Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Sum-of-Squares based computation of a Lyapunov function for proving stability of a satellite with electromagnetic actuation ⋆

Rahul Misra Rafał Wisniewski Özkan Karabacak *

* *Department of Electronic Systems, Automation and Control, Aalborg University, Fredrik Bajers Vej 7 C, 9220 Aalborg East, Denmark (e-mail: {rmi, raf, ozk}@ es.aau.dk).*

Abstract: This work focuses on the computation of a candidate Lyapunov function for a Low Earth Orbit satellite which is actuated using only magnetorquers. A satellite having only electromagnetic actuation is not controllable when the magnetic moment produced by the magnetorquers is parallel to the geomagnetic field. Further, the dynamics of the system are periodic due to the periodic nature of the geomagnetic field. Previously, a locally stable Proportional-Derivative control has been designed for such a satellite. In this work, we have found a polynomial candidate Lyapunov function for the resultant closed loop system using Sum-of-Squares (SoS) polynomials and Putinar's Positivstellensatz. Unlike previous applications of SoS techniques on rigid bodies, the kinematics have been defined using unit quaternions. The unit quaternions have a well-known advantage of being a singularity free representation of attitude kinematics with only a single constraint. The unit quaternion constraint has been ensured using semialgebraic sets. Furthermore, special emphasis has been placed on the verification of the candidate Lyapunov function and we have simulated the closed loop system with the candidate Lyapunov function.

Keywords: Nonlinear systems, Sum-of-Squares, Putinar's Positivstellensatz, quaternions, satellite, electromagnetic actuation, periodic systems

1. INTRODUCTION

In recent years, many control theorists have proposed using Sum-of-Squares (SoS) polynomials as a possible candidate Lyapunov function for proving stability of the solutions of dynamical systems (see Parrilo (2000), Papachristodoulou and Prajna (2005) and Sloth (2016)). This is due to the fact that checking global non-negativity of an arbitrary polynomial is an NP-hard problem (see Parrilo (2000)) and one way to avoid this is to search for an SoS polynomial (instead of an arbitrary non-negative polynomial) which is guaranteed to be non-negative. Furthermore, the problem of searching for a SoS polynomial can be posed as an semidefinite program (SDP) which can then be solved using SDP solvers like MOSEK or SeDuMi. Thus, we can relax the strict condition of non-negativity with an SoS condition which is referred to as an SoS relaxation. Suitable MATLAB based toolboxes have been developed for converting SoS problems into a standard SDP such as SOSTOOLS and YALMIP.

It is important to note that for multivariate polynomials, there can exist non-negative polynomials which are not SoS. This is relevant in the study of dynamical systems because a dynamical system having many states will have a multivariate polynomial candidate Lyapunov function, which is required to satisfy Lyapunov's stability theorem. Further, for a locally stable dynamical systems, we need to check whether the polynomial satisfies Lyapunov's stability theorem over the considered domain. Thus, we need representation theorems for writing polynomials non-negative on a semialgebraic set. Such theorems are referred to as Positivstellensatz in the literature (see Lasserre (2010)) and one such theorem is Putinar's Positivstellensatz which we have used in this work. The above methodology has been applied to a variety of dynamical systems such as motion planning for robots in Tedrake et al. (2010) and Van der Pol equations in Tan (2006).

Computing candidate Lyapunov function for a satellite having only electromagnetic actuation using the above methodology is non-trivial due to the periodic nature of the dynamical system and the lack of controllability whenever the satellite magnetic moment and geomagnetic field are parallel. Further, we have used unit quaternions which provide a singularity-free representation of kinematics at the cost of having to ensure the unit quaternion constraint. Previously such satellites have been studied in Martel et al. (1988) and Byrnes and Isidori (1991). The book Markley and Crassidis (2014) provides a good introduction to fundamental topics on the subject. This work is based on attitude control design for Ørsted satellite which was studied in Wisniewski (1996). SoS and Positivstellensatz have previously been applied for proving stability of satellite with kinematics defined using Modified Rodrigues Parameters instead of quaternions in Tobenkin et al. (2011). Modified Rodrigues Parameters

⋆ This work has been supported by the Independent Research Fund Denmark in the project DeBaTe.

are a projection of quaternions in $\mathbb{R}^3$ and they are not sufficient for a singularity free global representation of attitude. This is because global representation of attitude requires minimum 4 parameters Wertz (1978). As an alternative approach, the quaternion constraint can also be maintained by using Stengle's Positivstellensatz which can handle equality constraints. However, the resulting SDP may contain constraints in the form of a bilinear matrix inequality instead of constraints in the form of a standard linear matrix inequality. Proposition 2.1 in Tan (2006) mentions a relaxation for avoiding bilinear matrix inequality with Stengle's Positivstellensatz. In this work, however the authors focus on Putinar's Positivstellensatz instead.

To the best of the authors knowledge, the aforementioned methodology for finding candidate Lyapunov functions has not yet been applied on satellites with kinematics defined by unit quaternions and the goal of this work is to demonstrate that this can be done by carefully constructing suitable semialgebraic sets. We have also emphasized the verification of candidate Lyapunov function firstly, by checking whether the constraints of the SDP (formulated when searching for a candidate Lyapunov function) has been satisfied and secondly, by simulating the closed loop dynamical system with the obtained candidate Lyapunov function. The rest of the paper is organized as follows: The next section formally introduces the theorems used in this work. The third section defines the coordinate system and the model used for the satellite. Thereafter, we compute the candidate Lyapunov function in the fourth section. In the fifth section, we present the simulation results and lastly, we state concluding remarks.

2. PRELIMINARIES

In this section we introduce SoS polynomials, semialgebraic sets, Putinar's Positivstellensatz and Lyapunov's stability theorem for nonautonomous system.

2.1 SoS polynomials

Consider the state variables $X \in \mathbb{R}^n$. It should be noted that when we write a polynomial as an algebraic object, we will use X to denote variables but if we want to write a real vector, we will use x. Let $v_d(X)$ be a vector of monomials $v_d(X) = [1, X_1, X_2, .., X_n, X_1^2, X_1X_2, ...X_n^d]^T$ which can be generated upto a degree d. The dimension of this vector is $s(d) := \binom{n+d}{d}$. Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables X. A polynomial $p \in \mathbb{R}[X]$ is said to be SoS if it can be written as

$$p(X) = \Sigma_j p_j(X)^2, \text{ where } p_j(X) \in \mathbb{R}[X]. \tag{1}$$

A polynomial p has a SoS decomposition if and only if there exists a real, symmetric and positive semidefinite matrix $Q \in \mathbb{R}^{s(d)\times s(d)}$ such that

$$p(X) = v_d(X)^T Q v_d(X) \quad \forall X \in \mathbb{R}^n. \tag{2}$$

The problem of finding an SoS decomposition can be posed as the feasibility of the following SDP

$$\begin{aligned} \text{Find} \quad & Q \in \mathbb{R}^{s(d)\times s(d)} && \text{(3a)}\\ \text{such that} \quad & Q = Q^T, && \text{(3b)}\\ & Q \succcurlyeq 0, && \text{(3c)}\\ & trace(Q v_d v_d^T) = p. && \text{(3d)} \end{aligned}$$

2.2 Semialgebraic sets and Putinar's Positivstellensatz

Let

$$K := \{X \in \mathbb{R}^n : g_j(X) \geq 0, j = 1, \cdots, m\} \tag{4}$$

be a semialgebraic set, where $g_1, \cdots, g_m \in \mathbb{R}[X]$. We shall now define Putinar's Positivstellensatz as per Putinar (1993) and later use it for obtaining polynomial certificates of positivity on (4). We begin by defining quadratic modules $Q(g_1, \cdots, g_m)$ associated with $g_j \subset \mathbb{R}[X]$ as follows,

$$Q(g_1, \cdots, g_m) := \Big\{ q_0 + \sum_{j=1}^{m} q_j g_j : (q_j)_{j=0}^m \subset \Sigma[X] \Big\} \tag{5}$$

where $\Sigma[X]$ represents the set of SoS polynomials.

Assumption 2.1 There exists $u \in Q(g)$ such that the level set $\{X \in \mathbb{R}^n : u(X) \geq 0\}$ is compact.

Based on Assumption 2.1, we can state Putinar's Positivstellensatz as follows.

Theorem 1. Let the Assumption 2.1 hold. If $F \in \mathbb{R}[X]$ is strictly positive on K, then $F \in Q(g)$ that is,

$$F = s_0 + \sum_{j=1}^{m} s_j g_j \tag{6}$$

for some SoS polynomials $s_j \in \Sigma[X]$, $j = 0, 1, \cdots, m$.

2.3 Lyapunov stability theory for non-autonomous system

Consider the following non-autonomous system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2(t)), && \text{(7a)}\\ \dot{x}_2 &= f_2(x_2(t)), && \text{(7b)} \end{aligned}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $f_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^m \to \mathbb{R}^m$.

Theorem 2. Consider a solution of 7b, $\bar{x}_2(t)$ with $\bar{x}_2(0) = x_{2,0}$ and assume that $\bar{x}_1(t) \equiv 0$ is an equilibrium solution of (7a) for $x_2(t) = \bar{x}_2(t)$, i.e., $f_1(0, x_2) = 0 \forall x_2 \in \{\bar{x}_2(t)\}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x_1 = 0$ and suppose there exists $V : \mathcal{D} \to \mathbb{R}_0^+$ such that

$$\begin{aligned} k_1 \left\|x_1\right\|^2 \leq V(x_1) &\leq k_2 \left\|x_1\right\|^2 && \text{(8a)}\\ \frac{\partial V}{\partial x} f_1(x_1, x_2) &\leq -k_3 \left\|x_1\right\|^2 && \text{(8b)} \end{aligned}$$

for all $x_1 \in \mathcal{D}$ and $x_2 \in \{\bar{x}_2(t)\}$, where k_1, k_2, k_3 are positive constants. Then, for sufficiently small $x_{1,0}$, the solution of (7) for $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$ satisfies $x_1(0) \to 0$.

Proof. Consider the following non-autonomous system $\dot{x}_1 = f_1(x, \bar{x}_2(t)) =: f(t,x)$, $x \in \mathbb{R}^n$ and apply Theorem 7.5, Definition 7.1 and Definition 3.48 from Kloeden and Rasmussen (2011), while choosing a time invariant Lyapunov function $V(t,x) := V(x)$.

2.4 SoS program for finding candidate Lyapunov functions

Using Theorems 1 and 2, the following SoS program can be written for finding candidate Lyapunov functions valid in $\mathcal{D}$.

$$\text{Find} \quad V(x) \tag{9a}$$
$$\text{such that} \quad V(0) = 0, \tag{9b}$$
$$V(x) - \sum_{j=1}^{m} s_j g_j - k_1 \|x_1\|^2 \in \Sigma[x], \tag{9c}$$
$$- V(x) - \sum_{j=1}^{m} s_j g_j + k_2 \|x_1\|^2 \in \Sigma[x], \tag{9d}$$
$$- \frac{\partial V(x)}{\partial x} f(x) - \sum_{j=1}^{m} s_j g_j - k_3 \|x_1\|^2 \in \Sigma[x], \tag{9e}$$

where $\Sigma[x]$ represent SoS polynomials, $s_1, \cdots s_m$ generated upto degree of the candidate Lyapunov function and g represents the polynomial inequalities which generate K. The scalars k_1, k_2 and k_3 ensure that the polynomial remains positive over K (Putinar's Positivstellensatz alone ensures only the non-negativity of the polynomial over K).

3. MODELING OF ØRSTED SATELLITE

We begin by stating the general coordinate system (CS) and notation for the satellite.

Table 1. Coordinate system

Acronym	Description
BCS	CS built on principal axes
OCS	reference CS fixed in orbit
WCS	inertial right orthogonal CS

3.1 Kinematics

The attitude of the satellite is given as the orientation of BCS with respect to OCS. We have used unit quaternions for representing kinematics as follows

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2} {}^c\mathbf{\Omega}_{\text{co}} q_4 + \frac{1}{2} {}^c\mathbf{\Omega}_{\text{co}} \times \mathbf{q} \\ \dot{q_4} &= -\frac{1}{2} {}^c\mathbf{\Omega}_{\text{co}} \cdot \mathbf{q} \end{aligned} \tag{10}$$

The scalar part q_4 of attitude quaternion ${}^c_o\mathbf{q}$ is not unique but constrained as follows

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1. \tag{11}$$

The angular velocity in OCS is related to the angular velocity in WCS as follows

$${}^c\mathbf{\Omega}_{\text{co}} = {}^c\mathbf{\Omega}_{\text{cw}} - \omega_0 {}^c\mathbf{i}_o. \tag{12}$$

Table 2. Notation

Symbol	Description
${}^c v, {}^o v, {}^w v$	resolved in BCS, OCS, WCS
${}^c\mathbf{\Omega}_{\text{cw}}$	angular velocity of BCS w.r.t. WCS
${}^c\mathbf{\Omega}_{\text{co}}$	angular velocity of BCS w.r.t. OCS
ω_0	orbital rate
${}^{\text{cw}}\omega_x, {}^{\text{cw}}\omega_y, {}^{\text{cw}}\omega_z$	components of $\mathbf{\Omega}_{\text{cw}}$ about x, y and z axis respectively
I	inertia tensor of the satellite
I_x, I_y, I_z	moments of inertia about x, y and z axis respectively
$\mathbf{N}_{\text{ctrl}}$	control torque
$\mathbf{N}_{\text{gg}}$	gravity gradient torque
$\mathbf{m}$	magnetic moment used as control signal
$\mathbf{B}$	geomagnetic field vector
$\mathbf{B}_{\text{mag}}$	states of harmonic oscillator used for dipole model
v'	true anomaly measured from the ascending node
θ'_m	coelevation of the dipole model
ϕ'_m	east longitude of the dipole model
α_m	right ascension of the dipole model
a	equatorial radius of earth
r	geocentric distance of the satellite from center of the earth
g_1^0, g_1^1, h_1^1	gaussian coefficients from international geomagnetic reference field (IGRF) (2015)
${}^c_o\mathbf{q}$	attitude quaternion representing rotation of BCS relative to OCS
$\mathbf{q}, q_4$	vector part $\mathbf{q} = [q_1, q_2, q_3]$ and scalar part of ${}^c_o\mathbf{q}$
$\mathbf{i}_o, \mathbf{j}_o, \mathbf{k}_o$	unit vector along x, y and z-axis of OCS

3.2 Dynamics

The dynamics describe the relation between the satellite's angular momentum with the torques acting on the spacecraft.

$$\mathbf{I}{}^c\dot{\mathbf{\Omega}}_{\text{cw}}(t) = -{}^c\mathbf{\Omega}_{\text{cw}}(t) \times \mathbf{I}{}^c\mathbf{\Omega}_{\text{cw}}(t) + {}^c\mathbf{N}_{\text{ctrl}}(t) + {}^c\mathbf{N}_{\text{gg}}(t). \tag{13}$$

As all reference commands given to the satellite are in orbit frame, we are concerned with the dynamics of the satellite in orbit frame instead of control frame and they can be stated as follows

$$\begin{aligned} \mathbf{I}{}^c\dot{\mathbf{\Omega}}_{\text{co}}(t) = -{}^c\mathbf{\Omega}_{\text{cw}}(t) \times \mathbf{I}{}^c\mathbf{\Omega}_{\text{cw}}(t) + {}^c\mathbf{N}_{\text{ctrl}}(t) + \\ {}^c\mathbf{N}_{\text{gg}}(t) - \omega_0 \mathbf{I}{}^c\mathbf{\Omega}_{\text{co}}(t) \times {}^c\mathbf{i}_o. \end{aligned} \tag{14}$$

The magnetic control torque is generated as follows

$${}^c\mathbf{N}_{\text{ctrl}}(t) = {}^c\mathbf{m}(t) \times {}^c\mathbf{B}(t). \tag{15}$$

The geomagnetic field vector $\mathbf{B(t)}$ is obtained from the dipole model given in Wertz (1978). The cosine and sine components of the dipole model have been approximated by a harmonic oscillator for preserving the polynomial structure of the system. This is done specifically for application of SoS methodology. The harmonic oscillator is a 2^{nd} order differential equation which is as follows

$$\ddot{\mathbf{B}}_{\text{mag}} + \omega^2 \mathbf{B}_{\text{mag}} = 0, \tag{16}$$

where ω is frequency of the oscillations and B_{mag} are the oscillating states which shall be used later. It can be observed that the initial value of (16) changes the dipole model and consequently the dynamics. The dipole model in orbital frame can be stated as follows

$$\begin{aligned} {}^{o}\mathbf{B}_x &= \frac{a^3 H_0}{2R^3}\sin\theta'_m[3\cos(2v' - \alpha_m) + \cos\alpha_m], \\ {}^{o}\mathbf{B}_y &= \frac{a^3 H_0}{2R^3}\sin\theta'_m[3\sin(2v' - \alpha_m) + \sin\alpha_m], \\ {}^{o}\mathbf{B}_z &= -\frac{a^3 H_0}{2R^3}\cos\theta'_m, \end{aligned} \tag{17}$$

where $H_0 = [g_1^{0^2} + g_1^{1^2} + h_1^{1^2}]^{1/2}$ and $\alpha_m \approx \phi'_m$. Further, θ'_m and ϕ'_m are calculated as follows

$$\begin{aligned} \theta'_m &= \text{arc cos}\left(\frac{g_1^0}{H_0}\right), \\ \phi'_m &= \text{arc tan}\left(\frac{h_1^1}{g_1^1}\right). \end{aligned} \tag{18}$$

The solution of (16) approximates the cosine and sine components due to $2v' - \alpha_m$. Additional harmonic oscillators can be added for more accurate approximations at the cost of adding extra states to the system. The gravity gradient torque is also considered in the model and it can be stated as follows

$${}^{c}\mathbf{N}_{\text{gg}}(t) = 3\omega_0^2({}^{c}\mathbf{k}_o \times \mathbf{I}{}^{c}\mathbf{k}_o). \tag{19}$$

3.3 Proportional-Derivative (PD) controller

The existing PD control law is a conventional PD controller which takes into account that the control torque cannot be generated parallel to the geomagnetic field. The geomagnetic field is changing with time and attitude of the satellite and this controller is proven to be stable in Wisniewski and Blanke (1996). Let

$$\mathbf{m}(t) = {}^{c}\mathbf{B}(t) \times K_d {}^{c}\mathbf{\Omega}_{\text{co}} - {}^{c}\mathbf{B}(t) \times K_p \mathbf{q}, \tag{20}$$

where ${}^{c}\mathbf{B}(t)$ is the geomagnetic field in BCS, K_p is the proportional gain and K_d is the derivative gain. Due to the control law (20), the system (i.e. (10) and (14)) has an equilibrium point at $({}^{c}\mathbf{\Omega}_{\text{co}}, {}^{c}\mathbf{k}_o, {}^{c}\mathbf{i}_o) = (0, {}^{c}\mathbf{k}_o, {}^{c}\mathbf{i}_o))$. The attitude quaternion corresponding to this point can be either ${}^{c}_{o}q = [0, 0, 0, +1]$ or ${}^{c}_{o}q = [0, 0, 0, -1]$. This is due to the existence of a mapping $A : \mathbf{S}^3 \to SO_3(\mathbf{R})$ which maps $[0, 0, 0, +1]$ and $[0, 0, 0, -1]$ from $\mathbf{S}^3$ to same point on $SO_3(\mathbf{R})$. It should be noted that, we are concerned only with the stability of the state variables ${}^{c}\mathbf{\Omega}_{\text{co}}$ and $\mathbf{q}$ because value of q_4 will be automatically decided due to (11).

4. COMPUTATION OF CANDIDATE LYAPUNOV FUNCTION

We now apply the SoS procedure as per (9). It should be noted that the satellites kinematics (10) and dynamics (14) are represented by (7a) and the harmonic oscillator (16) for generating geomagnetic field vector are represented by (7b) in Theorem 2 defined previously. Firstly, we define the semialgebraic set using polynomial inequalities and the corresponding SoS constraints. Thereafter, we shall present the result obtained after solving the SDP. The semialgebraic set is constructed as follows

$$\begin{aligned} K =& \{x \in \mathbb{R}^8 : g_j(x) \geq 0, j = 1, \cdots, 8\}, && (21a) \\ \text{where} \quad g_1 =& \|{}^{c}_{o}q\|^2 - 1 + \epsilon, && (21b) \\ g_2 =& 1 - \|{}^{c}_{o}q\|^2 + \epsilon, && (21c) \\ g_3 =& -(q_4 + 1)(q_4 - 1 + \epsilon), && (21d) \\ g_4 =& -{}^{\text{co}}\omega_x^2 + 0.0025, && (21e) \\ g_5 =& -{}^{\text{co}}\omega_y^2 + 0.0025, && (21f) \\ g_6 =& -{}^{\text{co}}\omega_z^2 + 0.0025, && (21g) \\ g_7 =& B_{mag_1}^2 + B_{mag_2}^2 - 0.9, && (21h) \\ g_8 =& 1.1 - (B_{mag_1}^2 + B_{mag_2}^2). && (21i) \end{aligned}$$

In the above eq. (21), B_{mag_1}, B_{mag_2} are components of (16) and ϵ is a small number ($\approx 10^{-8}$). The semialgebraic set defines the region of state space where the candidate Lyapunov function is valid, i.e. the polynomial obtained after solving the SDP (9) satisfies the constraints (9b), (9c), (9d) and (9e). The polynomial inequalities $g_1, \cdots, g_6 \geq 0$ define the domain $\mathcal{D} \subset \mathbb{R}^n$ and the solution $\bar{x}_2(t)$ (see Theorem 2) is contained in the set defined by polynomial inequalities $g_7 \geq 0$ and $g_8 \geq 0$.

It is important to note that the quaternion constraint (11) is an equality constraint which is satisfied via polynomial inequalities $g_1 \geq 0, g_2 \geq 0$. This effectively ensures that q_4 is chosen such that the attitude quaternion ${}^{c}_{o}\mathbf{q}$ is constrained on the unit sphere (within a tolerance of ϵ). Consequently, the polynomial multipliers associated with g_1 and g_2 need not be constrained to be SoS. The polynomial inequality $g_3 \geq 0$ ensures that q_4 can only attain -1 and cannot attain $+1$ as both are equilibrium points. Lastly, the polynomial inequalities $g_4 \geq 0, g_5 \geq 0$ and $g_6 \geq 0$ ensure that the x, y and z components of angular velocity vector ${}^{c}\mathbf{\Omega}_{\text{co}}$ stay within $[-0.05, 0.05]$. Finally, it should be noted that the equality constraint (9b) has been implemented by setting the coefficients of x^0, in the vector of monomials $v_d(x)$ (specified for the candidate Lyapunov function) to zero prior to solving the SDP (9).

4.1 Computation result and Verification

Using (9), we have obtained the following candidate Lyapunov function.

$$\begin{aligned} &V^*(q_1, q_2, q_3, {}^{\text{co}}\omega_x, {}^{\text{co}}\omega_y, {}^{\text{co}}\omega_z) = 0.008680q_1^2 \\ &+ 0.0084885q_2^2 + 0.0134243q_3^2 + 0.00000106q_1q_2 \\ &- 0.0000756q_1q_3 + 0.0002412q_2q_3 - 0.0175008q_1{}^{\text{co}}\omega_x \\ &- 0.0175008q_2{}^{\text{co}}\omega_y - 0.0326737q_3{}^{\text{co}}\omega_z + 0.1283401{}^{\text{co}}\omega_x^2 \\ &+ 0.11860602{}^{\text{co}}\omega_y^2 + 0.0.03049558{}^{\text{co}}\omega_z^2. \end{aligned} \tag{22}$$

We have used the solver MOSEK which was interfaced by the YALMIP toolbox (see Löfberg (2009) and MOSEK (2016)). For this computation, $k_1 \approx 10^{-10}$, $k_2 \approx 10^{-5}$ and $k_3 \approx 10^{-6}$.

The standard method for verifying an SoS polynomial is by checking the error residuals if the primal problem has been solved or by checking the positive definiteness of the Q matrix if the dual problem has been solved. These methods are described in Löfberg (2009). Both of these procedures, essentially require us to check whether solution of SDP (9) satisfies all the SoS constraints specified. The obtained candidate Lyapunov function (22) satisfies the aforementioned constraints. However, it is desired to verify it in practice by simulation and the results are presented in the following section.

5. SIMULATION RESULTS

The closed loop model of Ørsted satellite has been simulated for an orbit time of 6000 sec and orbit height of 800 km. The satellite has the following moments of inertia when the boom is stowed $I_x = 3.428$, $I_y = 2.904$, $I_z = 1.28$. The simulations have been done for 10 orbits which is equivalent to 60000 sec. The initial angular velocity is ${}^{\text{co}}\omega_x = 0.0001$, ${}^{\text{co}}\omega_y = 0.0001$, ${}^{\text{co}}\omega_z = 0.0001$ m/sec. The initial quaternion states are $q_1 = 0.2517$, $q_2 = 0.2517$, $q_3 = 0.2517$ and $q_4 = -0.9$. We begin by simulating the dipole model of the geomagnetic field using (16) for approximating it as discussed previously. We have used IGRF (2015) coefficients from Thébault et al. (2015) and the initial phase of $\mathbf{B}_{mag}$ has been considered to be 45°

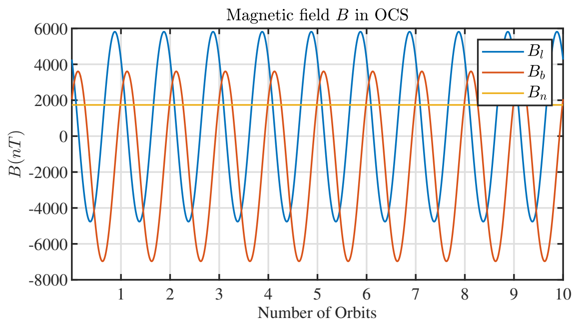

Fig. 1. Geomagnetic field as per IGRF (2015)

The quaternion states can be seen converging to the equilibrium point as follows

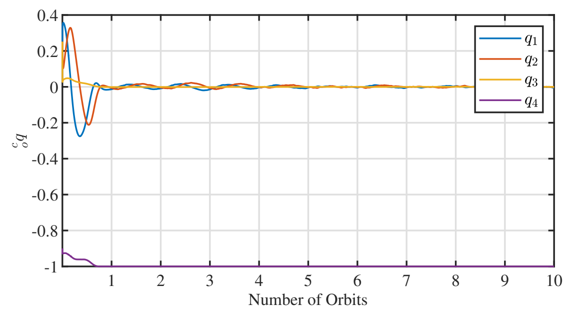

Fig. 2. ${}^c_o q$ converging to the equilibrium point

The angular velocity states can be seen converging to the equilibrium point in fig. 3.

We have also simulated the polynomial inequalities $g_1, \cdots, g_5 \geq 0$. From fig. 5, it can be seen that the quaternion constraint $g_1 \geq 0$ is violated slightly with the maximum violation being of the order of 10^{-6}.

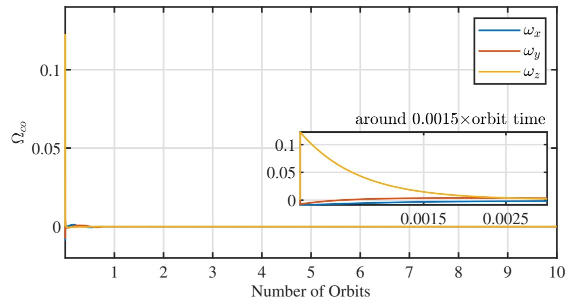

Fig. 3. ${}^c\mathbf{\Omega}_{\text{co}}$ converging to the equilibrium point

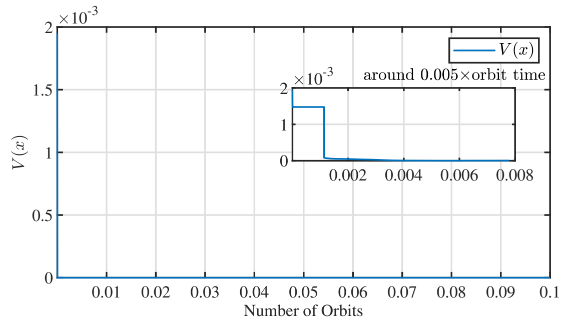

Fig. 4. Candidate Lyapunov function

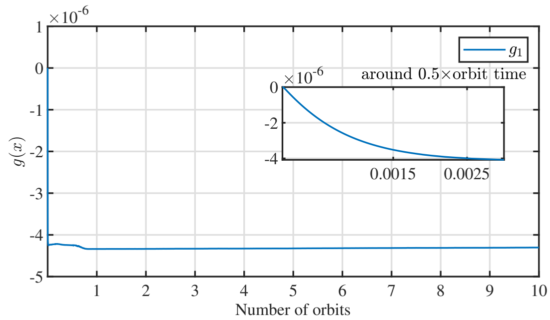

Fig. 5. Simulation of $g_1 = \|{}^c_o q\|^2 - 1$

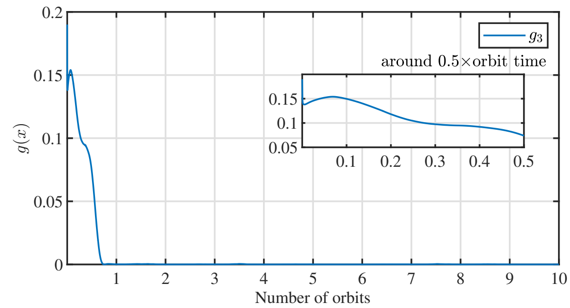

Fig. 6. Simulation of $g_3 = -(q_4 + 1)(q_4 - 1 + \epsilon) \geq 0$

From fig. 6, it can be seen that $g_3 \geq 0$ is satisfied throughout the simulation and this ensures uniqueness of q_4 as discussed previously. Furthermore, $g_4 \geq 0$ and $g_5 \geq 0$ are always satisfied. This means that the x and y components of ${}^c\mathbf{\Omega}_{\text{co}}$ always stay within $[-0.05, 0.05]$ m/sec. Lastly, from fig. 3, it can be seen that despite the state ${}^{\text{co}}\omega_z$ exceeding 0.05 m/sec in the beginning of the simulation, it still quickly converges to the origin. This is expected since the gravity gradient torque acts in the z

direction and therefore, the controller takes comparatively more time to stabilize in the z direction.

6. CONCLUSION

The aim of this work was to compute a candidate Lyapunov function for the satellite having only electromagnetic actuation while maintaining the quaternion constraint. This was done successfully and verified by simulations.

Putinar's Positivstellensatz is computationally efficient (compared to other Positivstellensatz) however, the size of the SDP grows rapidly with the number of states n and degree d of the candidate Lyapunov function (worst-case bound being n^d). In future, we would like to exploit the inherent sparsity in the system dynamics, for finding candidate Lyapunov functions such that the size of SDP can be reduced.

REFERENCES

Byrnes, C.I. and Isidori, A. (1991). On the attitude stabilization of rigid spacecraft. *Automatica*, 27(1), 87–95.

Kloeden, P.E. and Rasmussen, M. (2011). *Nonautonomous dynamical systems*. 176. American Mathematical Soc.

Lasserre, J.B. (2010). *Moments, positive polynomials and their applications*, volume 1. World Scientific.

Löfberg, J. (2009). Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5), 1007–1011.

Markley, F.L. and Crassidis, J.L. (2014). *Fundamentals of spacecraft attitude determination and control*, volume 33. Springer.

Martel, F., Pal, P., and Psiaki, M. (1988). Active magnetic control system for gravity gradient stabilized spacecraft.

MOSEK, A. (2016). Modeling cookbook.

Papachristodoulou, A. and Prajna, S. (2005). A tutorial on sum of squares techniques for systems analysis. In *Proceedings of the 2005, American Control Conference, 2005.*, 2686–2700. IEEE.

Parrilo, P.A. (2000). *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. Ph.D. thesis, California Institute of Technology.

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42(3), 969–984.

Sloth, C. (2016). On the computation of lyapunov functions for interconnected systems. In *2016 IEEE Conference on Computer Aided Control System Design (CACSD)*, 850–855. IEEE.

Tan, W. (2006). *Nonlinear Control Analysis and Synthesis using Sum-of-Square Programming*. Ph.D. thesis, University of California, Berkeley Berkeley, CA.

Tedrake, R., Manchester, I.R., Tobenkin, M., and Roberts, J.W. (2010). Lqr-trees: Feedback motion planning via sums-of-squares verification. *The International Journal of Robotics Research*, 29(8), 1038–1052.

Thébault, E., Finlay, C.C., Beggan, C.D., Alken, P., Aubert, J., Barrois, O., Bertrand, F., Bondar, T., Boness, A., Brocco, L., et al. (2015). International geomagnetic reference field: the 12th generation. *Earth, Planets and Space*, 67(1), 79.

Tobenkin, M.M., Manchester, I.R., and Tedrake, R. (2011). Invariant funnels around trajectories using sum-of-squares programming. *IFAC Proceedings Volumes*, 44(1), 9218–9223.

Wertz, J.R. (1978). *Spacecraft Attitude Determination and Control*, volume 73. Springer Science & Business Media.

Wisniewski, R. (1996). *Satellite attitude control using only electromagnetic actuation*. Ph.D. thesis, Aalborg University. Department of Control Engineering.

Wisniewski, R. and Blanke, M. (1996). Threeaxis satellite attitude control based on magnetic torquing. In *13th IFAC World Congress*, 1–36.