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Fast Stochastic Model Predictive Control of Unstable Dynamical Systems

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Abstract: Fast stochastic model predictive control (FSMPC) is a multivariable control algorithm that explicitly takes constraints and probabilistic parametric uncertainties into account while having low online computational cost for dynamical systems of high state dimension. This article extends FSMPC to be applicable to model uncertainty descriptions that include unstable dynamical systems. The proposed control structure, which embeds output feedback into past FSMPC formulations, is illustrated in a numerical example. Two different options for designing the embedded output feedback are compared and discussed.

Keywords: Predictive control, probabilistic robustness, control problems under conflict and/or uncertainties, control of constrained systems, process control, constrained control

1. INTRODUCTION

Model predictive control (MPC) is heavily used in industry to control multivariable processes, due to its ability to explicitly take constraints into account. The uncertainties that arise when modeling manufacturing systems have motivated extensions of MPC algorithms. The objective of *robust MPC* formulations is to optimize the worst-case scenario, which can result in sluggish closed-loop performance even in cases in which the worst-case behavior has vanishingly small probability of occurrence. The alternative approach of *stochastic MPC* explicitly takes probability distributions of the uncertain model parameters into account, which enables the optimization of the distribution of the controlled process outputs (Mesbah, 2016).

Another consideration when formulating MPC algorithms is that most manufacturing systems have high state dimension, with many thousands of states being common. The large number of states relative to the small number of inputs and outputs (typically on the orders of tens) means that the states in models for manufacturing systems are rarely observable, which makes the implementation of state-space MPC algorithms infeasible in practice. In industrial implementations, this situation is addressed by using input-output model formulations. For example, an MPC formulation widely applied in industrial manufacturing systems is dynamic matrix control (DMC), which was first developed at Shell Oil Company (Cutler and Ramaker, 1979). DMC uses an input-output step response model to optimize the control moves and the computational cost scales with the number of inputs and outputs. This is typically much lower than the number of system states, leading to low on-line computational cost, even for process systems of high state dimension.

Past work extended the DMC algorithm to explicitly take probabilistic parametric uncertainty into account (Paulson et al., 2014, 2018). The fast stochastic MPC (FSMPC) algorithm based on a realistic model of a manufacturing system with approximately 8000 states had an on-line computational cost of less than one second per sampling instance due to the input-output model formulation, which was much less than the sampling time of one minute. Extensions to the FSMPC formulation were derived that were proven to have zero steady-state error to step changes in the setpoints and disturbances (von Andrian and Braatz, 2019). The FSMPC formulations required that the step response coefficients are bounded for *all* realizations of uncertain model parameters, in other words, all models within the uncertainty set were required to be asymptotically stable. This assumption holds for most manufacturing systems, as the MPC algorithm is implemented on top of lower level regulatory control loops that have been designed to stabilize the individual unit operations. On the other hand, cases can arise in which the process model is unstable for some model parameter values within the probability distribution. Such cases can arise especially at the tails of the probability distributions for the model parameters. Despite having a very low probability of occurrence, the unstable system dynamics can dominate the step response model and make the model unusable in the design of FSMPC based on step response models.

This article proposes a combination of FSMPC with embedded output feedback to control systems in which the process model uncertainty set can include unstable dynamics. The article is organized as follows. A description of the control structure and synthesis of the FSMPC is followed by application to an example system with two different designs for the embedded controller. The article ends with a discussion and conclusion.

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2. METHODS

This article employs Polynomial Chaos Theory (PCT), which propagates parametric uncertainty through the process model to quantify the resulting uncertainty in process outputs (Xiu and Karniadakis, 2002). A stochastic MPC formulation with low on-line computational cost can be achieved by incorporating PCT to optimize the predicted expected value and variance of the outputs resulting from current and future control moves (Paulson et al., 2014; von Andrian and Braatz, 2019). The derivations of prior fast stochastic MPC (FSMPC) algorithms are available in these papers and are not repeated here due to space constraints. This section describes the changes made to extend the formulations to handle unstable process dynamics by embedding an additional output feedback within any of the past FSMPC formulations.

The structure of the FSMPC with output feedback is shown in Fig. 1. The control signal of the feedback controller u is complemented by an adjustment signal v, before being sent to the process as $u+v$. The adjustment signal v is calculated by the FSMPC to ensure constraint satisfaction while taking uncertainties into account. This article uses proportional integral (PI) control and unconstrained quadratic DMC (QDMC) as the feedback controller, although any other stabilizing output feedback controller could be used. The main idea behind this structure – combining MPC with other controllers – has been applied successfully with other MPC formulations (Lee and Park, 1991) to enable control of (partially) unstable processes with previously presented FSMPC.

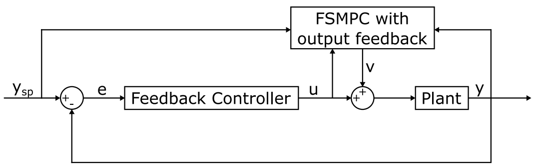

Fig. 1. Structure of FSMPC applicable to unstable systems, where y_{sp} is the setpoint, e is the control error, u is the feedback controller output, v is the feedback controller adjustment, and y is the process output.

At each time instance k, the optimization

$$\min_{\Delta\mathbf{v}(k)} \mathbf{v}(k)^\top \mathbf{W}_v \mathbf{v}(k) \tag{1}$$

$$\text{s.t.} \quad \mathbf{y}_{\min} \leq \mathbb{E}\left[\hat{\mathbf{y}}(k)\right] \leq \mathbf{y}_{\max}$$
$$\mathbf{u}_{\min} \leq \mathbb{E}\left[\hat{\mathbf{u}}(k)\right] + \mathbf{v}(k) \leq \mathbf{u}_{\max} \tag{2}$$
$$\Delta\mathbf{u}_{\min} \leq \mathbb{E}\left[\Delta\hat{\mathbf{u}}(k)\right] + \Delta\mathbf{v}(k) \leq \Delta\mathbf{u}_{\max},$$

is performed in the FSMPC, where

$$\mathbf{v}(k) \equiv \begin{bmatrix} v_1(k) \\ v_1(k+1) \\ \vdots \\ v_1(k+c-1) \\ v_2(k) \\ \vdots \\ v_{n_u}(k+c-1) \end{bmatrix} \in \mathbb{R}^{cn_u \times 1}$$

is the calculated input adjustment of all n_u inputs over the control horizon c at time instance k; $\mathbf{W}_v \in \mathbb{R}^{cn_u \times cn_u}$ is a diagonal matrix with positive weights selected according to the relative importance of each input; $\mathbf{y}_{\min}$ and $\mathbf{y}_{\max}$ $\in \mathbb{R}^{pn_y \times 1}$ are the output constraints of all n_y outputs over the prediction horizon p; $\mathbb{E}[\hat{\mathbf{y}}(k)] \in \mathbb{R}^{pn_y \times 1}$ is the expected value of all predicted process outputs over the prediction horizon with entries ordered analogously to $\mathbf{v}(k)$; $\mathbf{u}_{\min}$, $\mathbf{u}_{\max}$, $\Delta\mathbf{u}_{\min}$, $\Delta\mathbf{u}_{\max}$ are the constraints on the process inputs, which have to be fulfilled by the sum of expected value of predicted feedback controller output $\mathbb{E}[\hat{\mathbf{u}}(k)]$ and adjustment signal $\mathbf{v}(k)$ over the control horizon and the change of the respective signals $\mathbb{E}[\Delta\hat{\mathbf{u}}(k)] = \mathbb{E}[\hat{\mathbf{u}}(k)] - \mathbb{E}[\hat{\mathbf{u}}(k-1)]$ and $\Delta\mathbf{v} = \mathbf{v}(k) - \mathbf{v}(k-1)$, all $\in \mathbb{R}^{cn_u \times 1}$. The mathematical formulation implies that all setpoint tracking is done by the feedback controller and the FSMPC only ensures constraint satisfaction with minimal adjustment.

As with regular QDMC, at the heart of the FSMPC is a step response model of the Polynomial Chaos Expansion (PCE) coefficients, which provides information on how a step in controller input will influence process output. The formulation considers two input signals: the adjustment signal v and the desired process output y_{sp} which is treated as a measured disturbance in the FSMPC framework. For a step in each input, the corresponding output signals are recorded to form the step response model. In addition to the process output y, information about the feedback controller outputs u is needed to predict the future values, which are part of the constraints (2).

To derive the step response model in the original FSMPC formulation, PCE is applied to the uncertain process parameters $\boldsymbol{\theta}$, states $\mathbf{x}$, and outputs $\mathbf{y}$. Uncertainties are propagated through the process and feedback controller loop to the process outputs. PCE is applied additionally to the feedback controller input $\mathbf{e}$, possible feedback controller states, and the feedback controller output $\mathbf{u}$, which makes PCE also necessary for the process inputs.

The respective step response coefficients are stored in the matrices $\mathbf{G}_{v,y} \in \mathbb{R}^{pn_y \times cn_u}$, $\mathbf{G}_{v,u} \in \mathbb{R}^{cn_u \times cn_u}$, $\mathbf{G}_{ysp,y} \in \mathbb{R}^{pn_y \times pn_y}$, $\mathbf{G}_{ysp,u} \in \mathbb{R}^{cn_u \times pn_y}$ in the same triangular pattern as in QDMC. The matrix dimensions are given in terms of the number of input variables n_u, the number of output variables n_y, the length of the prediction horizon p, and the length of the control horizon c. In contrast to QDMC, a row of zeros is appended to the top of the matrices $\mathbf{G}_{v,u}$ and $\mathbf{G}_{ysp,u}$. These rows are inserted because at time instance k, from the perspective of the FSMPC, the feedback controller output $\mathbf{u}(k)$ is already calculated and only the next $c-1$ values need to be predicted.

Analogously to QDMC, the optimization (1)–(2) can be formulated as a quadratic program with linear constraints. The objective function (1) is expressed as a quadratic function in terms of the decision variables:

$$\min_{\Delta\mathbf{v}} \mathbf{v}^\top \mathbf{W}_v \mathbf{v} = \min_{\Delta\mathbf{v}} \frac{1}{2} \Delta\mathbf{v}^\top \mathbf{H} \Delta\mathbf{v} + \Delta\mathbf{v}^\top \mathbf{h}, \tag{3}$$

with the matrix defined by

$$\mathbf{H} \equiv (\mathbf{I}^{n_u} \otimes \mathbf{I}_L^c)^\top \mathbf{W}_v (\mathbf{I}^{n_u} \otimes \mathbf{I}_L^c) \in \mathbb{R}^{cn_u \times cn_u}, \tag{4}$$

where $\mathbf{I}$ is the identity matrix with the indicated size, $\mathbf{I}_L$ is a square lower triangular matrix with the indicated size with ones on and below the main diagonal, and $\otimes$ is the Kronecker product, and the vector defined by

$$\mathbf{h} \equiv (\mathbf{I}^{n_u} \otimes \mathbf{I}_L^c)^\top \mathbf{W}_v (\mathbf{I}^{n_u} \otimes \mathbf{1}^c) \mathbf{v}(k-1) \in \mathbb{R}^{cn_u \times 1}, \tag{5}$$

where $\mathbf{1}$ is a vector of ones with the indicated length, and $\mathbf{v}(k-1) \in \mathbb{R}^{n_u \times 1}$ is the vector of past input adjustments.

The expected value of the future process outputs $\mathbb{E}[\hat{\mathbf{y}}(k)]$ is predicted as a function of the decision variables $\Delta\mathbf{v}$, utilizing the step response matrices analogously to QDMC, with the difference that both, adjustment and setpoint signals are included. The expected value is equivalent to the 0^{th} PCE coefficient, with the 0 subscript omitted here for ease of notation. Note that

$$\hat{\mathbf{y}}(k) \equiv \begin{bmatrix} y_1(k+1|k) \\ y_1(k+2|k) \\ \vdots \\ y_1(k+p|k) \\ y_2(k+1|k) \\ \vdots \\ y_{ny}(k+p|k) \end{bmatrix} = (\mathbf{I}^{n_y} \otimes \mathbf{T})\mathbf{f}_y(k) + \tag{6}$$
$$\mathbf{G}_{v,y}\Delta\mathbf{v}(k) + \mathbf{G}_{ysp,y}\Delta\mathbf{y}_{sp}(k) + \mathbf{w}_y \in \mathbb{R}^{pn_y \times 1},$$

where the matrix $\mathbf{T} \in \mathbb{R}^{p\times n}$ shifts the free response $\mathbf{f}_y(k) \in \mathbb{R}^{n\times 1}$, as in QDMC. The integer n is the length of the step response model at which steady state is reached. In addition to the process output prediction, the expected value of the feedback controller output $\mathbb{E}[\hat{\mathbf{u}}(k)]$ is estimated into the future over the control horizon c to ensure constraint satisfaction. The PCE coefficient subscript 0 is omitted to simplify notation. Note that

$$\hat{\mathbf{u}}(k) \equiv \begin{bmatrix} u_1(k|k) \\ u_1(k+1|k) \\ \vdots \\ u_1(k+c-1|k) \\ u_2(k|k) \\ \vdots \\ u_{nu}(k+c-1|k) \end{bmatrix} = (\mathbf{I}^{n_u} \otimes \mathbf{T}_u)\mathbf{f}_u(k) + \tag{7}$$
$$\mathbf{G}_{v,u}\Delta\mathbf{v}(k) + \mathbf{G}_{ysp,u}\Delta\mathbf{y}_{sp}(k) + \mathbf{w}_u \in \mathbb{R}^{cn_u \times 1}.$$

For the prediction of the feedback controller output $\hat{\mathbf{u}}$, the free response $\mathbf{f}_u$ does not need to be shifted, only shortened, so the matrix $\mathbf{T}_u$ is an identity matrix concatenated with zeros: $\mathbf{T}_u \equiv [\mathbf{I}^c \;\; \mathbf{0}^{c\times(n-c)}] \in \mathbb{R}^{c\times n}$. For both $\hat{\mathbf{v}}$ and $\hat{\mathbf{u}}$, the free response is the respective prediction due to past input and setpoint changes, if no future input or setpoint changes occur. After the optimal trajectory of input adjustments is calculated, both free responses are updated, as in QDMC. The future setpoint changes over the prediction horizon is $\Delta\mathbf{y}_{sp}(k) \in \mathbb{R}^{pn_y\times 1}$ is, with the elements arranged in the same order as for $\hat{\mathbf{y}}(k)$. The bias terms $\mathbf{w}_y$ and $\mathbf{w}_u$ account for unmeasured disturbances and uncertainties, calculated as the difference between predicted and measured process and feedback controller outputs at time instance k, respectively. The bias is assumed to remain constant over the prediction horizon.

With (6)–(7), the constraints (2) can be written as linear inequalities in terms of the decision variables $\Delta\mathbf{v}$ as

$$\mathbf{A}\Delta\mathbf{v} \le \mathbf{b}, \tag{8}$$

with

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{G}_{v,y} \\ -\mathbf{G}_{v,y} \\ \mathbf{G}_{v,u} \\ -\mathbf{G}_{v,u} \\ (\mathbf{I}^{n_u} \otimes \mathbf{T}')\mathbf{G}_{v,u} + \mathbf{I}^{cn_u} \\ -(\mathbf{I}^{n_u} \otimes \mathbf{T}')\mathbf{G}_{v,u} - \mathbf{I}^{cn_u} \end{bmatrix} \in \mathbb{R}^{(2pn_y+4cn_u)\times cn_u},$$

$\mathbf{b} \in \mathbb{R}^{(2pn_y+4cn_u)\times 1}$ given in Fig. 2, $\mathbf{v}(k-1)$ and $\mathbf{u}(k-1)$ are the input adjustment and feedback controller output implemented at time $k-1$ respectively, $\mathbf{t}'$ is the vector $\mathbf{t}' \equiv [1\ 0\ 0 \ldots 0]^\top \in \mathbb{R}^{c\times 1}$, and

$$\mathbf{T}' \equiv \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{c\times c}.$$

To avoid the potential for infeasibility of the quadratic program (3) and (8), the output constraints are softened by adding slack variables (Ricker et al., 1988).

The difference between cascade control and the proposed control structure is that the FSMPC is sending a correction signal v directly to the process, rather than sending a setpoint to a lower level controller. To guarantee input (movement) constraint satisfaction, the sampling time of the FSMPC and the feedback controller are identical.

As in past FSMPC formulations, terms can be added to the objective function (1) to minimize not only the input adjustment v, but also to minimize the sum of the square of the expected value of the output error and/or the sum of the variance across different parameter realizations.

3. NUMERICAL EXAMPLE

Consider the single-input single-output system

$$\frac{dx}{dt} = -\theta x + u, \quad y = x, \tag{9}$$

with state x, time t, uncertain parameter θ, input u, and output y. The probabilistic parametric uncertainty of $\theta \sim \mathcal{U}(-0.5, 1.5)$ with nominal value $\theta_{\text{nom}} = 0.5$ results in a nominally stable system and the system is stable for $\theta > 0$ but is unstable for $\theta < 0$. Due to the instability for some values of the model parameter θ, FSMPC cannot be directly applied, because the step response of the expected value is dominated by the unstable system behavior and never reaches a steady state.

Applying a third-order PCE to the system gives the extended system for the PCE coefficients

$$\frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.5 & -1/3 & 0 & 0 \\ -1 & -0.5 & -0.4 & 0 \\ 0 & -2/3 & -0.5 & -3/7 \\ 0 & 0 & -0.6 & -0.5 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{10}$$
$$[y_0\ y_1\ y_2\ y_3]^\top = [x_0\ x_1\ x_2\ x_3]^\top,$$

which is used to generate the step response coefficients. This system is unstable due to a positive eigenvalue of the system matrix, which results from the unstable values for θ within the probabilistic uncertainty description.

Here two different feedback controllers are considered in Fig. 1: a PI controller and an unconstrained QDMC. PCE is applied to each controller to yield a closed-loop step response, the process output, and the feedback controller output PCE coefficients y_i and u_i, respectively.

3.1 *PI control as output feedback controller*

The discrete PI controller equation in velocity form is

$$u(k) = K_P(e(k) - e(k-1)) + \Delta t K_I e(k) + u(k-1), \tag{11}$$

$$\mathbf{b} \equiv \begin{bmatrix} (\mathbf{I}^{n_y} \otimes \mathbf{1}^p)\mathbf{y}_{max} - (\mathbf{I}^{n_u} \otimes \mathbf{T})\mathbf{f}_y(k) - \mathbf{G}_{ysp,y}\Delta \mathbf{y}_{sp}(k) - \mathbf{w}_y \\ -(\mathbf{I}^{n_y} \otimes \mathbf{1}^p)\mathbf{y}_{min} + (\mathbf{I}^{n_u} \otimes \mathbf{T})\mathbf{f}_y(k) + \mathbf{G}_{ysp,y}\Delta \mathbf{y}_{sp}(k) + \mathbf{w}_y \\ (\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\mathbf{u}_{max} - (\mathbf{I}^{n_u} \otimes \mathbf{T}_u)\mathbf{f}_u(k) - \mathbf{G}_{ysp,u}\Delta \mathbf{y}_{sp}(k) - \mathbf{w}_u - (\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\mathbf{v}(k-1) \\ -(\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\mathbf{u}_{min} + (\mathbf{I}^{n_u} \otimes \mathbf{T}_u)\mathbf{f}_u(k) + \mathbf{G}_{ysp,u}\Delta \mathbf{y}_{sp}(k) + \mathbf{w}_u + (\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\mathbf{v}(k-1) \\ (\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\Delta\mathbf{u}_{max} - (\mathbf{I}^{n_u} \otimes \mathbf{T}')\left((\mathbf{I}^{n_u} \otimes \mathbf{T}_u)\mathbf{f}_u(k) + \mathbf{G}_{ysp,u}\Delta \mathbf{y}_{sp}(k) + \mathbf{w}_u\right) + (\mathbf{I}^{n_u} \otimes \mathbf{t}')\mathbf{u}(k-1) \\ -(\mathbf{I}^{n_u} \otimes \mathbf{1}^c)\Delta\mathbf{u}_{min} + (\mathbf{I}^{n_u} \otimes \mathbf{T}')\left((\mathbf{I}^{n_u} \otimes \mathbf{T}_u)\mathbf{f}_u(k) + \mathbf{G}_{ysp,u}\Delta \mathbf{y}_{sp}(k) + \mathbf{w}_u\right) - (\mathbf{I}^{n_u} \otimes \mathbf{t}')\mathbf{u}(k-1) \end{bmatrix}$$

Fig. 2. Definition of the vector $\mathbf{b}$ in the quadratic program (8) that is solved online.

where K_P and K_I are the proportional and integral gains, $e = y_{sp} - y$ is the error, and Δt is the time interval. In order to obtain a closed-loop step response model of the PCE coefficients of both the process output and the PI controller output, PCE is applied to the PI controller:

$$u_0(k) = K_P(y_{sp}(k) - y_0(k) - y_{sp}(k-1) + y_0(k-1)) + \Delta t K_I(y_{sp}(k) - y_0(k)) + u_0(k-1) \tag{12}$$

$$u_i(k) = K_P(-y_i(k) + y_i(k-1)) + \Delta t K_I(-y_i(k)) + u_i(k-1),\ i = 1, 2, \dots, L \tag{13}$$

where $L+1$ is the number of PCE coefficients. An analysis of the PCE of the PI controller indicates that the PCE is almost identical for all coefficients, except that the higher order coefficients do not depend on y_{sp}. This structure can be interpreted as independently operating PI controllers with a setpoint of 0 for the higher order PCE coefficients. The controller drives the expected value (0^{th} PCE coefficient) to the setpoint y_{sp} and the variance, calculated from the higher order PCE terms, to zero, with the same tuning (K_P and K_I) used for all PCE coefficients.

The step response coefficients for the closed loop are calculated by simulating a unit step in v and y_{sp} respectively and recording the resulting change in y and u. The time interval is $\Delta t = 0.1$, the proportional constant is $K_P = 1$ and the integral constant is $K_I = 1$. The step response coefficients are shown in Fig. 3 and all step responses settle to a steady state, indicating a stable closed-loop response for all realizations of θ.

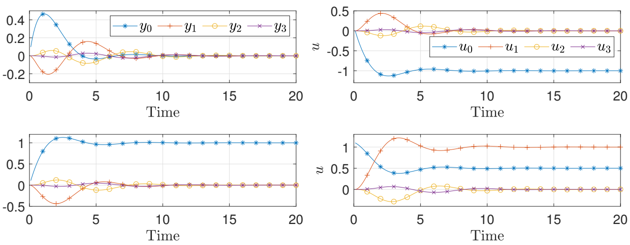

Fig. 3. Step response coefficients for process output y (left), resulting from a unit step in v (top) and y_{sp} (bottom) and for PI controller output u (right), resulting from a unit step in v (top) and y_{sp} (bottom).

The unit step in input adjustment v is treated as a disturbance by the PI controller and is quickly rejected, as can be seen by the return to zero of the expected value of the process output y_0 (Fig. 3 top left) and the step to -1 of the expected value of the PI controller output u_0 (Fig. 3 top right). This compensation happens for all realizations of the uncertain parameter θ, which can be seen by the higher order PCE coefficients returning to zero, indicating a variance of zero. However, at the beginning of the step, the process will behave differently for different realizations of θ, as evidenced by the nonzero values of the higher order PCE coefficients, indicating a nonzero variance across different realizations of θ.

The unit step in process output setpoint y_{sp} is followed by the controller, as can be seen by the step to 1 of the expected value of the process output y_0 (Fig. 3 bottom left). The new setpoint is reached by all realizations of the uncertain parameter θ, as shown by the higher order PCE coefficients returning to zero, indicating a variance of zero. The step in process output is achieved by a step in the PI controller output u, with this step being different for each realization of θ, because for each realization, a different process input u is necessary to achieve the same process output y (Fig. 3 bottom right). This can be seen by the first PCE coefficient u_1 not returning to zero, resulting in a nonzero variance across different realizations of θ.

The performance of the PI controller alone for a step in output setpoint y_{sp} and an input disturbance is shown in Fig. 4 for different realizations of θ, drawn from its uniform distribution. The closed-loop response is overdamped or underdamped, depending on the value of θ.

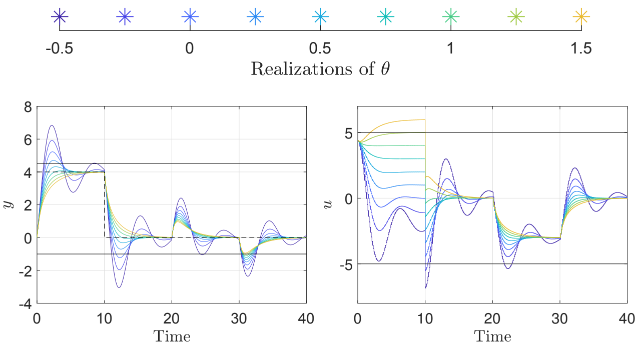

Fig. 4. PI control: process output y with constraints (left), and controller output u with constraints (right). Tested for a step in y_{sp} from 0 to 4 at $t = 0$ and back to 0 at $t = 10$ and a disturbance in process input (+3) between $t = 20$ and $t = 30$.

Combining the FSMPC with the PI controller, as shown in Fig. 1, allows input and output constraints to be taken explicitly into account. To demonstrate the effectiveness of the FSMPC, constraints are chosen that are violated when using only the PI controller. The input constraints are implemented as hard constraints, while the output constraints are implemented as soft constraints in order to avoid feasibility problems of the quadratic program. The tuning parameters and constraints are listed in Table 1.

The results of the PI controller together with the FSMPC are shown in Fig. 5, where it can be seen that both the PI controller output signal u and the FSMPC adjustment v individually violate the input constraints (bottom), but

Table 1. SQDMC parameters

Parameter	Value	Constraints	Value
Δt	0.1	$u_{\min}$	-5
p	199	$u_{\max}$	5
c	100	$\Delta u_{\min}$	-0.75
n	200	$\Delta u_{\max}$	0.75
w_v	1	$y_{\min}$	-1
w_ϵ	1000	$y_{\max}$	4.5
W_ϵ	1000		

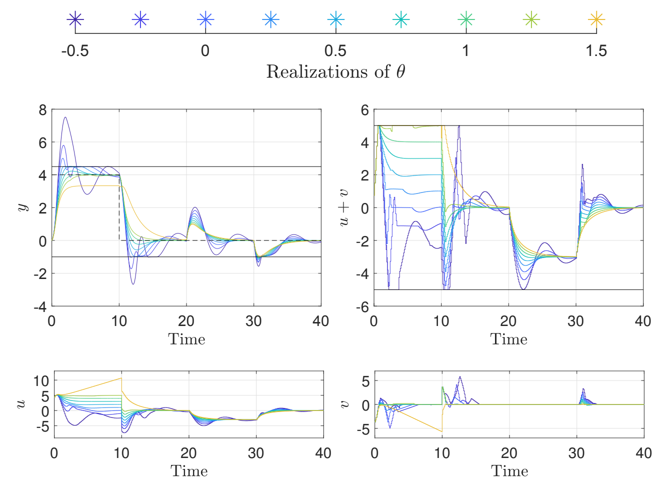

Fig. 5. FSMPC with PI as feedback controller: process output y with constraints (top left), and process input $u+v$ with constraints (top right). PI controller output u (bottom left) and FSMPC output v (bottom right). Tested for a step in y_{sp} from 0 to 4 at $t = 0$ and back to 0 at $t = 10$ and a disturbance in process input (+3) between $t = 20$ and $t = 30$.

the sum of both signals strictly satisfies the constraints (top right). For the realization of $\theta = 1.5$, this results in the PI controller continuously increasing the control signal between time 0 and 10 in an attempt to reach the setpoint y_{sp}, and the FSMPC adjusting for it to satisfy the input constraints. In this case, the input constraints prevent the process from reaching the output setpoint. During the setpoint changes and the input disturbances, the (soft) output constraints are violated, as can be seen in Fig. 5 (top left), which is caused by the underdamped closed loop response of the PI controller, for the particular realizations of θ. In those cases, the FSMPC is unable to adjust for the large PI controller action.

In comparison to PI control only (Fig. 4), the proposed combined approach (Fig. 5) is able to strictly satisfy the input constraints and reduce the number of realizations of θ, for which output constraint violations are observed.

3.2 Unconstrained QDMC as output feedback controller

For dynamical systems with multiple inputs and outputs, a multivariable controller has to be used as the feedback controller. Because the nominal process is stable, a QDMC can be designed, but PCE can be applied only to the closed-form solution of the optimization in the unconstrained QDMC. Therefore, no constraints are considered within the feedback controller, analogously to the PI controller. The unconstrained QDMC is designed with the same parameters in Table 1 and a weight for the error term of $\mathbf{W}_y = 10$ and a weight for the input movement of $\mathbf{W}_u = 1$. The closed-form solution for multiple inputs and multiple outputs can be expressed as

$$\mathbf{u}^*(k) = \alpha \mathbf{u}_{\text{past}}(k) + \beta \mathbf{e}(k), \tag{14}$$

where α and β are constant matrices. Like the PI controller, the unconstrained QDMC is linear in terms of $\mathbf{u}_{\text{past}}(k)$ and $\mathbf{e}(k)$, and the PCE is

$$\mathbf{u}_0^*(k) = \alpha \mathbf{u}_{0,\text{past}}(k) + \beta(\mathbf{y}_{sp} - \mathbf{y}_0(k)), \tag{15}$$

$$\mathbf{u}_i^*(k) = \alpha \mathbf{u}_{i,\text{past}}(k) - \beta \mathbf{y}_i(k)), i = 1, 2, \ldots, L, \tag{16}$$

which can be interpreted as $L+1$ individual controllers, bringing the expected process output (0^{th} PCE coefficient) to the setpoint for every realization of θ. This can be seen by the variance calculated from the higher order PCE coefficients being brought to zero. However, all $L+1$ controllers are designed equal, utilizing the nominal step response, which might not be an accurate description for each PCE coefficient.

The step response coefficients for the closed loop are calculated by simulating a unit step in v and y_{sp} respectively and recording the resulting change in y and u (Fig. 6). All step responses settle at a steady state, indicating a stable closed loop response for all realizations of θ.

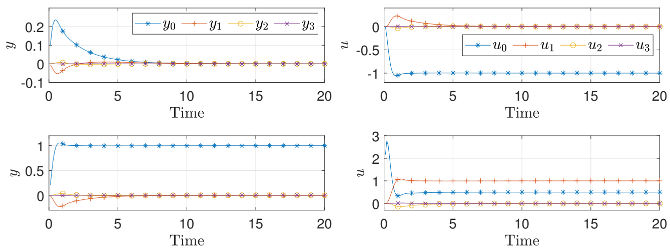

Fig. 6. Step response coefficients for process output y (left), resulting from a unit step in v (top) and y_{sp} (bottom) and for QDMC output u (right), resulting from a unit step in v (top) and y_{sp} (bottom), with $\Delta t = 0.1$.

The step responses of the PCE coefficients for the closed loop response with QDMC (Fig. 6) are similar to the closed-loop step response with the PI controller, discussed above, with the difference that the QDMC is tuned more aggressively, i.e. reaches a steady state sooner.

Fig. 7 shows the results of unconstrained QDMC together with FSMPC as outlined in Fig. 1. For this single-input single-output system, the result is similar to using a PI controller as the feedback controller, while also being applicable to multivariable control. Both the QDMC output signal u and the FSMPC adjustment v violate the input constraints individually (bottom), but the constraints are strictly satisfied by the sum of both signals $u + v$ (top right). For the realization of $\theta = 1.5$, the process output does not reach the desired setpoint y_{sp} between time 0 and 10, because the process input $u + v$ reaches the upper constraint. During that time, the QDMC increases the input u to reach the setpoint y_{sp}, but the FSMPC adjusts the signal to satisfy the input constraints, thus the input constraints prevent the process from reaching the output setpoint. The (soft) output constraints are violated during setpoint changes and input disturbances, Fig. 7 (top left). For those particular realizations of θ, the process gain is

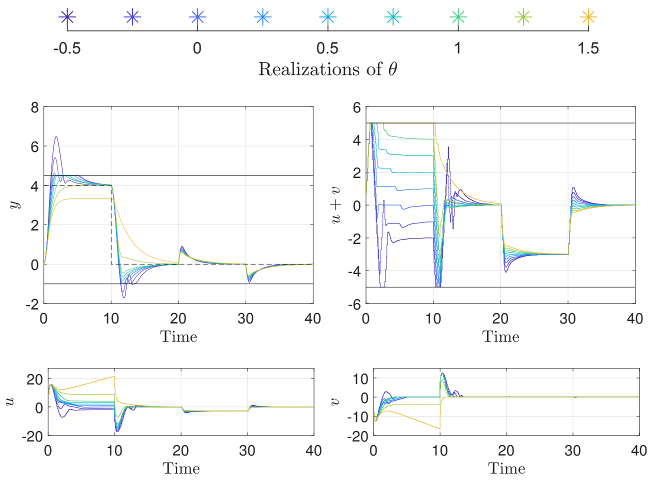

Fig. 7. FSMPC with QDMC as feedback controller: process output y with constraints (top left), and process input $u+v$ with constraints (top right). QDMC controller output u (bottom left) and FSMPC output v (bottom right). Tested for a step in y_{sp} from 0 to 4 at $t=0$ and back to 0 at $t=10$ and a disturbance in process input (+3) between $t=20$ and $t=30$.

particularly high and the FSMPC is unable to adjust the controller action as would be required.

3.3 Minimization of the expected error with unconstrained QDMC as output feedback controller

An error term is added to the objective function of the FSMPC, as shown at the end of Section 2, so that the magnitude of the adjustment signal v is minimized along with the predicted expected value of the process output error. The same unconstrained QDMC as before is used as feedback controller. The results in Fig. 8 show a similar performance for inputs and outputs in terms of constraint violations as before, and an improved setpoint tracking performance, particularly during time $t=5$ to $t=10$.

4. DISCUSSION

The proposed output feedback control scheme extends fast stochastic model predictive control to unstable dynamical systems by integration with a pre-stabilization of the complete set of uncertain systems by an unconstrained output feedback control. For either output feedback control designs – PI controller or unconstrained QDMC – including the error term in the FSMPC formulation resulted in improved overall closed-loop performance.

Although the control algorithms are demonstrated here for a system with one input and output, the unconstrained QDMC as feedback controller readily applies to multivariable control. Due to the input-output formulation of the controllers, the online computational cost is low and independent of the number of states in the system.

5. CONCLUSION

Fast stochastic MPC algorithms are extended to dynamical systems with probabilistic parametric uncertainties in which the process dynamics can be unstable for some

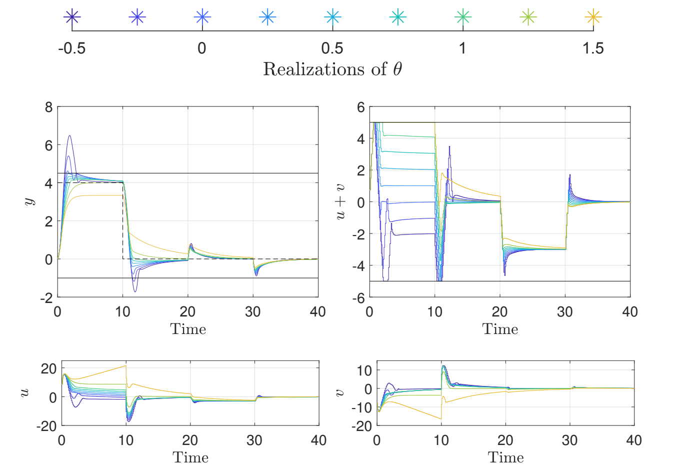

Fig. 8. FSMPC with QDMC as feedback controller and with additional error term in the objective function, resulting in tighter control compared to Fig. 7. Process output y with constraints (top left) and process input $u+v$ with constraints (top right). QDMC controller output u (bottom left) and FSMPC output v (bottom right). Tested for a step in y_{sp} from 0 to 4 at $t=0$ and back to 0 at $t=10$ and a disturbance in process input (+3) between $t=20$ and $t=30$.

parameter values. With the presented FSMPC and output feedback structure, such processes can be controlled successfully when the nominal system is stable and the system can be pre-stabilized by an output feedback controller for all uncertainty realizations. Then the FSMPC ensures constraint satisfaction while explicitly taking the probabilistic parameter uncertainties into account.

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