Challenges of Trajectory Planning with Integrator Models on Curved Roads

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Abstract: In the context of trajectory planning for autonomous vehicles, a widely used vehicle model relies on linear integrator dynamics. We consider planning with this model type, with a focus on the requirement to account for curved road topologies. As our analysis reveals, this generally gives rise to non-convex, coupled constraints on the vehicle’s states and inputs, which impedes computationally efficient planning. We propose a method to resolve this issue by modification of the non-convex constraints. This modification is based on inner approximations of sub-level sets of nonlinear functions, which are obtained by quantifier elimination. The efficacy of the method is demonstrated in two example scenarios.

Keywords: Autonomous driving, trajectory planning, vehicle dynamic systems, constrained control

1. INTRODUCTION

1.1 Motivation

A great challenge in the context of autonomous driving is to plan trajectories for a vehicle, i.e., to schedule its future positions and velocities without causing collisions with other vehicles or leaving the road. Due to its non-convex nature, this task constitutes a complex problem, which has found considerable attention in the fields of robotics, e.g. (LaValle, 2006), and autonomous driving, cf. (González et al., 2016). Many approaches formulate the problem as a discrete-time optimal control problem with the objective to minimize the value of a cost function \( J \). For a given initial state \( x_0 = x(t_0) \) of a dynamical system, the value of the cost function depends on the sequence of states \( x(\cdot) = (x(t_k)) \in \mathbb{R}^{n_x}_{k=0} \) resulting from a sequence of input signals \( u(\cdot) = (u(t_k)) \in \mathbb{R}^{n_u}_{k=0} \) as predicted over a horizon \( H \). The optimal input sequence results from:

Problem 1. (Planning Problem).

\[
\min_u J(x(t_h)) = \min_u \sum_{k=1}^{H} ||x(t_k) - x_{ref}||_Q^2 + ||u(t_{k-1})||_R^2,
\]

subject to dynamical constraints on the state:

\[
x(t_{k+1}) = f(x(t_k), u(t_k)) = Ax(t_k) + Bu(t_k).
\]  

Assume appropriate dimensions for \( A, B, Q, R, \) and \( C_\alpha \). Inputs and states are constrained to lie within sets \( U \subseteq \mathbb{R}^{n_u} \) and \( X \subseteq \mathbb{R}^{n_x} \), respectively. Also, coupling constraints:

\[
[x(t_k)^T u(t_k)^T]^T \in C \subseteq X \times U,
\]

may be imposed. Collision avoidance constraints require the system not to enter a set \( F(t_k) \subset X \) of forbidden states:

\[
x(t_k) \notin F(t_k),
\]

encoding, e.g., positions of other vehicles.

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Established methods exist for the implementation of (3), e.g. relying on mixed-integer programming (Williams, 1990). Since this is not the focus of this paper, we omit the details. Often (Qian et al., 2016; Eilbrecht and Stursberg, 2018; Hess et al., 2018; Burger and Lauer, 2018; Nilsson et al., 2015; Schürmann et al., 2017), the (discretized) dynamics (1) is based on the description of a vehicle’s dynamics similar to:

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{v}_x \\
\dot{v}_y \\
\dot{v}_r
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y \\
p_r \\
u_x \\
u_y \\
u_r
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
u_r
\end{bmatrix},
\]

with \( p_x, p_y, v_x, v_y, v_r, u_x, u_y, u_r \) denoting longitudinal \( (x) \) and lateral \( (y) \) positions, velocities, and accelerations. Additional constraints, e.g. on the rate of change of the inputs, can readily be incorporated and lead to higher-order integrator models or additional constraints on states and inputs.

Clearly, we (and certainly all others who use integrator models for planning) are well aware of the fact that actual vehicle dynamics are more complicated than the integrator dynamics in (4). This per se does not constitute a problem since every model is only a simplified description of the actual system dynamics. Low-level control of vehicles usually relies on the so-called bicycle model (Pacejka, 2005), which considers nonlinear coupling of lateral and longitudinal dynamics as well as (potentially nonlinear) tire dynamics, but neglects vertical, roll, and pitch dynamics. These degrees of freedom, in contrast, are considered by multi-body vehicle models (Blundell and Harty, 2004) used for driving dynamics simulation. But even detailed multi-body vehicle models are often based on rigid body assumptions, neglecting complex elastic deformations of the car body, and so on.

Thus, the mere existence of more detailed vehicle models does not imply that a specific model is not suited for a...
certain task. The following arguments promote the usefulness of integrator models for planning: At first, planning is understood as generation of reference trajectories, which are then fed to lower-layer controllers, which are based on more detailed vehicle models, such that vehicle operation is not directly based on the simple integrator models. Then, planning procedures can be combined with methods to verify the safety of a plan. This has been demonstrated by Schirmann et al. (2017) for plans generated based on integrator models. Finally, models with integrator structure result from feedback linearization of nonlinear systems, such that integrator models may actually reflect nonlinear dynamics. That being said, the ultimate purpose of this paper is not to investigate into the appropriateness of integrator models. Instead, we will assume its appropriateness and analyze which challenges arise in this context if curved roads have to be considered. To our knowledge, this has not been explicitly considered in approaches relying on integrator models.

1.2 Related Work

The question of how to represent constraints arising from the topology of a road has found different answers in the literature: A significant part of existing approaches to trajectory planning for autonomous vehicles plans in a Cartesian coordinate frame. These approaches either rely on local linearization, i.e., assume a straight road over the planning horizon, e.g. (Nilsson et al., 2015; Elbrecht and Stursberg, 2018; Qian et al., 2016), or use approximations to the road boundaries, cf. (Ziegler et al., 2014). Clearly, the validity of the first kind of approach depends on the curvature of the road and the planning horizon. While the second type of approach is flexible and allows to extract information from maps easily, it leads to computationally demanding, non-convex optimization problems.

Other approaches do not use a global, Cartesian coordinate system, but plan in a Frenet frame (Perantoni and Limebeer, 2014; Hess et al., 2018), cf. Sec. 2. This allows to represent constraints arising from the road topology as convex constraints. However, it complicates the equations of motion of a vehicle and the imposed constraints, which generally requires to employ computationally very demanding algorithms (Perantoni and Limebeer, 2014). This problem can be circumvented by simplifying assumptions (Hess et al., 2018), however at the cost of reduced planning accuracy.

2. PRELIMINARIES

We define a planning model to be a tuple: 
\[ \mathcal{M} = (f, \mathcal{C}) \],
(5)
consisting of equations of motion \( f \) as in (1) and constraints \( \mathcal{C} \) as in (2). Equations of motion are derived based on kinematic and kinetic considerations, in which the kinematics describe a motion, while the kinetics are concerned with the description of the forces causing the motion. In addition, a coordinate system is required to relate a motion to a global coordinate system. For planning, apart from the global coordinate system, two additional coordinate systems are widely used: a body-fixed one and a so-called Frenet frame. These are described here for the sake of completeness and for use in subsequent derivations. While different coordinate systems can give the same information, some may be more appropriate for certain tasks than others. For example, dynamical constraints can be best expressed in a body-fixed reference frame, while it is easier to enforce constraints resulting from the road topology in a Frenet frame.

2.1 Notations and Coordinate Systems

In the following, boldface letters denote vectors, e.g. \( \mathbf{r} \). A vector can be expressed by a tuple of coefficients referred to in a certain basis, e.g.: 
\[ \mathbf{r} = r_1 \mathbf{e}_1 + r_2 \mathbf{e}_2 + r_3 \mathbf{e}_3 = [r_1 \ r_2 \ r_3]^T \mathbf{B}_{123}, \]
(6)
in which the basis is a tuple of basis vectors: 
\[ \mathbf{B}_{123} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3]^T \]
and is assumed to be ordered and orthonormal. Three coordinate systems will be used, cf. Fig. 1:

Global Inertial Reference Frame The planar position of a vehicle in a global coordinate system with the basis \( \mathbf{B}_g = [\mathbf{e}_{g,x} \ \mathbf{e}_{g,y} \ \mathbf{e}_{g,z}]^T \) is given by: 
\[ \mathbf{r}(t) = [p_x(t) \ p_y(t) \ 0]^T \mathbf{B}_g. \]
(7)
Frenet Frame In addition, a moving reference frame (Perantoni and Limebeer, 2014) with the basis \( \mathbf{B}_{tn} = [\mathbf{e}_t \ \mathbf{e}_n \ \mathbf{e}_{g,z}]^T \) will be used. Its basis moves along a path characterized by the path coordinate \( s \) and the angle \( \theta(s) \) between the tangent at \( s \) and the basis vector \( \mathbf{e}_{g,x} \) of the global reference frame. Positions are then given by the path coordinate \( s \) and an offset \( n \) normal to the path at \( s \). A relation between \( \mathbf{B}_{tn} \) and \( \mathbf{B}_g \) can be established based on: 
\[ \frac{dp_x}{ds} = d \cos \theta(s) - n(s) \sin \theta(s), \]
\[ \frac{dp_y}{ds} = d \sin \theta(s) + n(s) \cos \theta(s), \]
(8)
(9)
which leads to:
\[ p_x(s) = \int_{s_{\min}}^s \cos \theta(\sigma) d\sigma + p_x(s_{\min}) - n(s) \sin \theta(s), \]
\[ p_y(s) = \int_{s_{\min}}^s \sin \theta(\sigma) d\sigma + p_y(s_{\min}) + n(s) \cos \theta(s). \]
(10)
(11)
Defining the curvature \( C = \frac{d\theta}{ds} \), the tangent angle is:
\[ \theta(s) = \int_{s_{\min}}^s C(\sigma) d\sigma + \theta(s_{\min}). \]
(12)
Thus, knowledge of the trajectories \( C(s) \) and \( n(s) \) enables one to transform every point given in the moving reference frame into the global reference frame, in which \( n(s) = 0 \) gives the shape of the path itself. A trajectory \( s(t) \) then determines trajectories over time for all other quantities, e.g. \( C = C(s(t)) \). The benefit from using this reference frame is that constraints resulting from the road topology can be accounted for conveniently.

Body-fixed Coordinate System The third considered coordinate system has the basis \( \mathbf{B}_b = [\mathbf{e}_{b,x} \ \mathbf{e}_{b,y} \ \mathbf{e}_{b,z}]^T \), which is fixed at the center of gravity of a considered vehicle. The basis vectors \( \mathbf{e}_{b,x} \) and \( \mathbf{e}_{b,y} \) span the road plane and \( \mathbf{e}_{b,x} \) is aligned with the longitudinal axis of the vehicle. A vector in this coordinate system can be expressed in the Frenet frame:
Fig. 1. Coordinate Systems.

\[ v^T B_b = v^T \begin{bmatrix} \cos(\xi) & \sin(\xi) & 0 \\ -\sin(\xi) & \cos(\xi) & 0 \\ 0 & 0 & 1 \end{bmatrix} B_{tn}. \] (13)

The body-fixed frame makes it possible to easily express constraints resulting from the vehicle dynamics.

### 2.2 Kinematics

In order to describe the motion of a vehicle in one of the coordinate systems, information about its position, velocity, and acceleration is required — this is what kinematics is concerned with. A vector \( r \) can be expressed combining the global and the Frenet frame (cf. Fig. 1):

\[ r = r_{tn} + r_b = r_{tn}^T B_g + r_b^T B_{tn}, \] (14)

in which \( r_{tn} \) indicates the position of the origin of the basis \( B_{tn} \) and \( r_b \) the connection from there on to the position of the vehicle. If \( r = r(t) \), differentiating (14) gives the velocity vector:

\[ \dot{r} = \dot{r}_{tn} + \dot{r}_b. \] (15)

Alternatively, the velocity vector can be described conveniently in the body-fixed reference frame:

\[ \dot{r} = v^T B_h, \] (16)

which is rotated against the Frenet frame by the angle \( \xi = \psi - \theta \). The latter indicates the heading of the vehicle as measured in the Frenet frame, and \( v = [v_x \, v_y \, 0]^T \). Next, we have that:

\[ \dot{r}_{tn} = v_r^T B_g = [\dot{s} \, 0 \, 0] B_g \] (17)

and, with \( \dot{\theta} \) as angular velocity of \( B_{tn} \):

\[ \dot{r}_b = \dot{r}_{tn}^T B_h + (\dot{\theta} \times r_b)^T B_{tn} = [0 \, \dot{n} \, 0] B_{tn}^+. \] (18)

From (15) and (16), it follows that:

\[ \dot{r} = \dot{r}_{tn} + \dot{r}_b = v^T B_h, \] (19)

and using \( \dot{\theta} = \frac{d\theta}{dt} \dot{s} = C(s) \dot{s} \) and (13), we have (cf. Perantoni and Limebeer (2014)):

\[ \begin{bmatrix} \dot{s} \left(1 - nC(s)\right) & v_x \cos(\xi) - v_y \sin(\xi) \\ \dot{n} & v_x \sin(\xi) + v_y \cos(\xi) \end{bmatrix} B_{tn} = \begin{bmatrix} \dot{s} \dot{n} \theta s \theta & 0 \\ 0 & 0 \end{bmatrix} B_{tn}. \] (20)

Differentiating (19) gives:

\[ \ddot{r} = \ddot{r}_{tn} + \ddot{r}_b = \ddot{v}^T B_h + (\ddot{\psi} \times \dot{r}_b)^T B_h, \] (21)

in which \( \dot{\psi} \) is the angular velocity of \( B_h \), and:

\[ \ddot{r}_{tn} = \begin{bmatrix} \dot{s} & 0 & 0 \\ 0 & \dot{n} & 0 \\ 0 & 0 & 0 \end{bmatrix} B_{tn}^+, \]

\[ \ddot{r}_b = \begin{bmatrix} -\dot{n} \dot{\theta} - \dot{n} \dot{\theta} & \dot{s} & \dot{s} \\ \dot{n} & \dot{n} \dot{\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} B_{tn}^+, \]

such that \( \ddot{r} = \begin{bmatrix} \dot{s} - 2\dot{n} \dot{\theta} - \dot{n} \dot{\theta}^2 & \dot{s} + \dot{n} \dot{\theta}^2 & 0 \end{bmatrix} B_{tn} \), with:

\[ C(s) = \frac{dC(s)}{ds} = \frac{dC(s)}{ds} \dot{s} = C'(s) \dot{s}, \]

it shows that \( \dot{\theta} = \frac{d\theta}{ds} = C'(s) \dot{s}^2 + C(s) \dot{s} \), such that

\[ \ddot{r} = \begin{bmatrix} \dot{s} (1 - nC(s)) - 2nC(s) \dot{s} - nC'(s) \dot{s}^2 + C(s) \dot{s} & \dot{s} + \dot{n} \dot{\theta}^2 \end{bmatrix} B_{tn} = \begin{bmatrix} a_{x,tn} & a_{y,tn} \end{bmatrix} B_{tn}, \] (22)

with the external accelerations \( a_{x,tn} \) and \( a_{y,tn} \) in the Frenet frame. The right hand side of (21) reads:

\[ \ddot{r} = \begin{bmatrix} \dot{v}_x & \dot{v}_y & 0 \\ 0 & 0 & 0 \end{bmatrix} B_h = \begin{bmatrix} a_{x,b} \dot{v}_y & a_{y,b} \dot{v}_y \end{bmatrix} B_h = \begin{bmatrix} a_{x,b} & a_{y,b} \end{bmatrix} B_h, \]

in which \( a_{x,b} \) and \( a_{y,b} \) are the accelerations in the body-fixed reference frame. For planning purposes, a state-space model is required. By introducing the yaw acceleration \( a_\psi \), the state vector \( x_b = [p_x \, p_y \, \psi \, v_x \, v_y \, \dot{\psi}]^T \), and \( a_b = [a_{x,b} \, a_{y,b} \, a_\dot{\psi}]^T \), the model reads:

\[ \dot{x}_b = f_b(x_b, a_b, a_\dot{\psi}) = \begin{bmatrix} v_x \cos(\psi) - v_y \sin(\psi) \\ v_x \sin(\psi) + v_y \cos(\psi) \\ \dot{\psi} + a_{\dot{\psi}} \dot{\psi} \end{bmatrix} \]

(23)

in the body-fixed reference frame. Note that the accelerations are independent of the position and the orientation. Alternatively, due to its ability to conveniently handle constraints arising from the road topology, a state-space model can be derived from the kinematics (22) in the Frenet frame:

\[ \dot{x}_{tn} = f_{tn}(x_{tn}, a_{x,tn}, a_{y,tn}, a_\psi) = \begin{bmatrix} \dot{s} \\ \dot{n} \\ \dot{\psi} - C(s) \dot{s} \\ a_{x,tn} + 2nC(s) \dot{s} - nC'(s) \dot{s}^2 \\ a_{y,tn} - C(s) \dot{s}^2 (1 - nC(s)) \\ a_{\dot{\psi}} \end{bmatrix} \]

(24)

with the state vector: \( x_{tn} = [s \, n \, \xi \, s \, \dot{s} \, \psi] \) and \( a_{tn} = [a_{x,tn} \, a_{y,tn}]^T \).

### 3. DERIVATION OF AN INTEGRATOR MODEL

#### 3.1 Equations of Motion

The basis of planning in Frenet coordinates is the kinematic equation (24). In this section, we show that a simple model of the form (4) can be derived from (24) by
appropriate assumptions. Kinematic equations describe a vehicle’s motion without considering the forces causing it and generally hold for all vehicles. Kinetics, in contrast, provide a model of these forces or external accelerations, typically expressed in the body-fixed reference frame.

Usually, the acceleration depends on both the vehicle’s velocities (e.g. due to wind resistance) and its inputs (e.g. steering and throttle commands), while we assume independence of the position and orientation. Clearly, this relation may differ from vehicle to vehicle. For the model class at hand, it is assumed that the external accelerations $a_b$ acting on the vehicle in the body-fixed reference frame can be directly controlled, i.e.:

$$ u = \left[ a_{x,b} \ a_{y,b} \ a_{\phi} \right]^T. $$

(25)

This assumes that questions concerning tire dynamics or air resistance have been accounted for, e.g. by lower-level feedback controllers and proper choice of constraints on $a_b$.

Another crucial assumption concerns the orientation $\xi$ relative to the road. In the case of integrator models, this is assumed to be always aligned with the orientation of the road, i.e.:

$$ \xi = \psi - \theta = 0. $$(26)

This makes the considered model less a vehicle model, but a model of a road-aligned box which may be occupied by a vehicle at a certain point in time. According to (13), this equates the accelerations in the body-fixed frame and the Frenet frame, i.e.:

$$ a_b = a_{tn}, $$

and fixes the rotational velocity: $\psi = \dot{\theta} = C(s) \dot{s}$ as well as rotational acceleration:

$$ \ddot{\psi} = a_{\phi} = C'(s) \dot{s}^2 + C(s) \ddot{s}, $$

(28)

eliminating $\xi$ and $\dot{\psi}$ as a states from (24), leaving $x_1 := \left[ s \ n \ \dot{s} \right]^T$. Despite these simplifying assumptions, the resulting equations of motion are still nonlinear:

$$ \dot{x}_1 := f_1(x_1, a_{tn}) = \begin{bmatrix} \dot{s} \\ \dot{n} \\ a_{x,tn} - C(s) \dot{s} \dot{n} \\ a_{y,tn} - C(s) \ddot{s} \dot{n} \end{bmatrix}. $$

(29)

Further simplifications can be obtained by noting that (24) is affine in the accelerations $a_{x,tn}$ and $a_{y,tn}$, while the states are arranged in an integrator chain. Therefore, we introduce artificial system inputs similar to feedback linearization of input affine systems (Khalil, 1996):

$$ \ddot{u} := \begin{bmatrix} u_t \\ u_n \end{bmatrix} = \begin{bmatrix} a_{x,tn} + 2n C(s) + n C'(s) \dot{s}^2 \\ a_{y,tn} - C(s) \ddot{s} \dot{n} \end{bmatrix}, $$

(30)

such that (29) becomes:

$$ \dot{x}_2 := f_2(x_2, \ddot{u}) = \left[ \ddot{s} \ \dot{n} \ u_t \ u_n \right]^T, $$

(31)

where $x_2 \equiv x_1$. The similarity to (4) is obvious.

### 3.2 Constraints

A model often not only consists of equations of motion, but also of constraints. Even though planning requires all constraints to be defined in the Frenet frame, some may initially be defined in different coordinate systems and have to be transformed. This will be addressed in this section.

Constraints on a vehicle’s velocity are typically imposed in the body-fixed coordinate system as interval bounds:

$$ \begin{bmatrix} v_{x \text{min}} \\ v_{y \text{min}} \end{bmatrix} \leq \begin{bmatrix} v_x \\ v_y \end{bmatrix} \leq \begin{bmatrix} v_{x \text{max}} \\ v_{y \text{max}} \end{bmatrix}. $$

(32)

The transformation of (32) from the body-fixed frame to the Frenet frame results according to (13), which, considering that $\xi = 0$ as per (26), leads to:

$$ \begin{bmatrix} v_{x \text{min}} \\ v_{y \text{min}} \end{bmatrix} \leq \begin{bmatrix} \hat{s}(1 + n C(s)) \\ \hat{n} \end{bmatrix} \leq \begin{bmatrix} v_{x \text{max}} \\ v_{y \text{max}} \end{bmatrix}. $$

(33)

State constraints resulting from the road topology are typically expressed in the Frenet frame. Combining these with the transformed velocity constraints (33) yields the state constraint set:

$$ \mathcal{X}_{tn} := \left\{ x_{tn} \mid C_{\text{min}} \leq C(s) \leq C_{\text{max}}, \ 0 \leq \hat{s} \leq \hat{s}_{\text{max}}(s), \ n_{\text{min}}(s) \leq n \leq n_{\text{max}}(s), \ \xi_{\text{min}} \leq \xi \leq \xi_{\text{max}}, \right\}. $$

(34)

The interval bounds on $n$ and $\dot{s}$ depend on $s$ in order to account for changes in the road width or speed limits. Note that requirements from collision avoidance with other vehicles are not in the focus of this paper. They can be formulated in either the Frenet frame or the global coordinate system without affecting any of the subsequent derivations.

The accelerations are coupled and constrained by the so-called friction ellipsoid (Pacejka, 2005):

$$ a_{x,b}^2 + a_{y,b}^2 \leq \text{const}. $$

(35)

In addition, each acceleration is confined to an interval:

$$ a_{b,\text{min}} \leq a_b \leq a_{b,\text{max}}. $$

(36)

By combining (35) and (36) and by transforming the accelerations from the body-fixed to the Frenet frame according to (27), the following set of constraints on the accelerations results:

$$ \mathcal{U}_{tn} := \left\{ a_{tn} \left| a_{x,tn}^2 + a_{y,tn}^2 \leq \text{const}, \ a_{b,\text{min}} \leq a_{tn} \leq a_{b,\text{max}} \right\}. $$

(37)

The planning model (31) is not based on the actual accelerations, but on artificial inputs $\ddot{u}$, for which a constraint set must be specified. According to (30), $\ddot{u}$ depends on both $a_{tn}$ and $x_{tn}$. Therefore, it must be chosen such that both (34) and (37) are not violated. Solving (30) for $a_{tn}$ yields:

$$ g(x_{tn}, \ddot{u}) = \begin{bmatrix} (1 - n C(s)) \dot{n} - (2n C(s) \dot{s} + n C'(s) \dot{s}^2) \\ u_n + C(s) \dot{s}^2 \cdot (1 - n C(s)) \end{bmatrix}. $$

(38)

Note that inserting (38) into (37) leads to nonlinear, mixed state-input constraints. Collecting (34) and (37) (combined with (38)) then yields the constraints:

$$ \mathcal{Z} = \left\{ x_{tn} \mid C_{\text{min}} \leq C(s) \leq C_{\text{max}}, \ n_{\text{min}} \leq n \leq n_{\text{max}}, \ v_{x,\text{min}} \leq \dot{s}(1 + n C(s)) \leq v_{x,\text{max}}, \ \dot{\psi}_{\text{min}} \leq C(s) \dot{s} \leq \dot{\psi}_{\text{max}}, \ a_{x,b,\text{min}} \leq C(s) \dot{s}^2 + C(s) u_n \leq a_{x,b,\text{max}}, \ a_{b,\text{min}} \leq g(x_{tn}, \ddot{u}) \leq a_{b,\text{max}} \cdot ||g(x_{tn}, \ddot{u})||^2 \leq \text{const} \right\}. $$

(39)
3.3 Constraint Simplification

Clearly, while (24) has been substantially simplified by introduction of the artificial inputs $\hat{u}$, the new constraint (39) is much more complex than the original constraints on the accelerations (37). To be precise, (39) is defined by level sets of nonlinear functions, such that it generally can be non-convex or even non-connected. In general, these properties cannot even be checked efficiently (Weber and Reissig, 2014). All the more, (39) depends on the longitudinal position $s$, which introduces further nonlinearities into the planning. For computationally efficient planning, in contrast, simple, if possible polytopic constraints are required. Thus, instead of using (39) directly for planning, this section is concerned with the following problem:

**Problem 2.** Find a polytopic inner approximation to (39):

$$Z = \left\{ \begin{bmatrix} x_{tn} \\ \hat{u} \end{bmatrix} \big| N \begin{bmatrix} x_{tn} \\ \hat{u} \end{bmatrix} \leq b \right\} \subseteq Z$$

by appropriate choice of a matrix $N$ and a vector $b$.

In a first step, we obtain a polytopic inner approximation for the convex, but nonlinear constraint (35). For simplicity, we employ a rectangle, which decouples the interdependence of $a_{x,b}$ and $a_{y,b}$. Thus, the input constraints essentially reduce to the type (39) with appropriately tightened bounds $a_{p,\min}$ and $a_{b,\max}$.

In a next step, we aim to resolve the position dependence of (39). This is achieved by elimination of $s, C(s), n$ and $\hat{u}$ in order to further reduce the dimensionality of the resulting set) from (39) through computation of a set:

$$\tilde{Z} = \left\{ \begin{bmatrix} \hat{s} \\ u_t \\ \hat{u} \end{bmatrix} \big| s, C(s) \in Z \forall \begin{bmatrix} s \\ n \\ \hat{u} \end{bmatrix} \in \begin{bmatrix} s_{\min}, s_{\max} \\ C_{\min}, C_{\max} \\ n_{\min}, n_{\max} \end{bmatrix} \right\}.$$  

Here, a typical road topology allows to assume $C(s)$ to be linear or constant, such that $C'(s)$ is constant or zero, i.e., not a variable anymore. The set $\tilde{Z}$ is an inner approximation of the projection of $Z$ and can be determined by elimination of the quantifier $\forall$ by using algorithms as implemented in Mathematica (Wolfram, 1991) or Maple (Heck, 1993). Note that this is in general a computationally very demanding procedure. In the given case, however, it turns out to be tractable.

The resulting set has been reduced to three dimensions, namely $\hat{s}, u_t$, and $\hat{u}$. This allows to gain insight into its shape, for example to assess connectivity by visual inspection. If the set is convex, but not polytopic, standard algorithms as implemented in Mathematica (Wolfram, 1991) or Maple (Heck, 1993) can be readily employed. Otherwise, for arbitrarily complex, but simply connected sets, the procedure from Xue et al. (2016) can be used if an automated approach is required. However, as we demonstrate in Sec. 4, the sets resulting for typical scenarios are relatively simple even in the non-convex case, such that human intuition quickly leads to sufficiently accurate approximations. The set $\tilde{Z}$ can then be obtained as follows:

$$Z = \tilde{Z} \times [s_{\min}, s_{\max}] \times [n_{\min}, n_{\max}] \times [v_{y,\min}, v_{y,\max}].$$

3.4 Resulting Models

In the previous sections, we have derived several equations of motion and corresponding sets of (mixed) states and input constraints. Different combinations of these lead to different planning models:

Combining the nonlinear equations of motion (29) with the input and state constraints (34) and (37) gives:

$$M_1 = (f_1, X_{tn} \times U_{tn}).$$

The second model is obtained by combination of the simplified, linear dynamics (31) with the nonlinear, mixed state-input constraints (39):

$$M_2 = (f_2, Z).$$

Combining the same dynamics with the linearized constraints (40) gives the third model:

$$M_3 = (f_2, Z).$$

For comparison purposes, we also introduce a fourth model, combining the linear dynamics (31) with the original input and state constraints (34) and (37):

$$M_4 = (f_2, X_{tn} \times U_{tn}).$$

This model clearly is naive in so far as that it ignores the constraint coupling resulting from (38). Based on these models, our main result is summarized as follows:

**Proposition 1.** Planning with the linear model $M_3$ generates trajectories which are feasible for the nonlinear model $M_1$. More formally, given the constraint set $Z$ from (45) and the original constraint sets $X_{tn}$ and $U_{tn}$ from (43),

$$[x_{tn}, \hat{u}]^T \in Z \Rightarrow x_{tn} \in X_{tn} \land \exists a_{tn} \in U_{tn} : a_{tn} \text{ and } \hat{u} \text{ fulfill } (43).$$

**Proof.** Due to (39), $[x_{tn}, \hat{u}]^T \in Z \Rightarrow [x_{tn}, \hat{u}]^T \in Z$. Since $Z$ is defined as the conjunction of (34) and (37) (mapped to $\hat{u}$), the result is immediate. □

3.5 Strategies to Reduce Conservatism

Clearly, the procedure described in Sec. 3.3 introduces conservatism: first by imposing conditions for all possible values of certain variables, and second by inner approximating the resulting set. While finer inner approximations obviously lead to less conservatism, a remedy to the first issue is less obvious. We propose to partition roads into segments of a certain length, in which the ranges of $C$ and $n$ are small enough to introduce only little conservatism when eliminating these variables. Also, a constant value for $C'$ is used. This introduces again position dependence, but on a coarser scale, constituting a good compromise between computational tractability and conservatism. In addition, maneuver-dependent bounds on $\hat{u}$ can be used, e.g. when the vehicle simply has to follow a lane or has to overtake.

4. EXAMPLE

We demonstrate the efficacy of our approach by planning trajectories for a vehicle driving on the road depicted in Fig. 2. The track consists of two circles connected by so-called Euler spirals (road segments of linearly varying curvature) – two common shapes in road design. Fig. 3a shows the curvature values, which represent a relatively
Based on these constraints, for each segment, a set $\mathcal{E}^*$ is computed according to Sec. 3.3. Projections of the resulting sets on the $s$ - $u_1$ and $s$ - $u_2$ - space are shown in Fig. 4, in which dotted lines denote the inner approximations (where none are visible, these coincide with the bounds of the original set). Note that the inner approximation can be obtained by inner-approximating the projections of the set because the accelerations are independent of each other. The obtained constraints $\mathcal{E}^*$ are then used to define model $M_3$ in each segment. Depending on the segment in which the vehicle is predicted to be located in, different constraints are activated, which is implemented using binary variables and the Big-M-method, cf. (Williams, 1990). In the following, the equations of motion and constraints of two different models from Sec. 3.4 will be used when solving Problem 1: Model $M_3$, which is the result of the proposed method, and model $M_4$, which represents a naive approach to the planning problem.

### 4.1 Scenario 1

We first demonstrate that solving Problem 1 based on the proposed method, i.e., model $M_3$, allows excessive lateral motion on curved roads without violating constraints, while the naive model $M_4$ leads to immediate failure. To that end, we impose a sinusoidal reference trajectory for the lateral motion of the vehicle while traversing the northern circle. In order to follow that motion, lateral inputs as shown in Fig. 5 are applied, which correspond to lateral accelerations as given in Fig. 6. Clearly, the inputs as determined by both methods are admissible. However, in the naive approach, the resulting accelerations violate the imposed constraints as indicated by the gray lines. The proposed method, in contrast, yields admissible accelerations due to the employed state-dependent input constraints. Certainly, admissible accelerations could also be obtained based on the naive model $M_4$ by introducing speed limits. However, as the next scenario shows, constant speed limits may not be sufficient to guarantee constraint compliance.

### 4.2 Scenario 2

Secondly, we focus on the aspect of varying curvature, while the vehicle keeps a constant lateral position during its journey along the track in Fig. 2. As the previous section has demonstrated, failure of the naive approach on roads with high curvature is foreseeable if no speed limits are present. Therefore, in order to give the naive approach a chance, the maximal admissible longitudinal velocity is reduced to 70 km h$^{-1}$ in the smaller circle of the track and in the first segments preceding and following it. For the proposed approach, it is not necessary to enforce this constraint, because one can rely on its ability to automatically choose admissible speeds.

The longitudinal input trajectories resulting from both planning procedures are given in Fig. 7. Use of both models leads to feasible inputs, i.e., the input constraints are not violated, which are constant in the case of the naive planner and vary with the speed in the proposed method. The longitudinal and lateral accelerations resulting for the vehicle from these inputs are given in Fig. 8. Despite overall similar appearance, subtle but crucial differences demonstrate the value of our approach: Some longitudinal accelerations are not admissible in the naive case (cf. markings). The proposed method, in contrast, automatically limits the longitudinal inputs as shown in Fig. 7 at the critical instances, leading to admissible accelerations.
This is not viable for autonomous vehicles, because traveling at a prescribed speed does not operate the vehicle at the bounds of the admissible region. This is not sufficient for autonomous vehicles, whose speed limits are the standard mean to prevent dynamically infeasible situations, this is not sufficient for autonomous vehicles, whose accelerations by tuning of its cost function or by introducing additional constraints (such as further reducing the maximal speed). While in human driving, or by introducing additional constraints (such as further reducing the maximal speed). While in human driving, the acceleration constraints in both scenarios, it can be noted that the degree of violation is relatively small. Moreover, one could argue that the acceleration constraints in both scenarios, it can be noted that the degree of violation is relatively small. Moreover, one could argue that the degree of violation is relatively small. Moreover, one could argue that the degree of violation is relatively small.

5. CONCLUSION

While planning based on the naive model \( \mathcal{M}_4 \) violates the acceleration constraints in both scenarios, it can be noted that the degree of violation is relatively small. Moreover, one could argue that \( \mathcal{M}_4 \) could certainly return admissible accelerations by tuning of its cost function or by introducing additional constraints (such as further reducing the maximal speed). While in human driving, speed limits are the standard mean to prevent dynamically infeasible situations, this is not sufficient for autonomous vehicles, because traveling at a prescribed speed does not imply compliance with dynamical constraints, which also depends on the other states of the vehicle and the chosen inputs. In order to select these inputs properly, a human driver relies on experience and typically tries to not operate the vehicle at the bounds of the admissible region. This is not viable for autonomous vehicles, whose...
planning algorithms do not have human experience. Also, an optimization-based planner will often find an optimum on the boundary of a given admissible set, not leading to cautious driving. Therefore, the tuning effort may be significant.

In addition, tuning goes along with significant testing effort in order to reduce the likelihood of failure, without being able to ever completely exclude it. Our method, in contrast, is able to guarantee compliance with the original constraints based on a theoretically sound approach. While it is conservative, constraints tailored to a certain vehicle can still lead to less conservative behavior than simple speed limits imposed on a road segment, which do not account for the dynamics of a certain vehicle. This becomes clear when considering the velocity trajectories resulting from both planning procedures as given in Fig. 9: while the naïve planner reduces the vehicle’s longitudinal velocity according to the speed limits, the planner based on the proposed method does not have to account for a global speed limit. Thus, it only reduces the vehicle’s speed based on its speed-dependent input constraints, leading to overall less reduction, while still complying with the acceleration constraints (unlike the more conservative naïve approach).

Fig. 9. Resulting velocities in the body-fixed frame (dashed blue: naïve approach; solid red: proposed method).

REFERENCES


