin{bmatrix} s_0 \\ u_0 \end{bmatrix}^\top \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} \begin{bmatrix} s_0 \\ u_0 \end{bmatrix} + s_1^\top P s_1 \\ \text{s.t.} \quad & s_0 - x = 0, \\ & s_1 = A_{T_d} s_0 + B_{T_d} u_0, \end{aligned} \tag{57}$$

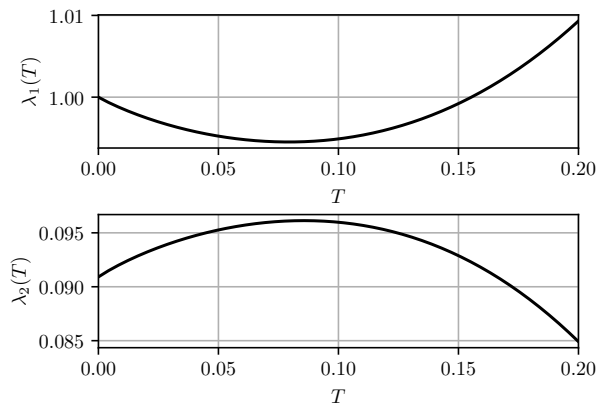

Fig. 2. Eigenvalues as a function of the sampling time T for problem (57). For sufficiently short sampling times, the auxiliary system in Definition 21 is asymptotically stable.

where the discrete-time dynamics are obtained using an exact discretization with piece-wise constant parametrization of the control trajectories:

$$A_{T_d} := \exp(A_c T_d),\ B_{T_d} := \left(\int_0^{T_d} \exp(A_c \tau) \mathrm{d}\tau \right) B_c. \tag{58}$$

Finally, the symmetric positive-definite matrix P that defines the terminal cost for the discretized problem is computed by solving the discrete-time algebraic Riccati equation

$$\begin{aligned} P \quad &= A_{T_d}^\top P A_{T_d} - \\ & (A_{T_d}^\top P B_{T_d})(R + B_{T_d}^\top P B_{T_d})^{-1}(B_{T_d}^\top P A_{T_d}) + Q, \end{aligned} \tag{59}$$

where $R := T_d R_c$ and $Q := T_d Q_c$. After elimination of s_0, the first-order optimality conditions of problem (57) read

$$H u_0 + G x = 0, \tag{60}$$

where

$$H := (T_d R + B_{T_d}^\top P B_{T_d}), \quad G := B_{T_d}^\top P A_{T_d}. \tag{61}$$

We solve (60) with the following real-time gradient descent method:

$$u_{0,+} = -\tilde{H}^{-1} \left((H - \tilde{H}) u_0 + G x_+ \right), \tag{62}$$

where $\tilde{H} = \rho \mathbb{I}$ for some positive constant $\rho > 1$. Using standard arguments from convergence theory for Newton-type methods (see e.g (Diehl, 2016)), it is easy to show, that, for a fixed value of the parameter x, the following contraction estimate holds:

$$\|u_{0,+} - \bar{u}_0(x)\| \leq \hat{\kappa} \|u_0 - \bar{u}_0(x)\|, \tag{63}$$

where

$$\hat{\kappa} := \|\tilde{H}^{-1}(H - \tilde{H})\|. \tag{64}$$

Since $V(x) = x^\top P x$ and $\bar{u}_0(x) = -H^{-1} G x$, we can compute exactly the constants $\mu = 2\lambda_{\max}(H)$ and $\sigma = \|H^{-1} G\|$. Let

$$A_T := \exp(A_c T),\ B_T := \left(\int_{\tau=0}^{T} \exp(A_c \tau) \mathrm{d}\tau \right) B_c \tag{65}$$

describe the dynamics discretized according to the sampling time T. Given that

$$x_+ = A_T x + B_T u = x + \frac{T}{T}((A_T - \mathbb{I})x + B_T u), \tag{66}$$

we can compute the constants

$$L_{\psi,x} = \frac{1}{T}\|A_T - \mathbb{I}\| \quad \text{and} \quad L_{\psi,u} = \frac{1}{T}\|B_T\|. \tag{67}$$

Following the definitions in Propositions 16 and 17, we have $\hat{\gamma} = \frac{1}{\sqrt{a_1}}\sigma\hat{\kappa}\eta$ and $\kappa = \hat{\kappa}(1 + T\sigma\theta)$, where $\eta = L_{\psi,u} + L_{\psi,x}\sigma$ and $\theta = L_{\psi,u}$. In order to validate Assumption 3 and compute an estimate for constant a_3, we compute the largest eigenvalue $\lambda_{\max}(\Delta P)$ of the matrix

$$\Delta P(T) := \frac{1}{T}((x^+)^\top P x^+ - x^\top P x), \tag{68}$$

where

$$x^+ = (A_T + B_T K)x \tag{69}$$

and $K = -H^{-1}G$. Figure 1 shows the estimated decrease rate compared with the minimum eigenvalue of P. Choosing

$$a_3 = \min_T \lambda_{\max}(\Delta P(T)), \tag{70}$$

we obtain a value of a_3 that satisfies (14) for any T such that $0 \leq T \leq T_d$. Finally, we can compute the constants in (10a) as $a_1 = \lambda_{\min}(P)$ and $a_2 = \lambda_{\max}(P)$.

Given that we can numerically compute all the constants involved in the Assumptions of Theorem 23, it is possible to compute the longest sampling time for which the auxiliary system in (50) is asymptotically stable. Figure 2 shows the eigenvalues of the auxiliary system as a function of T.

6. CONCLUSIONS AND OUTLOOK

In this paper we present an asymptotic stability proof for real-time methods for unconstrained NMPC. We extend the well-known attractivity results derived in (Diehl et al., 2005) and (Diehl et al., 2007) under similar settings. The interaction between system and optimizer is analyzed using an auxiliary system whose states are the square root of the optimal value and the numerical error associated with the approximate solution. Asymptotic stability of such auxiliary system is established under the assumption that the sampling time is sufficiently short. Ongoing research involves the extension of the results to the case where the optimal control formulation includes inequality constraints and the derivation of practical conditions under which Assumptions 3 and 5 hold, using arguments along the lines of Remarks 4 and 6. Similarly, Assumption 8 will be adapted to the more general setting described in Remark 9.

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