

Stability analysis of discrete-time LPV switched systems [★]

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Abstract: This paper addresses the stability problem for discrete-time switched systems under autonomous switching. Each mode of the switched system is modeled as a Linear Parameter Varying (LPV) system, the time-varying parameters can vary arbitrarily fast and are represented in a polytopic form. The Lyapunov theory is employed to get new conditions in the form of parameter-dependent LMIs. The constructed Lyapunov function takes advantage of using an augmented state vector with shifted states in its construction. In this sense, the Lyapunov function employed in this paper can be viewed as a discrete-time LPV switched Lyapunov function. Numerical experiments illustrate the efficacy of the technique in providing stability certificates.

Keywords: Hybrid systems, time-varying parameters, LMIs.

1. INTRODUCTION

In the last decades great attention has been paid to the study of hybrid systems (Goebel et al., 2012). This is due the fact that this class of systems may be used to represent several dynamics systems. The switched systems are a particular class of hybrid systems. A switched system is composed by a number of modes and each one of them can be active individually at each time. The transition between two different modes may be ruled by time, states or it can be autonomous, meaning that a transition may occur at any time (Liberzon, 2003).

Stability is a fundamental issue in the study of dynamical systems, including the ones with switching dynamics. In this sense, the Lyapunov theory has been successfully employed to provide stability certificates for switched systems. The Lyapunov theory allows the conditions to be written in the form of Linear Matrix Inequalities (LMIs) that can be solved via semidefinite programming (Boyd et al., 1994). Concerning discrete-time systems with autonomous switching one may cite (Daafouz et al., 2002) that used a switched Lyapunov function for stability analysis and design of an output-feedback control. In Lee and Dullerud (2006) stability conditions based on a path-dependent Lyapunov function have been exploited. The problem of stability for switched systems with time-varying delays has been investigated in Hetel et al. (2006b). In Jungers et al. (2017) different sets of LMIs that may be used to certify stability of switched discrete-time systems are presented. Recently, a new class of switched Lyapunov functions based on the use of an augmented state vector was presented in Gomide and Lacerda (2018).

Even with a growing number of studies focused on stability analysis for switched systems, a small part of these studies consider the presence of uncertainties and time-varying parameters in the subsystems. Therefore, there is still great potential for the development of new less conservative and more efficient methods for switched systems. It is well known that the presence of uncertainties and time-varying parameters may affect the performance of the systems. In fact, Linear Parameter Varying (LPV) systems have been extensively studied in the last years (Briat, 2015; Chesi, 2013, 2014; Mohammadpour and Scherer, 2012). Thus, when analyzing switched systems it is important to consider the presence and effect of uncertainties and time-varying parameters in the stability analysis and in control design (Binazadeh and Bahmani, 2017; Binazadeh and Shafiei, 2014; Binazadeh and Bahmani, 2016). Different approaches to the representation of uncertainties can be found in the literature, among them one can cite the polytopic uncertainties (Kermani and Sakly, 2014; Niamsup and Rajchakit, 2013; Rajchakit et al., 2012), norm bound uncertainties (Sun et al., 2006; Zhang and Yan, 2015) and uncertainties in affine form (Baleghi and Shafiei, 2018).

This paper proposes new stability conditions for discrete-time LPV switched systems under arbitrary switching. Each mode of the switched system is modeled as a LPV system in a polytopic domain. The time-varying parameters can vary arbitrarily fast and there is no information about their rates of variation. Stability will be guaranteed by means of a Lyapunov function composed by an augmented state vector. This class of function allows to introduce the switched dynamics of the system and the LPV feature in the Lyapunov function. In this sense, the Lyapunov function employed in this paper can be viewed as a discrete-time LPV switched Lyapunov function. This methodology is based upon the methods presented in Go-

^{*} This work is supported by the Brazilian agencies CNPq grants 402830/2016-4, 425800/2018-0 and FAPEMIG grant APQ-00692-17.

midé and Lacerda (2018), concerned with stability problem for precisely known switched systems, and in Lacerda and Gomide (2020), where the stability and stabilizability problem have been considered. The use of structured Lyapunov functions with non-monotonic terms was explored to deal with the stability problem for uncertain systems in Lacerda and Seiler (2017), moreover, stability and performance for uncertain systems were investigated using an augmented state-vector in the Lyapunov function (Pessim et al., 2018, 2019). The main objective of this paper is to propose less conservative conditions to guarantee stability of discrete-time switched LPV systems. The key feature in this paper is the use of shifted states, for instance $x(k+1) = A_{\sigma(k)}(\alpha_k)x(k)$, implying that $x(k+2) = A_{\sigma(k+1)}(\alpha_{k+1})x(k+1)$ or simply $x(k+2) = A_{\sigma(k+1)}(\alpha_{k+1})A_{\sigma(k)}(\alpha_k)x(k)$. Note that both the time varying parameter α_k and the switching rule $\sigma(k)$ are evaluated in different instants. This fact have been investigated in Daafouz and Bernussou (2001) for LPV systems, in Daafouz et al. (2002) for switched systems and in Hetel et al. (2006a) for switched LPV systems considering only two different instants. The approach addressed in this paper admits the use of a generic number of shifted states and consequently a generic number of instants in the switched rule and also in the LPV parameter. Numerical examples borrowed from the literature are employed to illustrate the advantages of the proposed technique when compared to existing approaches.

This paper is organized as follows. Preliminary results are presented in Section 2, Section 3 details the main contributions of the paper. The performance of the method is illustrated via numerical experiments in Section 4, while Section 5 concludes the paper.

2. PRELIMINARIES

2.1 System description

Consider the following switched discrete-time LPV system

$$x(k+1) = A_{\sigma(k)}(\alpha_k)x(k) \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $A_{\sigma(k)}(\alpha_k) \in \mathbb{R}^{n \times n}$ is the dynamic matrix, $\sigma(k)$, belongs to a finite set \mathcal{P} that denotes the switching rule $\mathcal{P} = \{1, \dots, m\}$, α_k is the time-varying parameter that belongs to a polytopic domain parameterized in terms of a vector of time-varying parameters. Although each mode could be subject to a different time-varying parameter, to simplify the developments, let us consider that all the modes present the same number of vertices and are affected by the same time-varying parameter α_k .

For a specific mode $\sigma(k)$ it is possible to write

$$A_{\sigma(k)}(\alpha_k) = \sum_{i=1}^V \alpha_{k,i} A_{\sigma(k),i}, \quad \alpha_k \in \Lambda_V$$

where $A_{\sigma(k),i}$, $i = 1, \dots, V$, are the vertices of the polytope and Λ_V is the unit simplex given by

$$\Lambda_V = \left\{ \alpha_k \in \mathbb{R}^V : \sum_{i=1}^V \alpha_{k,i} = 1; \alpha_{k,i} \geq 0, i = 1, \dots, V \right\}.$$

Only one mode of the matrix $A_{\sigma(k)}$ is active at a time. The indicator function will be used to describe such a behavior. Consider $\xi(k) = [\xi_1(k), \dots, \xi_m(k)]^T$

$$\xi_i(k) = \begin{cases} 1, & \text{if } \sigma(k) = i \\ 0, & \text{otherwise.} \end{cases}$$

In this way, system (1) can be written as

$$x(k+1) = A(\xi(k), \alpha_k)x(k). \quad (2)$$

2.2 Stability analysis

Stability of system (1) can be certified by the existence of a radially unbounded Lyapunov function $V(k, x(k))$ satisfying the following criteria (Vidyasagar, 1993)

$$V(k, 0) = 0, \quad V(k, x(k)) > 0, \quad \forall x(k) \neq 0, \quad (3)$$

$$\Delta V(k, x(k)) < 0, \quad \forall x(k) \neq 0, \quad (4)$$

where $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$. The Lyapunov function satisfies

$$\beta_1 \|x(k)\|^2 \leq V(k, x(k)) \leq \beta_2 \|x(k)\|^2 \quad (5)$$

for all $x(k) \in \mathbb{R}^n$ and $k \geq 0$ with β_1 and β_2 positive scalars. Moreover, $\Delta V(k, x(k)) < -\beta_3 \|x(k)\|^2$, where β_3 is a sufficiently small positive scalar. If such a Lyapunov function exists, then system (2) is GUAS (Globally Uniformly Asymptotically Stable).

In Hetel et al. (2006a) a set of conditions for robust stability analysis of switched systems is proposed, where each switching mode is described by a polytopic domain represented by a vector of time-varying parameters. The following lemma presents the main result of such paper.

Lemma 1. If there exist symmetric positive definite matrices $S_i(\alpha_k), S_j(\alpha_{k+1})$ and matrices $G_i(\alpha_k)$ of appropriate dimensions such that

$$\begin{bmatrix} G_i(\alpha_k) + G_i(\alpha_k)^T - S_i(\alpha_k) & \star \\ A_i(\alpha_k)G_i(\alpha_k) & S_j(\alpha_{k+1}) \end{bmatrix} > 0 \quad (6)$$

$\forall \alpha_k \in \Lambda_V, \alpha_{k+1} \in \Lambda_V, i \in \mathcal{P}, j \in \mathcal{P}$, then system (2) is GUAS.

Proof. Since

$$G_i(\alpha_k)^T S_i(\alpha_k)^{-1} G_i(\alpha_k) \geq G_i(\alpha_k) + G_i(\alpha_k)^T - S_i(\alpha_k),$$

condition (6) implies

$$\begin{bmatrix} G_i(\alpha_k)^T S_i(\alpha_k)^{-1} G_i(\alpha_k) & \star \\ A_i(\alpha_k)G_i(\alpha_k) & S_j(\alpha_{k+1}) \end{bmatrix} > 0.$$

Pre- and post-multiplying the latter condition respectively by $\text{diag}(G_i(\alpha_k)^{-T}, S_j^{-1}(\alpha_{k+1}))$ and its transpose, and setting $S_i^{-1}(\alpha_k) = P_i(\alpha_k)$, results in

$$\begin{bmatrix} P_i(\alpha_k) & \star \\ P_j(\alpha_{k+1})A_i(\alpha_k) & P_j(\alpha_{k+1}) \end{bmatrix} > 0.$$

The application of a Schur complement (Boyd et al., 1994) yields

$$A_i(\alpha_k)^T P_j(\alpha_{k+1}) A_i(\alpha_k) - P_i(\alpha_k) < 0. \quad (7)$$

Multiplying (7) by $\xi_i(k)^2$, $i = 1, \dots, m$, and summing up gives

$$A(\xi(k), \alpha_k)^T P_j(\alpha_{k+1}) A(\xi(k), \alpha_k) - P(\xi(k), \alpha_k) < 0. \quad (8)$$

Multiplying (8) by $\xi_j(k+1)$, $j = 1, \dots, m$, and summing up results in

$$A(\xi(k), \alpha_k)^T P(\xi(k+1), \alpha_{k+1}) A(\xi(k), \alpha_k) - P(\xi(k), \alpha_k) < 0, \quad (9)$$

which is equivalent to condition (4) with

$$V(k, x(k)) = x(k)^T P(\xi(k), \alpha_k) x(k).$$

Since $P(\xi(k), \alpha_k) > 0$, the Lyapunov function $V(k, x(k))$ is also positive definite, concluding the proof.

The condition presented in Lemma 1 depends on additional slack variables that, although reducing the conservativeness,

increase the computational cost to solve the problem. In the following section, an alternative way to assess the stability of discrete-time LPV switched systems is proposed, based on the utilization of augmented Lyapunov functions.

3. MAIN RESULTS

This paper employs a class of structured Lyapunov functions to provide stability certificates for switched discrete-time LPV systems. This class of Lyapunov functions introduces the dynamics of the system in its construction. To better illustrate our approach, firstly we will provide a formulation based on a particular case.

Lemma 2. If there exist symmetric matrices $P_1 \in \mathbb{R}^{n \times n}$ and $P_2 \in \mathbb{R}^{n \times n}$ such that

$$P_1 + A_i(\alpha_k)^T P_2 A_i(\alpha_k) > 0 \quad (10)$$

$$A_i(\alpha_k)^T P_1 A_i(\alpha_k) + A_i(\alpha_k)^T A_j(\alpha_{k+1})^T P_2 A_j(\alpha_{k+1}) A_i(\alpha_k) - (P_1 + A_i(\alpha_k)^T P_2 A_i(\alpha_k)) < 0 \quad (11)$$

$\forall \alpha_k \in \Lambda_V, \alpha_{k+1} \in \Lambda_V, i \in \mathcal{P}, j \in \mathcal{P}$, then system (2) is GUAS.

Proof. By multiplying (11) by $\xi_i(k)^2, i = 1, \dots, m$, and summing up one has

$$A(\xi(k), \alpha_k)^T P_1 A(\xi(k), \alpha_k)^T + A(\xi(k), \alpha_k)^T A_j(\alpha_{k+1})^T P_2 A_j(\alpha_{k+1}) A(\xi(k), \alpha_k) - (P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k)) < 0 \quad (12)$$

Multiplying (12) by $\xi_j(k+1)^2, j = 1, \dots, m$, and summing up one has

$$A(\xi(k), \alpha_k)^T P_1 A(\xi(k), \alpha_k)^T + \Upsilon^T P_2 \Upsilon - (P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k)) < 0 \quad (13)$$

with

$$\Upsilon = A(\xi(k+1), \alpha_{k+1}) A(\xi(k), \alpha_k).$$

Pre- and post-multiplying (13) by $x(k)^T$ and $x(k)$ respectively and considering the dynamics of the system, i.e., $x(k+1) = A(\xi(k), \alpha_k)x(k)$ and $x(k+2) = A(\xi(k+1), \alpha_{k+1})x(k+1)$ yields

$$V(x(k+1)) - V(x(k)) < 0$$

with $V(x(k)) = x(k)^T (P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k)) x(k)$. Note that, by multiplying (10) by $\xi_i(k)^2, i = 1, \dots, m$, and summing up one has

$$P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k) > 0, \quad (14)$$

ensuring that the Lyapunov function $V(x(k))$ is positive definite. Moreover, one may choose

$$\beta_1 = \min_{i \in \mathcal{P}, \alpha_k \in \Lambda_V} \lambda_{\min} (P_1 + A_i(\alpha_k)^T P_2 A_i(\alpha_k))$$

$$\beta_2 = \max_{i \in \mathcal{P}, \alpha_k \in \Lambda_V} \lambda_{\max} (P_1 + A_i(\alpha_k)^T P_2 A_i(\alpha_k))$$

to guarantee that (5) is satisfied and

$$\beta_3 = \min_{i, j \in \mathcal{P}, \alpha_k, \alpha_{k+1} \in \Lambda_V} \lambda_{\min} (A_i(\alpha_k)^T P_1 A_i(\alpha_k) + A_i(\alpha_k)^T A_j(\alpha_{k+1})^T P_2 A_j(\alpha_{k+1}) A_i(\alpha_k) - P_1 - A_i(\alpha_k)^T P_2 A_i(\alpha_k))$$

to ensure $\Delta V(k, x(k)) < -\beta_3 \|x(k)\|^2$, concluding the proof.

Remark 3. Note that even considering constant matrices P_1 and P_2 , the Lyapunov function

$$V(x(k)) = x(k)^T (P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k)) x(k)$$

depends upon the switching modes $\xi(k)$ and the LPV parameter α_k . Moreover, there is no sign constraints imposed to the symmetric matrices P_1 and P_2 individually. It is also simple to verify that, considering $P_2 = 0$ in Lemma 2, allow us to recover the results presented in (7).

In what follows the result presented in Lemma 2 will be extended to the more general case making use of N symmetric matrices P_i . Before introducing the main results let us define some notation. Consider

$$\Phi_0 = I$$

$$\Phi_1 = A_{i_1}(\alpha_k)$$

$$\Phi_2 = A_{i_2}(\alpha_{k+1}) A_{i_1}(\alpha_k)$$

$$\Phi_R = A_{i_R}(\alpha_{k+R-1}) A_{i_{R-1}}(\alpha_{k+R-2}) \cdots A_{i_1}(\alpha_k).$$

Moreover, a multi-simplex domain composed by the cartesian product of N different simplex sets, each of them with V vertices, is denoted by Λ_V^N . In other words

$$\Lambda_V^N = \underbrace{\Lambda_V \times \Lambda_V \times \dots \times \Lambda_V}_{N\text{-times}}$$

In the same way

$$\mathcal{P}^N = \underbrace{\mathcal{P} \times \mathcal{P} \times \dots \times \mathcal{P}}_{N\text{-times}}$$

is the cartesian product of finite sets \mathcal{P} .

Theorem 4. If there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}, i = 1, \dots, N$, such that

$$\sum_{j=0}^{N-1} \Phi_j^T P_{j+1} \Phi_j > 0 \quad (15)$$

$$\forall (i_1, \dots, i_{N-1}) \in \mathcal{P}^{N-1}, \forall (\alpha_k, \dots, \alpha_{k+N-2}) \in \Lambda_V^{N-1}$$

$$\sum_{z=1}^N \Phi_z^T P_z \Phi_z - \sum_{j=0}^{N-1} \Phi_j^T P_{j+1} \Phi_j < 0 \quad (16)$$

$$\forall (i_1, \dots, i_N) \in \mathcal{P}^N, \forall (\alpha_k, \dots, \alpha_{k+N-1}) \in \Lambda_V^N$$

then system (2) is GUAS.

Proof. By multiplying (16) successively by $\xi_{i_j}(k+j-1)^2, j = 1, \dots, N, i_j \in \mathcal{P}$ and summing up yields

$$A(\xi(k), \alpha_k)^T M_{k+1} A(\xi(k), \alpha_k) - M_k < 0 \quad (17)$$

with

$$M_k = P_1 + A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k) + \Psi_2(k)^T P_3 \Psi_2(k) + \dots + \Psi_{N-1}(k)^T P_N \Psi_{N-1}(k)$$

where

$$\Psi_2(k) = A(\xi(k+1), \alpha_{k+1}) A(\xi(k), \alpha_k)$$

$$\Psi_3(k) = A(\xi(k+2), \alpha_{k+2}) A(\xi(k+1), \alpha_{k+1}) A(\xi(k), \alpha_k)$$

$$\vdots$$

$$\Psi_N(k) = A(\xi(k+N-1), \alpha_{k+N-1}) \times A(\xi(k+N-2), \alpha_{k+N-2}) \cdots A(\xi(k), \alpha_k).$$

Pre- and post multiplying (17) by $x(k)^T$ and $x(k)$ respectively, and considering the dynamics of the system $x(k+1) = A(\xi(k), \alpha_k)x(k)$ one can write

$$x(k+1)^T M_{k+1} x(k+1) - x(k)^T M_k x(k) < 0$$

that is equivalent to $V(x(k+1)) - V(x(k)) < 0$ with $V(x(k)) = x(k)^T M_k x(k)$. Note that (15) guarantees that M_k is positive definite. The same procedure adopted in Lemma 2 can be used to choose the scalars β_1, β_2 and β_3 , concluding the proof.

Remark 5. The number of scalar decision variables (N_V) spent by Theorem 4 can be computed as

$$N_V = \frac{Nn(n+1)}{2}$$

where n is the number of states and N is the number of employed matrices P_i . The number of LMI rows (N_R) can be computed as

$$N_R = nm^{N-1} \left(\frac{(V+1)!}{2!(V-1)!} \right)^{N-1} + nm^N \left(\frac{(V+1)!}{2!(V-1)!} \right)^N.$$

If the system is precisely known, the conditions presented in Theorem 4 recover the results presented in (Gomide and Lacerda, 2018, Theorem 5). The conditions presented in Theorem 4 can be easily adapted to consider time-invariant uncertainties. For this end, it suffices to consider $\alpha_{k+\theta} = \alpha$, for all values of θ .

To reduce the conservativeness of Theorem 4 it is possible to introduce parameter dependent matrices $P_i(\alpha_k)$.

Corollary 6. If there exist symmetric matrices $P_i(\alpha_k) \in \mathbb{R}^{n \times n}$, $i = 1, \dots, N$, such that

$$\sum_{j=0}^{N-1} \Phi_j^T P_{j+1}(\alpha_k) \Phi_j > 0 \quad (18)$$

$$\forall (i_1, \dots, i_{N-1}) \in \mathcal{P}^{N-1}, \forall (\alpha_k, \dots, \alpha_{k+N-2}) \in \Lambda_V^{N-1}$$

$$\sum_{z=1}^N \Phi_z^T P_z(\alpha_{k+1}) \Phi_z - \sum_{j=0}^{N-1} \Phi_j^T P_{j+1}(\alpha_k) \Phi_j < 0 \quad (19)$$

$$\forall (i_1, \dots, i_N) \in \mathcal{P}^N, \forall (\alpha_k, \dots, \alpha_{k+N-1}) \in \Lambda_V^N$$

then system (2) is GUAS.

Proof. The proof follows the same steps presented in the proof of Theorem 4.

All the conditions presented until this point are in the form of parameter-dependent LMIs that depends upon α_{k+N} . In order to get a finite set of LMIs, in terms of the vertices of each switched mode, the ROLMIP package was employed (Aguilhari et al., 2019). To illustrate the process employed to write the LMIs, the conditions of Lemma 2 will be presented in a finite form.

Lemma 7. If there exist symmetric matrices $P_1 \in \mathbb{R}^{n \times n}$ and $P_2 \in \mathbb{R}^{n \times n}$ such that, $\forall i \in \mathcal{P}, j \in \mathcal{P}$, one has

$$P_1 + A_{i,\ell}^T P_2 A_{i,\ell} > 0, \quad \ell = 1, \dots, V, \quad (20)$$

$$2P_1 + A_{i,\ell}^T P_2 A_{i,q} + A_{i,q}^T P_2 A_{i,\ell} > 0, \\ \ell = 1, \dots, V-1, \quad q = \ell+1, \dots, V, \quad (21)$$

$$A_{i,\ell}^T P_1 A_{i,\ell} + A_{i,\ell}^T A_{j,r}^T P_2 A_{j,r} A_{i,\ell} - (P_1 + A_{i,\ell}^T P_2 A_{i,\ell}) < 0, \\ \ell = 1, \dots, V, \quad r = 1, \dots, V \quad (22)$$

$$2A_{i,\ell}^T P_1 A_{i,\ell} + A_{i,\ell}^T A_{j,r}^T P_2 A_{j,p} A_{i,\ell} + A_{i,\ell}^T A_{j,p}^T P_2 A_{j,r} A_{i,\ell} \\ - (2P_1 + 2A_{i,\ell}^T P_2 A_{i,\ell}) < 0, \\ \ell = 1, \dots, V, \quad r = 1, \dots, V-1, \quad p = r+1, \dots, V, \quad (23)$$

$$A_{i,\ell}^T P_1 A_{i,q} + A_{i,q}^T P_1 A_{i,\ell} + A_{i,\ell}^T A_{j,r}^T P_2 A_{j,r} A_{i,q} + A_{i,q}^T A_{j,r}^T P_2 A_{j,r} A_{i,\ell} \\ - (2P_1 + A_{i,\ell}^T P_2 A_{i,q} + A_{i,q}^T P_2 A_{i,\ell}) < 0, \\ \ell = 1, \dots, V-1, \quad q = \ell+1, \dots, V, \quad r = 1, \dots, V \quad (24)$$

$$2A_{i,\ell}^T P_1 A_{i,q} + 2A_{i,q}^T P_1 A_{i,\ell} + A_{i,\ell}^T A_{j,r}^T P_2 A_{j,p} A_{i,q} \\ + A_{i,q}^T A_{j,r}^T P_2 A_{j,p} A_{i,\ell} + A_{i,\ell}^T A_{j,p}^T P_2 A_{j,r} A_{i,q} + A_{i,q}^T A_{j,p}^T P_2 A_{j,r} A_{i,\ell} \\ - (4P_1 + 2A_{i,\ell}^T P_2 A_{i,q} + 2A_{i,q}^T P_2 A_{i,\ell}) < 0, \\ \ell = 1, \dots, V-1, \quad q = \ell+1, \dots, V, \\ r = 1, \dots, V-1, \quad p = r+1, \dots, V, \quad (25)$$

then system (2) is GUAS.

Proof. Multiplying (20) by $\alpha_{k,\ell}^2$ and (21) by $\alpha_{k,\ell} \alpha_{k,r}$, adding both results and summing up variables ℓ and r in the respective domains yields (10). Multiplying (22) by $\alpha_{k,\ell}^2 \alpha_{k+1,r}^2$, (23) by $\alpha_{k,\ell}^2 \alpha_{k+1,r} \alpha_{k+1,p}$, (24) by $\alpha_{k,\ell} \alpha_{k,q} \alpha_{k+1,r}^2$ and (25) by $\alpha_{k,\ell} \alpha_{k,q} \alpha_{k+1,r} \alpha_{k+1,p}$, adding all the results and summing up variables ℓ, r, p and q in the respective domains yields (11), concluding the proof.

Remark 8. In the proof of Lemma 7 it is considered that α_k is independent of α_{k+1} , i.e., the variation rate is arbitrary. If bounded variation rates are to be considered, then one should properly relate the parameters α_k and α_{k+1} .

4. NUMERICAL EXPERIMENTS

In this section a comparative analysis among the conditions proposed in this paper and the available results from the literature are presented. The routines were implemented in Matlab R2015a, by using the packages YALMIP (Löfberg, 2004), ROLMIP (Aguilhari et al., 2019) and the solver SeDuMi (Sturm, 1999).

Example 1

Consider the following switched discrete-time LPV system borrowed from Hetel et al. (2006a) with matrices

$$\hat{A}_\sigma(k) = A_{0\sigma} + D_\sigma F(k) E_\sigma$$

where $F(k) = \rho(k)$ and $\rho(k) \in [-1, 1]$

$$A_{\sigma 1} = A_{0\sigma} + \rho D_\sigma E_\sigma$$

$$A_{\sigma 2} = A_{0\sigma} - \rho D_\sigma E_\sigma$$

with

$$A_{01} = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.1 & -0.5 \\ 0.8 & 0 & -0.1 & -0.3 & 0.3 \\ 0 & -0.3 & -0.4 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0.3 & 0.5 \\ -0.2 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

$$A_{02} = \begin{bmatrix} -0.7 & -0.7 & 0 & 0 & 0.2 \\ 0.5 & 0.3 & 0.3 & -0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0.6 & 0.3 \\ 0.3 & -0.8 & 0 & 0 & 0 \\ 0.1 & -0.7 & 0.1 & -0.3 & 0.3 \end{bmatrix}$$

$$\sigma \in [1, 2]$$

$$D_1^T = [0.2 \ 0.5 \ -0.1 \ 0.3 \ 0.2]$$

$$D_2^T = [-0.5 \ 0.38 \ 0.5 \ 0.2 \ 0.5]$$

$$E_1 = [-0.3 \ -0.3 \ -0.5 \ 0.2 \ 0.3]$$

$$E_2 = [-0.2 \ 0.1 \ -0.1 \ -0.05 \ 0.7]$$

In this way the switched discrete-time LPV system can be written as:

$$A_1(\alpha_k) = \alpha_{k,1}A_{11} + \alpha_{k,2}A_{12}$$

$$A_2(\alpha_k) = \alpha_{k,1}A_{21} + \alpha_{k,2}A_{22}$$

For this example, Lemma 2 is able to provide a solution with the use of 30 scalar decision variables and 210 LMI rows. On the other hand, the result presented in Hetel et al. (2006b) makes use of 160 scalar decision variables and 160 LMI rows. It is also important to emphasize that the method proposed in Xie et al. (2003) fail to find a solution in this case.

The Lyapunov function obtained from Lemma 2 is composed by two components $V(x(k)) = V_1 + V_2$ with

$$V_1 = x(k)^T P_1 x(k)$$

$$V_2 = x(k)^T A(\xi(k), \alpha_k)^T P_2 A(\xi(k), \alpha_k) x(k)$$

Figure 1 depicts the evolution of the Lyapunov function $V(x(k))$ (solid red line), V_1 (dashed blue line), and V_2 (black dotted line), along the trajectories of the LPV discrete-time switched system. Note that V_2 is not monotonically decreasing along the trajectories. It is important to remember that the switched system is subjected to the action of the time-varying parameters. Figure 2 shows the behavior of the time-varying parameter $\alpha_{k,1}$ over time. The switching rule may be arbitrary, but in this case it has been considered to change each iteration, starting in mode 1.

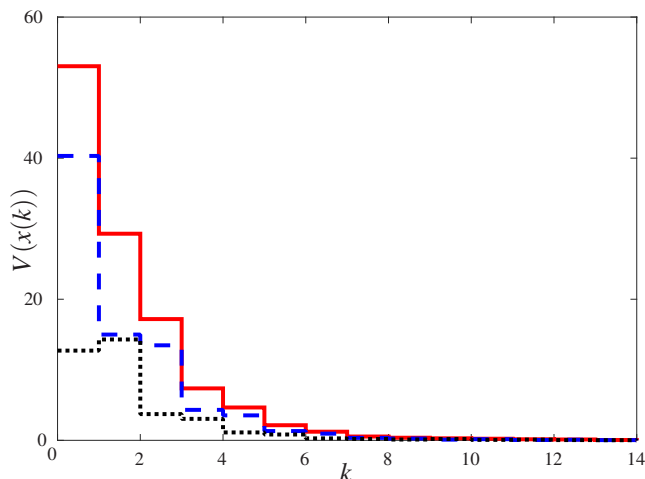


Fig. 1. Time evolution of the Lyapunov function $V(x(k))$ (solid red line) and its components V_1 (dashed blue line), and V_2 (black dotted line), along the trajectories of the LPV discrete-time switched system

Example 2

This example is adapted from Lee and Dullerud (2006). Consider the switched discrete-time LPV system

$$A_1(\alpha) = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix}, \quad A_2(\alpha) = \begin{bmatrix} -\beta & 0 \\ \beta & -\beta \end{bmatrix}$$

where β is the time-varying parameter $\beta \in [-\theta, \theta]$. The main goal is to find the maximum value of θ such that it is possible to certify the stability of the system. For this end, Theorem 4 and Corollary 6 will be employed with different values of N . Table 1 presents the maximum values of θ , as well as the number of scalar decision variables N_V and LMI rows N_R obtained for each method and different values of N .

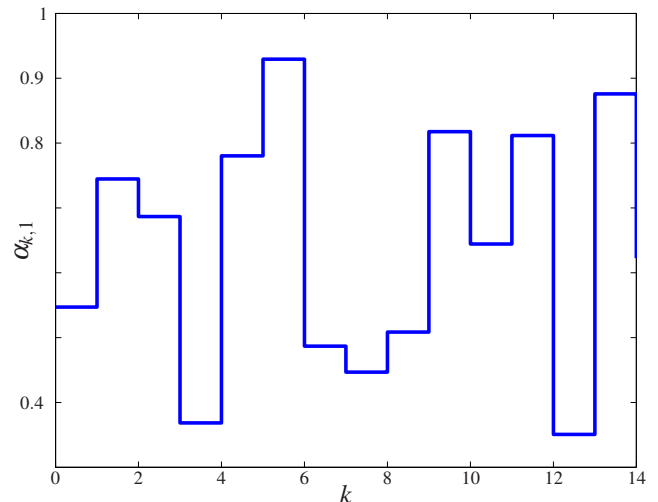


Fig. 2. Temporal evolution of the time-varying parameter $\alpha_{k,1}$.

Table 1. Maximum values for θ , number of scalar decision variables N_V , and number of LMI rows N_R when considering different values of N in Theorem 4 and in Corollary 6.

Theorem 4				
N	2	3	4	5
θ_{Max}	0.7413	0.7430	0.7430	0.7430
N_V	6	9	12	15
N_R	84	504	3024	18144
Corollary 6				
N	2	3	4	5
θ_{Max}	0.7547	0.7714	0.7723	0.7723
N_V	12	18	24	30
N_R	144	864	5184	31104

It can be seen that higher values of N provide less conservative results, notably when using Corollary 6. However, the best results come with a greater computational burden. The technique (Hetel et al., 2006b, Theorem 3) is also applied to the current example, resulting in $N_V = 28$, $N_R = 64$ and $\theta_{Max} = 0.7548$. In this sense, the method presented in Corollary 6 is able to assess the stability with a broader interval for the uncertainty β with a smaller number of scalar decision variables.

5. CONCLUSIONS

New stability conditions for discrete-time LPV switched systems have been proposed in this paper. The system is supposed to be affected by arbitrary switching, where each mode depends on time-varying parameters lying within a polytopic domain. The proposed conditions stem from the application of Lyapunov functions depending not only on the current states, but also on shifted states. Numerical experiments illustrate the advantages of the proposed method, which is capable of certifying the stability of LPV switched systems by using less variables than other techniques from the literature. Additionally, the proposed Lyapunov function may depend on an arbitrary number of shifted states, and increasing such number leads to less conservative conditions, as shown in the experiments. As

future research the authors are investigating the stabilization problem for LPV switched systems.

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