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A Simplified Implementation of Tube-Enhanced Multi-Stage NMPC

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Abstract: In a previous work, multi-stage NMPC and tube-based NMPC schemes were combined into a single framework called tube-enhanced multi-stage NMPC with the goal of achieving an improved trade-off between simplicity and performance. In tube-enhanced multi-stage NMPC, the large uncertainties are handled using a multi-stage primary controller and the small uncertainties are handled using a multi-stage ancillary controller that tracks the predictions of the primary controller. In this work, we propose the replacement of the multi-stage ancillary controller by a single scenario NMPC that tracks the predicted trajectories of one of the scenarios of the multi-stage primary controller. The scenario that will be tracked by the ancillary controller as well as the ancillary controller model are time varying and are adapted to the current plant dynamics. The benefits of the new formulation are demonstrated on the benchmark Williams-Otto Continuous Stirred Tank Reactor (CSTR) example.

Keywords: Nonlinear Model-predictive control, Robust NMPC, Multi-stage NMPC, Process control

1. INTRODUCTION

Robust Model Predictive Control (NMPC) approaches were introduced to deal with the presence of uncertainties in the available process models. Min-max MPC tackles the presence of uncertainties by minimizing the worst-case value of the objective while satisfying the constraints for all realizations of the uncertainty (Campo and Morari, 1987). However, min-max MPC is conservative because the presence of feedback information in the predictions is not considered in the problem formulation. This can easily lead to situations where the constraints cannot be satisfied for all possible uncertainties. To reduce the conservatism, feedback min-max MPC was proposed for linear systems in (Scokaert and Mayne, 1998) where the decision problem was formulated by considering the feedback information in the predictions while minimizing the worst-case objective. Several formulations for feedback min-max NMPC were proposed such as in (Fontes and Magni, 2003) and (Limon et al., 2006). However, the proposed formulations result in optimization problems which are very difficult to solve.

Multi-stage MPC was proposed in (Muñoz de la Peña et al., 2005) for the linear case. For the nonlinear case it was initially proposed by (Dadhe and Engell, 2008) and elaborated in (Lucia and Engell, 2012) and (Lucia et al., 2013). It provides a computationally tractable robust scheme that considers a scenario tree for the future evolution of the uncertainty and reflects the presence of feedback information explicitly by formulating the optimization problem as a multi-stage problem where the future inputs constitute the recourse variables by which the reaction to the future information is incorporated. As shown in Figure 1, the branches represent the different realizations of the uncertainties that give rise to different state predictions which are the tree nodes. An inherent drawback of the multi-stage NMPC approach is that the size of the scenario tree and consequently the optimization problem grows rapidly with the number of uncertainties considered and with the prediction horizon. The restriction of the scenario tree to the so-called robust horizon N_R after which the uncertainties are assumed to be constant alleviates this problem to some extent. However, the problem size can still be large despite the use of a small robust horizon, if many uncertainties are considered in the scenario tree.

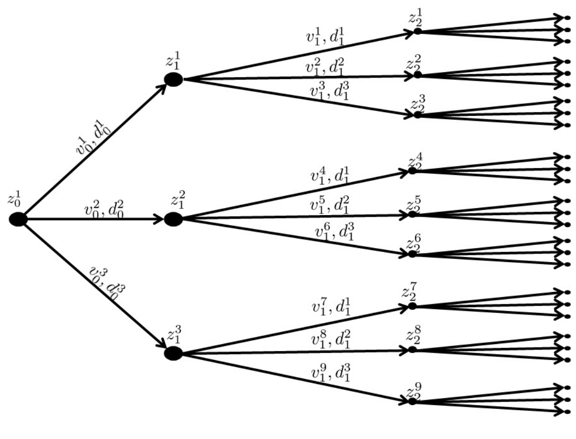

Fig. 1. Scenario tree representation for multi-stage NMPC.

Another family of robust NMPC schemes, is tube-based NMPC. Tube-based NMPC mimics the pragmatic engi-

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neering approach of lower-layer controllers by which the nominal trajectory is followed despite the uncertainty in the behaviour of the plant. Different formulations of tube-based NMPC schemes were developed in (Cannon et al., 2011), (Mayne et al., 2011), (Yu et al., 2013) and (Villanueva et al., 2017). In (Mayne et al., 2011), an NMPC is employed as a primary controller with tightened constraints to minimize a given objective and an ancillary controller is employed to keep the system trajectories close to the predicted nominal trajectories such that robust constraint satisfaction is achieved. A key feature of this approach is that the problem complexity remains close to that of nominal NMPC. However, since only the nominal model is used, the tube-based NMPC can be conservative for large uncertainties.

Recently in (Subramanian et al., 2018), multi-stage NMPC and tube-based NMPC were combined into a single framework called tube-enhanced multi-stage NMPC. Tube-enhanced multi-stage NMPC uses two controllers similar to the tube-based NMPC scheme proposed in (Mayne et al., 2011). However, unlike tube-based NMPC, robust multi-stage NMPC controllers are employed as primary and ancillary controllers. The uncertainties with significant effect are handled using the multi-stage NMPC framework and the small disturbances and state estimation errors are handled using the tube-based NMPC concept, where a multi-stage ancillary controller was employed to track the predicted trajectories of the primary controller for all the realizations of the uncertainty that are considered in the primary controller. As a result, the growth in problem size due to the small uncertainties is eliminated with small or even insignificant loss of performance.

In this paper, a simplified variant of tube-enhanced multi-stage NMPC is proposed and evaluated. We propose to use a single scenario NMPC ancillary controller to track one of the predicted trajectories of the multi-stage primary controller at each time step. The scenario (reference trajectory) that will be tracked by the ancillary controller and the ancillary controller model can change according to the plant measurements or states estimates. This results in a significant reduction of the computational effort while maintaining the robustness properties of the original scheme. We demonstrate how stability of the system controlled by the proposed scheme can be established from the results in the robust NMPC literature. The capabilities of the proposed controller are demonstrated using the benchmark Williams-Otto CSTR.

2. SYSTEM UNDER CONSIDERATION

We consider a discrete time uncertain nonlinear system described by:

$$x_{t+1} = f(x_t, u_t, d_t) + w_t, \tag{1a}$$
$$y_t = h(x_t, u_t) + \delta_t \tag{1b}$$

where $x_t \in \mathbb{X} \subset \mathbb{R}^{n_x}$, $u_t \in \mathbb{U} \subset \mathbb{R}^{n_u}$ are the state and input vectors respectively, $\mathbb{X}$ and $\mathbb{U}$ are the state and input constraints, $d_t \in \mathbb{D} \subset \mathbb{R}^{n_d}$ denotes the parametric uncertainties, $w_t \in \mathbb{W} \subset \mathbb{R}^{n_x}$ is the vector of additive disturbances, $y_t \in \mathbb{R}^{n_y}$ is the output vector, $\delta_t \in \Delta \subset \mathbb{R}^{n_y}$ is the measurement noise. The sets $\mathbb{X}$, $\mathbb{U}$, $\mathbb{D}$, $\mathbb{W}$ and Δ are compact. The nonlinear map $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_x}$ represents the system dynamics, and the system output is described by (1b).

An estimator is assumed to be employed to estimate the states from the available model and the measurement information. The initial state estimation error $e_0 = x_0 - \hat{x}_0$ is assumed to lie in the compact set $\mathbb{E}_0$, and subsequent estimation errors $e_t = x_t - \hat{x}_t$ lie in the compact sets $\mathbb{E}_t$, $\forall t > 0$, where $\hat{x}_t$ is the state estimate at time step t. We assume that the parametric uncertainties d_t have a significant influence on the evolution of the states, while the effects of w_t and e_t are assumed to be relatively small.

We want to control the system described by (1) either to track a reference or to achieve the best possible economic objective along with state and input constraints satisfaction.

3. TUBE-ENHANCED MULTI-STAGE NMPC

The subscript t denotes the current time step, and with some abuse of notation the subscript k will denote future predictions from the current time step t, and hence it is implicit that x_k means the predicted state at the future time step $t + k$ while x_t means the current state.

Tube-enhanced multi-stage NMPC (Subramanian et al., 2018) employs two controllers, a multi-stage primary NMPC which optimizes the original objective and takes into account the future evolution of the large uncertainties (assumed to be the parametric uncertainties d_k) at each time step and a multi-stage ancillary NMPC that tracks the optimal predictions of the primary controller for the considered discrete realizations of the parametric uncertainties to counteract the effects of the small uncertainties (assumed to be the additive disturbances and state estimation errors). Hence it inherits properties from both the tube-based and the multi-stage NMPC schemes.

By considering only the large uncertainties in the primary controller formulation, the growth in problem size with respect to small uncertainties is eliminated. An ancillary controller of the same tree structure as the primary controller is employed to track the predictions of the primary controller.

In order to have a primary system state z that is affected only by the parametric uncertainties, the primary controller is initialized at each time step t with the one step ahead predicted tree node (predicted at the previous time step $t - 1$ by the primary controller) which has the minimum Euclidean distance to the current state estimate $\hat{x}_t$. The ancillary controller is initialized at each time step t with the current state estimate $\hat{x}_t$.

The primary system state and input vectors are denoted by z and v respectively, hence the evolution of the state of the model used in the primary controller is given by:

$$z_{t+1} = f(z_t, v_t, d_t). \tag{2}$$

The evolution of the state of the plant is given by:

$$x_{t+1} = f(x_t, u_t, d_t) + w_t, \tag{3}$$

which is considered to be unknown (due to state estimation errors) but lies in a compact set centered at $\hat{x}_t$.

4. SIMPLIFIED TUBE-ENHANCED MULTI-STAGE NMPC

The idea proposed in this paper is to replace the multi-stage ancillary NMPC by a single scenario ancillary NMPC that employs only one plant model at each time step. The task of the ancillary controller is to track the predicted input and state trajectories of one of the scenarios of the primary controller which has the same model as the ancillary controller. This is motivated by the fact that plant measurements when used with (uncertain) process models provide some information about the uncertainty that has realized in the plant (in the previous time step) and the fact that parametric uncertainties d are usually slowly varying.

The primary controller is the same as the one proposed in (Subramanian et al., 2018). At each time-step t, the optimization problem of the primary controller is formulated as follows:

$$\min_{z_{k+1}^j, v_k^j \ \forall (j,k)\in I} \bar{V}_N(z_t) \tag{4a}$$

subject to

$$z_{k+1}^j = f(z_k^{p(j)}, v_k^j, d_k^{r(j)}), \quad \forall (j, k+1) \in I, \tag{4b}$$

$$z_{k+1}^j \in \mathbb{Z}, \qquad \forall (j, k+1) \in I, \tag{4c}$$

$$v_k^j \in \mathbb{V}, \qquad \forall (j, k) \in I, \tag{4d}$$

$$v_k^j = v_k^l, \text{if } z_k^{p(j)} = z_k^{p(l)}, \ \forall (j,k), (l,k) \in I, \tag{4e}$$

$$\bar{V}_N(z_t) = \sum_{i=1}^{N_s} \omega_i \tilde{V}_i, \tag{5}$$

where I denotes the set of indices of the scenario-tree, N denotes the prediction horizon, N_s denotes the number of scenarios which depends on the considered discrete realizations of the uncertainties and the robust horizon, ω_i is the respective scenario weight, z_{k+1}^j is the predicted primary system state which is determined by (4b) and depends on its parent primary system state $z_k^{p(j)}$, the primary system input v_k^j and the realization of the uncertainty $d_k^{r(j)}$. The scenario cost $\tilde{V}_i = \sum_{k=0}^{N-1} \ell(z_{k+1}^j, v_k^j)$, $\forall z_{k+1}^j$, v_k^j in scenario i. The constraints on the primary system states and inputs are defined by (4c) and (4d), where $\mathbb{Z} \subseteq \mathbb{X}$ and $\mathbb{V} \subseteq \mathbb{U}$. To represent causality in the decision problem formulation, decisions based on the same information must be the same. This is enforced by the non-anticipativity constraint (4e). The scenario tree of the primary controller considers s possibilities for the uncertainty realizations. Hence, $d_k^{r(j)} \in \mathbb{D}^d \subseteq \mathbb{D}$, where $\mathbb{D}^d = \{d_1, d_2, \cdots, d_s\}$. The optimal predicted inputs and states of the primary control problem are denoted by v_k^{*j} and z_k^{*j} respectively, $\forall (j,k) \in I$.

At $t = 0$, the primary controller is initialized with the initial state estimate $\hat{x}_0$. For all $t \geq 1$, the discrete set $Z_r(t-1)$ defined as $Z_r(t-1) \hat{=} \{z_1^{*j} | (j,1) \in I\}$ defines the optimal one step ahead predicted primary states at the previous time step $t-1$. At each time step $\forall t \geq 1$, the primary system state in $Z_r(t-1)$ with the minimum Euclidean distance from the current state estimate $\hat{x}_t$ will act as the initial primary tree node. The ancillary controller is always initialized with the state estimate $\hat{x}_t$.

The ancillary controller optimization problem is formulated as follows:

$$\min_{\hat{x}_{k+1}, u_k, \forall 0 \leq k \leq N-1} V_N(\hat{x}_t) \tag{6a}$$

subject to

$$\hat{x}_{k+1} = f(\hat{x}_k, u_k, d^n), \quad 0 \leq k \leq N-1 \tag{6b}$$

$$u_k^j \in \mathbb{U}, \qquad 0 \leq k \leq N-1 \tag{6c}$$

$$V_N(\hat{x}_t) = \sum_{k=0}^{N-1} \ell\left(\hat{x}_{k+1} - z_{k+1}^{*i}, u_k - v_k^{*i}\right), \tag{7}$$

where $d^n \in \mathbb{D}$ is the realization of the uncertainty that is considered in the ancillary controller, and

$$z_{k+1}^{*i} = f(z_k^{*p(i)}, v_k^{*i}, d^n). \tag{8}$$

We propose to select d^n at each time step t as the uncertainty value which was estimated to have resulted in the primary system state z_t. Hence, d^n at each time step t belongs to the discrete set of uncertainties considered by the scenario tree of the primary controller ($d^n \in \mathbb{D}^d$).

In tube-based NMPC (Mayne et al., 2011), only additive disturbances on the plant state were considered. The purpose of the ancillary controller was to keep the plant state in a close neighborhood of the primary system trajectory. For the case of a more generic form of uncertainty, a tube-based NMPC scheme was proposed in (Falugi and Mayne, 2011).

The key difference in our proposed scheme is that the ancillary controller does not track a fixed reference trajectory. Instead, it may track a different scenario at each time step depending on the actual (estimated) realization of the plant uncertainty. The reference scenario tracked by the ancillary controller is generated by the primary controller using the same model (same uncertainty d) as the model used in the ancillary controller. The multi-stage primary controller implicitly optimizes (with respect to d) for the required back-offs for each scenario from the primary system constraints, for each current initial state of the primary system. If the primary controller uses only one model, the constraint tightening has to be done manually for all initial states and for all possible uncertainty sequences (w, e and d), which would lead to sub-optimal plant operation. Hence, in our proposed approach, constraint tightening for the primary controller is only with respect to the small uncertainties (w and e).

As explained in (Mayne et al., 2011) and (Subramanian et al., 2018), the computation of the required constraint tightening (the sets $\mathbb{Z}$ and $\mathbb{V}$) rigorously is a formidable task, and simulation studies have to be conducted to determine $\mathbb{Z}$ and $\mathbb{V}$.

Assume that at time step t, the ancillary controller model was based on uncertainty d^i, where $d^i \in \mathbb{D}^d$ which was the actual plant uncertainty up to time step $t-1$. If at time step t, the plant uncertainty changes from d^i to d^j, where $d^j \in \mathbb{D}^d$, the actual plant state at time step $t+1$ will be different from what was predicted by the ancillary controller due to the model mismatch in the ancillary controller at time step t. However, the scenario tree of the primary controller considers all $d \in \mathbb{D}^d$, and hence we expect the plant state x_{t+1}, although different from what was predicted by the ancillary controller, to be close to $z_{t+1} = z^{*j} \in Z_r(t)$ (which is one of the nodes of the scenario tree of the primary controller). At $t+1$, the ancillary controller will use the correct model (which corresponds to d^j) because the state of the primary system is $z_{t+1} = z^{*j}$. Therefore, although the ancillary controller uses only one model at each time step, robustness against the uncertainty d is achieved by the primary controller. By the ancillary controller, the plant state is kept in a vicinity of the primary system state. After a change of the large uncertainty at some point in time, convergence of the state of the plant to a robust positive invariant set around the state of the primary system can be guaranteed as will be discussed in the following section. The controller can be implemented as per Algorithm 1.

Algorithm 1 Controller Implementation

Require:	$\mathbb{Z}$, $\mathbb{V}$, $\hat{x}_t$.
If $t = 0$:	Initialize the primary controller with $\hat{x}_0$. Solve (4) and apply v_0^{*1} to the plant. Store the elements of $Z_r(0)$ (z_1^{*j}, $\forall (j,1) \in I$).
else:	
Step 1	Estimate the current state $\hat{x}_t$.
Step 2	Determine $z_1^{*j} \in Z_r(t-1)$ which has the minimum distance to $\hat{x}_t$ and use it as the root node for the primary controller (z_0).
Step 3	Determine the parameter $d \in \mathbb{D}^d$ which resulted in $z_1^{*j} \in Z_r(t-1)$ which has the minimum distance to $\hat{x}_t$ and store it as d^n.
Step 4	Solve (4) and store the optimal solution sequences $\{z_k^{*j}\}$, $\{v_k^{*j}\}$.
Step 5	Store the elements of $Z_r(t)$.
Step 6	Determine the reference scenario by using d^n in (8) from $k = 0$ to $k = N-1$.
Step 7	Solve (6) using d^n in the ancillary controller model.
Step 8	Apply u_0^{*1} to the plant, set $t = t+1$, and go to step 1 at the next sampling instant.

The advantage of the proposed scheme over the original tube-enhanced multi-stage NMPC (with muti-stage primary and ancillary controllers) is a significant reduction in the computation time, as only a standard NMPC problem is solved for the ancillary controller. In the process industry, the sampling time is usually large, and hence the primary controller optimization problem can be solved within the sampling interval as many times as the number of considered discrete realizations of the uncertainty, with root nodes $z_1^{*j} \in Z_r(t-1)$, $\forall (j,1) \in I$. When the state estimate $\hat{x}_t$ arrives at time step t, the primary controller solution that corresponds to $z_1^{*j} \in Z_r(t-1)$ which has the minimum Euclidean distance to $\hat{x}_t$ can be used to select the ancillary controller model and the reference trajectory (8) needed in (6) and (7). This results in a robust NMPC scheme that has the same response time as an NMPC based on a nominal model.

5. DISCUSSION ON THE STABILITY OF THE PROPOSED SCHEME

In this section, we sketch briefly how convergence of the state trajectory of the plant to a robust positive invariant set around the state trajectory of the primary system can be established. We will not present a detailed stability proof, but we discuss in a conceptual manner how the presented scheme (when extended by stabilizing ingredients) can guarantee stability of the controlled plant. We will invoke already established assumptions and results from the tube-based NMPC and multi-stage NMPC literature only when needed for the sake of our discussion.

Throughout, we assume that the actual realization of the uncertainty in the plant belongs to the discrete set $\mathbb{D}^d$ which is taken into account by the primary controller. We consider the full state information case ($x_t = \hat{x}_t$, $\forall t \geq 0$) for simplicity. The following is assumed for the remainder of this section.

Assumption 1.

i. The plant dynamics $f(\cdot,\cdot,\cdot)$ is Lipschitz continuous with Lipschitz constant L_f.
ii. The solution to the ancillary controller optimization problem is unique.

5.1 Primary Controller

A robust positive invariant terminal constraint set $\mathbb{Z}_f$ and a terminal penalty function satisfying the assumptions in (Lucia, 2014) can be added to the optimization problem of the primary controller, which guarantees the existence of a feasible terminal control law $\kappa_f(z)$, $\forall d \in \mathbb{D}^d$, $\forall z \in \mathbb{Z}_f$.

The scenario tree of the primary controller is initialized at each time step t by one of the tree nodes from the set $Z_r(t-1)$. Since the terminal constraint set is robust positive invariant, a feasible solution for the primary controller optimization problem always exists $\forall t \geq 0$. Input-to-state practical stability (ISpS) of the primary system state then follows as in (Lucia, 2014).

5.2 Ancillary Controller

We assume in the sequel that the actual plant uncertainty d_t at time step t changes from d^i to d^j, where $d^i, d^j \in \mathbb{D}^d$. Due to the way the primary controller is initialized, and since $d_t = d^j \in \mathbb{D}^d$, at time step $t+1$, the plant state and the primary system state are given by

$$x_{t+1} = f(x_t, u_t, d_t) + w_t, \tag{9a}$$

$$z_{t+1} = f(z_t, v_t, d_t), \tag{9b}$$

where $z_{t+1} \in Z_r(t)$ will be the root node of the scenario tree of the primary controller at $t+1$.

We assume quadratic stage costs of the form

$$\ell(x-z, u-v) = (x-z)^T Q (x-z) + (u-v)^T R (u-v)$$

for the ancillary controller, where the matrices Q and R are positive definite.

It follows from proposition 2 in (Mayne et al., 2011), that there exists a constant $c_1 > 0$ such that,

$$V_N(x_t) \leq c_1 |x_t - z_t|^2 \tag{10}$$

Therefore, from (10) and the positive definiteness of Q and R, there exists a constant $L_1 > 0$ such that,

$$|u_t - v_t| \leq L_1 |x_t - z_t|. \tag{11}$$

From (9), (11), Assumption 1.i and the triangle inequality, there exists a constant L_2 such that,

$$|x_{t+1} - z_{t+1}| \leq L_2 |x_t - z_t| + |w_t|. \tag{12}$$

Inequality (12) shows that if the plant uncertainty changed at time step t, the control error at time step $t+1$ ($|x_{t+1} - z_{t+1}|$) is bounded and the bound depends on the previous control error ($|x_t - z_t|$) and the additive disturbances $|w_t|$.

Since the ancillary controller has no state constraints, recursive feasibility is guaranteed by construction, and from time step $t+1$ on, convergence of the state of the plant to a robust positive invariant set around the the state of the primary system can be established in a similar fashion as in (Mayne et al., 2011). Therefore, the plant state and the ancillary controller input are always in a neighborhood of the primary system state and input. However, unlike in (Mayne et al., 2011), the terminal ingredients have to be satisfied $\forall d \in \mathbb{D}^d$ because there exists s possibilities for the model in the ancillary controller, where s is the cardinality of the set $\mathbb{D}^d$.

6. CASE STUDY

The core unit in the Williams-Otto process (Williams and Otto, 1960) is the CSTR in which the following reactions occur:

$$A + B \rightarrow C, \qquad k_1 = 1.6599 \times 10^6 e^{-\frac{6666.7}{T_R+273.15}}\ s^{-1},$$

$$B + C \rightarrow P + E, \qquad k_2 = 7.2117 \times 10^8 e^{-\frac{8333.3}{T_R+273.15}}\ s^{-1},$$

$$C + P \rightarrow G, \qquad k_3 = 2.6745 \times 10^{12} e^{-\frac{11111}{T_R+273.15}}\ s^{-1}$$

where $k_i = a_i \times e^{-\frac{b_i}{T_R+273.15}}$ is the general form of the reaction rate. We consider the CSTR model presented in their work with the modifications of removing the recycle flows and adding an ODE for modelling the jacket temperature. Consequently, the model dynamics is governed by:

$$W_R\dot{X}_A = F_A - (F_A + F_B)X_A - k_1 X_A X_B W_R$$

$$W_R\dot{X}_B = F_B - (F_A + F_B)X_B - (k_1 X_A X_B + k_2 X_B X_C)W_R$$

$$W_R\dot{X}_C = -(F_A + F_B)X_C + (2k_1 X_A X_B - 2k_2 X_B X_C - k_3 X_C X_P)W_R$$

$$W_R\dot{X}_E = -(F_A + F_B)X_E + 2k_2 X_B X_C W_R$$

$$W_R\dot{X}_G = -(F_A + F_B)X_G + 1.5k_3 X_C X_P W_R$$

$$W_R\dot{X}_P = -(F_A + F_B)X_P + (k_2 X_B X_C - 0.5k_3 X_C X_P)W_R$$

$$W_R C_p \dot{T}_R = 2k_1 X_A X_B W H_1 + 3k_2 X_B X_C W H_2 + 1.5 X_C X_P W H_3 - h_w A_w (T_R - T_J) - F_A C_p (T_R - T_A) - F_B C_p (T_R - T_J)$$

$$W_w C_p \dot{T}_J = F_w C_p (T_{Jin} - T_J) + h_w A_w (T_R - T_J)$$

where X_A, X_B, X_C, X_E, X_G and X_P are the mass fractions of the 6 components, T_R and T_J are the reactor and jacket temperatures. The process inputs are the flow rates of the pure inlet components A and B which are F_A and F_B, and the jacket cooling water inlet temperature T_{Jin}. The mass of the liquid inside the reactor W_R is considered to be constant and locally controlled. W_w is the mass of the water inside the jacket. H_1, H_2 and H_3 are the heat of reactions for each of the 3 reactions taking place. The complete details for the values of the model parameters can be found in (Williams and Otto, 1960).

The exponential term b_1 in the first reaction rate is considered to be uncertain by $\pm 6\%$ and its nominal value is 6666.7. We consider this as the main parametric uncertainty which is modelled in the scenario tree of the primary controller. Hence, $d_t \in [0.94, 1.06]$, $\forall t \geq 0$. Random but bounded additive disturbances are added to the solution of the differential equations at each time step. The values of the additive disturbances are $\pm 5 \times 10^{-4}$ for the concentrations, ± 0.2 for T_R and ± 0.01 for T_J.

The mass fractions of X_E, X_G and X_P are measured with a measurement error ± 0.05. Also T_R and T_J are measured with a measurement error of $\pm 0.3°C$. An extended Kalman filter (EKF) is used to estimate the the 8 states and the unknown parameter b_1. Initial state estimation errors $e_0 \in [-0.05, 0.05]$ for components A and B, $e_0 \in [-0.03, 0.03]$ for components C and E, $e_0 \in [-0.01, 0.01]$ for components G and P and $e_0 \in [-2.5, 2.5]$°C for T_R and T_J are assumed.

The sampling time for all controllers and for the EKF is $T_s = 30$ seconds. The prediction horizon for all controllers is $N = 20$, and all the multi-stage controllers have a robust horizon $N_R = 1$.

The state and input constraints are, 60 °C $\leq T_R \leq$ 90, 0.5 kg s^{-1} $\leq F_A \leq$ 10 kg s^{-1}, 0.5 kg s^{-1} $\leq F_B \leq$ 10 kg s^{-1}, 15 °C $\leq T_{Jin} \leq$ 100 °C.

We consider three controllers; the tube-enhanced multi-stage controller with primary and ancillary multi-stage controllers (TEMS NMPC) from (Subramanian et al., 2018), the newly proposed tube-enhanced multi-stage with a single scenario NMPC ancillary controller (STEMS NMPC) which considers $d = 1$ in its model for the first time step, and $\forall t \geq 1$ selects its model as detailed in section 4, and a controller that only considers the nominal parameter realization ($d = 1$) for the primary and the ancillary controllers, which we call tube NMPC in the sequel. The multi-stage primary and ancillary controllers for the TEMS NMPC and the multi-stage primary controller for the STEMS NMPC consider the minimum, nominal and maximum values of d, which are $\{0.94, 1.0, 1.06\}$.

All the simulations presented below were implemented using CasADi (Andersson et al., 2019) for automatic differentiation, and IPOPT (Wächter and Biegler, 2006) for solving the resulting nonlinear optimization problems.

We first show the results of the three controllers for tracking a reference temperature of 90 $°C$. The constraints for the primary and ancillary controllers are shown in Table 1. As can be seen, the temperature of 90 $°C$ is an unreachable set point for all the primary controllers because of the tightened constraints on T_R. Both TEMS NMPC and STEMS NMPC required the same constraint tightening for the primary controllers, while tube NMPC required tightening the constraint for the reactor temperature by 15.5 $°C$ more in order to achieve robust constraint satisfaction $\forall d \in [0.94, 1.06]$. As shown in Figure 2, for the highest reaction rate ($d = 0.94$) all the controllers satisfy the plant constraints. TEMS NMPC and STEMS NMPC resulted in similar results, while tube NMPC results in a slower behaviour. The behavior of the reactor temperature for the nominal ($d = 1.0$) and lowest ($d = 1.06$) reaction rates shows that tube NMPC resulted in a significantly

Table 1. Constraints for the tracking objective.

Primary Controller	TEMS, STEMS	Tube	Units
F_A	$(0.6 - 9.7)$	$(0.6 - 9.7)$	kg s^{-1}
F_B	$(0.6 - 9.7)$	$(0.6 - 9.7)$	kg s^{-1}
T_{Jin}	$(19 - 97)$	$(19 - 97)$	°C
T_R	$(60.5 - 88.5)$	$(60.5 - 73)$	°C
Ancillary Controller	TEMS, STEMS	Tube	Units
F_A	$(0.5 - 10)$	$(0.5 - 10)$	kg s^{-1}
F_B	$(0.5 - 10)$	$(0.5 - 10)$	kg s^{-1}
T_{Jin}	$(15 - 100)$	$(15 - 100)$	°C

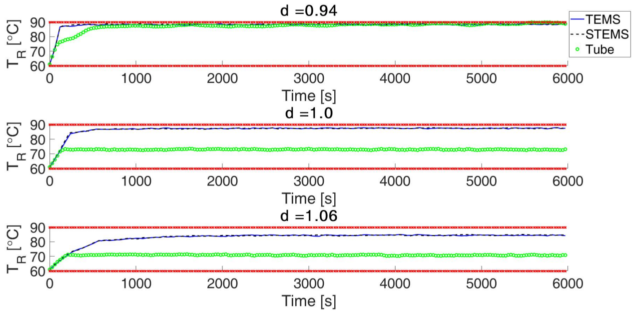

Fig. 2. Behavior of the reactor temperature T_R, for the CSTR controlled by TEMS NMPC, STEMS NMPC, tube NMPC, for the uncertainty realization $d = 0.94$ (Upper figure), $d = 1.0$ (Middle figure), $d = 1.06$ (Lower figure)

larger steady state error compared to the TEMS NMPC and STEMS NMPC due to the significant conservatism introduced by the required constraint tightening.

In what follows, we show the results of the three controllers for an economic objective. It is desired to control the Williams-Otto CSTR with the uncertain model parameter b_1 such that instantaneous profit given by:

$$(5554.1X_P + 125.91X_E)(F_A + F_B) - 370.3F_A - 555.42F_B \ \$ \ \mathrm{s}^{-1},$$

is maximized, while the input and state constraints are satisfied all the time. The ancillary controllers are tuned to track the primary system states and inputs with the following stage cost:

$$10\Delta X_P^2 + 2\Delta X_E^2 + 5\Delta F_A^2 + 10\Delta T_R^2 + \Delta F_B^2 + 10^{-3}\Delta T_{Jin}^2,$$

where Δx is the deviation of the state or input of the plant from the primary system state or input.

500 simulations were conducted, scanning the parameter uncertainty range with a step size of 0.005, and random but bounded generation of the additive disturbances, measurements noise and initial state estimation errors as explained earlier in this section.

Figure 3 shows a large spread of X_P (the mass fraction of the most profitable component) for the system controlled by the tube NMPC. The steady state value of X_P is as low as 0.07 in some of the simulations. This is because the controller tries to bring the reactor temperature close to the significantly tightened nominal reactor temperature using sub-optimal (with respect to the actual plant) inputs, because it is based on an incorrect model in the primary and ancillary controllers. This resulted in a reduced average steady state profit and even steady state loss in some of the simulations as shown in Figure 3. Figure 4 shows the inputs generated by the tube NMPC.

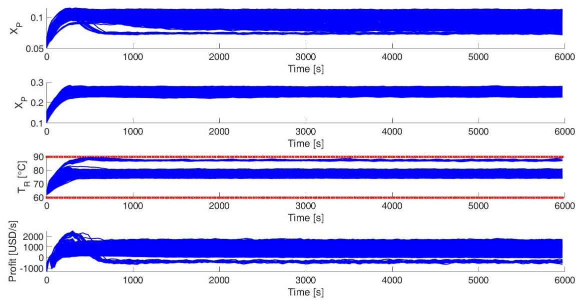

Fig. 3. Tube NMPC: Trajectories for the mass fractions X_E, X_P, reactor temperature T_R and the profit dynamics for the 500 simulations

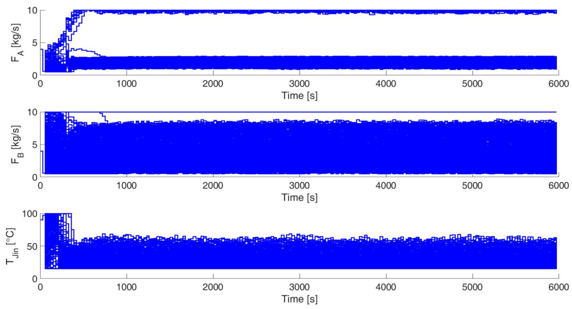

Fig. 4. Tube NMPC: Trajectories for the inlet flow rates F_A, F_B and the inlet cooling water temperature T_{Jin} for the 500 simulations

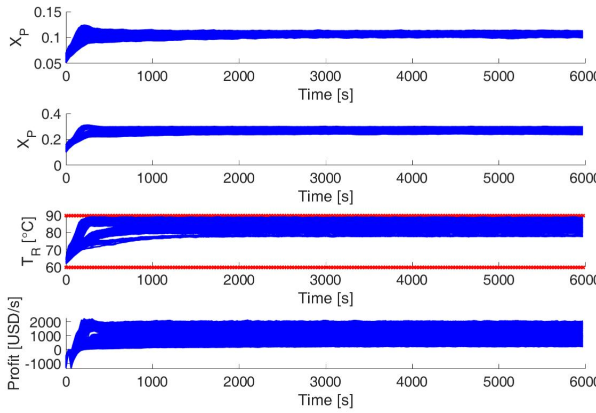

Fig. 5. STEMS NMPC: Trajectories for the mass fractions X_E, X_P, reactor temperature T_R and the profit dynamics for the 500 simulations

As can be seen from the small spread of X_P in Figure 5, this loss of performance did not occur for the system controlled by the proposed STEMS NMPC. The steady state value of X_P did not fall below 0.1 in any of the simulations and the steady state profit was never below 250 $\$s^{-1}$. Figure 6 shows the generated inputs by the STEMS NMPC. Similar to the tracking objective case illustrated earlier in this section, the STEMS NMPC and the TEMS NMPC (plots eliminated here for brevity) generated similar trajectories.

Table 2 shows the average steady state profits over the 500 simulations, where the TEMS NMPC and STEMS NMPC resulted in approximately 27 % increase of profit over tube NMPC. Table 3 shows the average computation times for the primary and the ancillary controllers of the 3

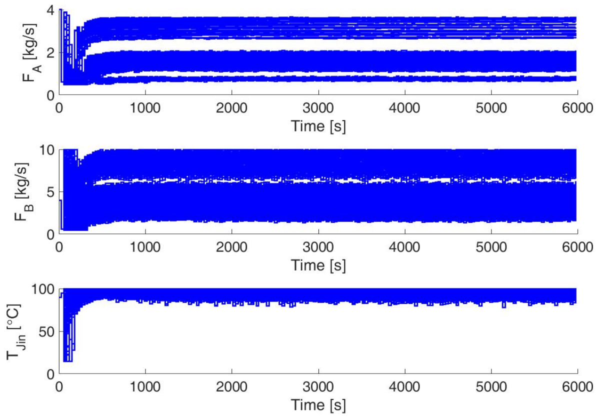

Fig. 6. STEMS NMPC: Trajectories for the inlet flow rates F_A, F_B and the inlet cooling water temperature T_{Jin} for the 500 simulations

Table 2. Average steady state profit.

TEMS	STEMS	Tube	Units
955	962	751	\$ s^{-1}

Table 3. Average computation time.

Controller	TEMS	STEMS	Tube	Units
Primary	380	380	100	ms
Ancillary	220	50	50	ms

control schemes. The computation time for the ancillary controller of the STEMS NMPC and the tube NMPC are the same and much lower than that of the TEMS NMPC. As mentioned earlier, the primary controller can be solved three times within the 30 seconds sampling time for the three future predicted states, and only the ancillary controller optimization problem needs to be solved after receiving the measurements. Hence, STEMS NMPC can be implemented with the same input delay as nominal NMPC.

7. CONCLUSION

In this paper we proposed a new variant of tube-enhanced multi-stage NMPC, in which the multi-stage ancillary controller is replaced by a single scenario NMPC ancillary controller. The simulation study showed that the proposed NMPC scheme has a potential of achieving similar performance results as the original tube-enhanced NMPC scheme with a significant reduction in the computational effort and in the reaction time of the controller.

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