Robust Error Feedback Sliding Mode Regulator for Nonlinear Systems in Regular Form ⋆

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Abstract: In this paper, the problem of designing a nonlinear Sliding Mode (SM) regulator is addressed for nonlinear affine control systems in Regular form subject to unmodeled disturbance. In particular, the error feedback SM regulator problem is defined, taking the concepts related to the zero output tracking submanifold as a starting point. Applying the internal model concept to the time-invariant SM equation, the solvability conditions to the problem are derived. A Proportional-Integral (PI) nonlinear observer is proposed, and using the observer state, a sliding manifold on which the tracking error is ultimately bounded, is formulated. A SM control algorithm is proposed to ensure the designed manifold to be attractive, achieving robustness with respect to allowed uncertainties. The effectiveness of the proposed method is demonstrated by the application to the Pendubot system.

Keywords: Sliding Mode Control, Nonlinear Regulation, Error Feedback Regulator, Nonlinear Systems, Regular Form, Electro-mechanical systems, Pendubot.

1. INTRODUCTION

One of the many control techniques used for trajectory tracking of nonlinear systems is associated with the term of regulation. The regulation problem for nonlinear systems is defined as finding a feedback law that is capable of forcing the system trajectories to track a predefined reference, the latter provided by an external system, or exosystem. In the classical regulation theory, the solution is found by performing a mathematical analysis that leads to the solvability of the Francis-Isidori-Byrnes equations. The internal model principle, for regulation based on the output of the system is presented. Both procedures will generate the feedback control law that produces the desired steady state behaviour, Isidori (1995). For a more realistic approach, the presence of unmodeled disturbances in the plant, that is, perturbations not generated by the exosystem, could be considered, nevertheless, the corresponding regulator equations cannot be solved since the perturbation term appears explicitly and is unknown. An alternative approach for dealing with this problem is to combine the regulation theory with SM control techniques, Utkin et al. (2009), which allows to decompose and simplify the regulator design procedure and impose robustness properties with respect to at least matched perturbations, Draženović (1969), El-Ghezawi et al. (2007). The combination of these two techniques has been broadly studied in the last two decades by several authors (see, among others, Jeong and Utkin (1999), Elmali and Olgac (1992), DZ et al. (2001), Zheng and Zhong (2013), Govindaswamy et al. (2014)), for *minimum phase* systems in general form. Few works were addressed to *non-minimum phase* systems, see for example, Jeong and Utkin (1999), Utkin and Utkin (2014), Bonivento et al. (2001).

In Loukianov et al. (2018), a robust SM state-feedback regulator is proposed for systems in general and Regular form, which is able to compensate matched time-varying perturbations, and using the equivalent control technique Utkin et al. (2009), the autonomous nonlinear regulator equation as in the classic regulation theory, can be used for the SM dynamics. The error-feedback regulator is left out of this work.

On the other hand, for a specific class of nonlinear systems, that includes electro-mechanical sub-actuated non-minimum phase systems, the transformation to Regular form eases the synthesis of the SM and the corresponding regulator equations. Furthermore, it allows to perform the analysis of the SM behaviour in an explicit form, since to derive the SM dynamics, the equivalent control Utkin et al. (2009) is not needed.

In this paper, the error feedback SM regulator problem for nonlinear systems in Regular form subject to matched unmodeled perturbations is formulated and the solution existence conditions are derived using the regulator equation that correspond to the SM equation. The solution to this equation is used to define the control error and its dynamics, as well as a local center manifold on which

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the output error is zeroed. To estimate the unmeasured control error and exosystem state variables, a nonlinear Proportional-Integral (PI) observer, based on Beale and Shafai (1988), is designed. Using the estimated control errors, a sliding manifold is formulated, and a reaching discontinuous control law is proposed. Additionally, a stability analysis of the complete closed-loop system is presented.

The rest of this work is organized as follows. The error feedback SM regulator problem for nonlinear systems in Regular form is presented in Section 2, along with the conditions for the solution of the problem. A nonlinear PI observer, sliding manifold, and discontinuous control action are designed in Section 3; along with the stability analysis of the closed-loop dynamics. To show the effectiveness of the proposed method, the SM regulator is designed for an electro-mechanical system, the so-called Pendubot in Section 4. Finally, Section 5 concludes this work.

2. PROBLEM STATEMENT

Consider a time-varying nonlinear perturbed system described by

$$\dot{x} = f(x) + B(x)u + D(x)w + \Delta(x,t) \tag{1}$$
$$e = h(x) - q(w) \tag{2}$$

where $x \in X \subset \Re^n$ is the state of the system, $u \in \Re^m$ is the control input, $e \in \Re^p$ represents the output tracking error and $\Delta(x,t)$ comprises the perturbation due to plant parameter variations, unmodeled dynamics and external disturbance, rank $B(x) = m$; the exogenous signal $w \in W \subset \Re^q$ represents the modeled disturbance and desired reference signal $q(w)$ generated by the exosystem

$$\dot{w} = s(w). \tag{3}$$

The control objective is to design an error feedback control law such that a solution of the closed-loop system is locally stable and the regulated output error $e(t)$ (2), uniformly asymptotically approaches zero or at least uniformly ultimately bounded in presence of perturbation, $\Delta(x,t)$. This problem is called Output Regulation Problem.

In the *classical* setup Isidori (1995), in absence of the perturbation, that is, $\Delta(x,t) = 0$, it has been shown that the solvability of the Output Regulation Problem can be stated in terms of the existence of a pair of mappings $x = \pi(w)$ and $u = c(w)$ with $\pi(0) = 0$ and $c(0) = 0$ which solve the following Regulator Equation:

$$\frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w)) + B(\pi(w))c(\pi(w)) + D(\pi(w))w$$
$$0 = h(\pi(w)) - q(w).$$

and the classical control provided by a continuous error feedback can stabilize the system (1) in the first approximation ensuring the output error $e(t)$ (2) decays to zero as time tends to infinity.

In presence of $\Delta(x,t)$, the system (1) becomes time-varying; therefore, it is assumed that there exist smooth functions $\pi_s(w,t)$ and $c_s(w,t)$ with $\pi_s(0,t) = 0$ and $c_s(0,t) = 0$ such that the following expression holds, Yang and Huang (2012):

$$\frac{d\pi_s(w,t)}{dt} = f(w,t) + B(w,t)c_s(w,t) + D(w,t)w + \Delta(w,t)$$
$$0 = h(\pi_s(w,t)) - q(w).$$

It can be seen that these equations are impossible to solve since $\Delta(\pi(w),t)$ is unmodeled and unknown. To overcome this problem, in Loukianov et al. (2018), the SM technique was used and a state feedback SM regulator was designed under assumption that the state vectors x and w are available for measurement.

In this paper, we consider a more realistic situation when only the components of the error e are available for measurement. Defining constant matrices A and B as

$$A = \left[\frac{\partial f}{\partial x}\right]_{(0)} \qquad B_0 = B(0)$$

the following assumptions are introduced:

- **A1**. The pair A, B is stabilizable.
- **A2**. The matrix $S = \left[\frac{\partial s(w)}{\partial w}\right]_{(0)}$ has all its eigenvalues on the imaginary axis.
- **A3**. The perturbation $\Delta(x,t)$ satisfies the following matching condition:
$$\Delta(x,t) = B(x)\delta(x,t), \delta \in \Re^m. \tag{4}$$
- **A4**. For a specific class of systems, namely, the ones where the conditions of Frobenius' theorem are satisfied for the corresponding Pfaffian system, Lukyanov and Utkin (1981), a local diffeomorphism $z = T(x)$ can be obtained in order to represent the system (1)-(2) in Regular form, as
$$\dot{z}_1 = f_1(z_1,z_2) + D_1(z_1,z_2)w \tag{5}$$
$$\dot{z}_2 = f_2(z) + D_2(z)w + B_2(z)(u + \delta(z,t)) \tag{6}$$
$$e = h(z_1,z_2) - q(w), \tag{7}$$
where $z \in Z \subset \Re^n$, $z = [z_1,z_2]^T$, $z_1 \in Z_1 \subset \Re^{n-m}$, $z_2 \in Z_2 \subset \Re^m$, and rank $B_2(z) = m$.

The Error Feedback SM Regulator Problem can be defined as finding a dynamic discontinuous controller

$$\dot{\xi} = \eta(\xi,u,e) \tag{8}$$
$$u = \begin{cases} u^+(\xi) & \text{if } \sigma(\xi) > 0 \\ u^-(\xi) & \text{if } \sigma(\xi) < 0 \end{cases} \tag{9}$$

with $\xi \in \Xi \subset \Re^n$, and the sliding manifold

$$\sigma(\xi) = 0, \qquad \sigma = [\sigma_1, \ldots, \sigma_m)]^T \tag{10}$$

such that the following conditions are satisfied:

- **C1**. Finite-time convergence of the closed-loop system states to the sliding manifold $\sigma(\xi) = 0$.
- **C2**. Asymptotic stability of the SM dynamics in the first approximation in absence of the perturbation.
- **C3**.
  - **A.** For the case of $\|\delta(z,t)\| \leq \gamma_1, \gamma_1 > 0$, there are $b > 0, T_0 > 0$ and a neighborhood $Z_0 \subset Z \times W \times \Theta$ of the origin where
  $$\|e(t)\| \leq b, \quad \forall\, t \geq T_0.$$
  - **B.** For the case of $\|\delta(z,t)\| \leq \gamma_2\|x\|$, $\gamma_2 > 0$, there is a neighborhood $Z_0 \subset Z \times W \times \Theta$ of the origin where
  $$\lim_{t\to\infty} e(t) = 0.$$

3. SOLUTION OF PROBLEM FOR NONLINEAR SYSTEMS IN REGULAR FORM

In this section, a solution for the Error Feedback SM Regulator problem for nonlinear systems in Regular form (5)-(6) will be proposed. First, a control error dynamics is defined, and then we propose an observer that will produce the estimated states which are used to design a sliding manifold.

3.1 Control error dynamics

Let us introduce the mappings $z_i = \pi_i(w)$, $i = 1, 2$, so we can define the control error or local center manifold, Isidori (1995), as

$$\varepsilon_1(z_1, w) = 0 \qquad \varepsilon_1 = z_1 - \pi_1(w) \tag{11}$$

$$\varepsilon_2(z_2, w) = 0 \qquad \varepsilon_2 = z_2 - \pi_2(w), \tag{12}$$

From (5)-(6), the control error dynamics become

$$\dot{\varepsilon}_1 = f_1(\varepsilon_1, \varepsilon_2, w) + D_1(\varepsilon_1, \varepsilon_2, w)w - \frac{\partial \pi_1(w)}{\partial w}s(w) \tag{13}$$

$$\begin{aligned}\dot{\varepsilon}_2 = &f_2(\varepsilon, w) + D_2(\varepsilon, w)w - \frac{\partial \pi_2(w)}{\partial w}s(w)\\ &+ B_2(\varepsilon, w)(u + \delta(\varepsilon, w, t))\end{aligned} \tag{14}$$

$$e = h(\varepsilon_1, \varepsilon_2, w) - q(w). \tag{15}$$

The linear approximation of (13)-(15) is presented as

$$\begin{bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{w}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & H_1\\ A_{21} & A_{22} & H_2\\ 0 & 0 & S\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ w\end{bmatrix} + \begin{bmatrix}0\\ B_2\\ 0\end{bmatrix}u + \begin{bmatrix}\phi_1(\varepsilon, w)\\ \phi_2(\varepsilon, w)\\ \phi_w(w)\end{bmatrix} \tag{16}$$

$$e = [C_1 \;\; C_2 \;\; (C_1\Pi_1 + C_2\Pi_2 - Q)]\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ w\end{bmatrix} + \phi_e(\varepsilon, w) \tag{17}$$

where linear matrices A_{11}, A_{12} were previously defined, and $A_{2i} = \left[\frac{\partial f_2}{\partial z_i}\right]_{(0,0)}$, $H_i = A_{i1}\Pi_1 + A_{i2}\Pi_2 - \Pi_i S + D_i$, $\Pi_i = \left[\frac{\partial \pi_i}{\partial w}\right]_{(0)}$, $D_i = D_i(0)$, $C_i = \left[\frac{\partial h}{\partial z_i}\right]_{(0,0)}$, for $i = 1, 2$; and $B_2 = B_2(0)$, $Q = \left[\frac{\partial q}{\partial w}\right]_{(0)}$. Here, ϕ_1, ϕ_2, ϕ_w, and ϕ_e contain nonlinear terms and they vanish at the origin, along with their first derivatives.

Let us define the following matrices:

$$\bar{A} = \begin{bmatrix}A_{11} & A_{12} & H_1\\ A_{21} & A_{22} & H_2\\ 0 & 0 & S\end{bmatrix} \tag{18}$$

$$\bar{C} = [C_1 \;\; C_2 \;\; (C_1\Pi_1 + C_2\Pi_2 - Q)], \tag{19}$$

therefore, the following assumption can be stated:

- **A5**. The pair $\bar{A}$, $\bar{C}$ is detectable.

3.2 SM dynamics

With the purpose of designing a discontinuous control law, the sliding manifold is formulated as

$$\sigma = \varepsilon_2 + \sigma_0(\varepsilon_1). \tag{20}$$

When $\sigma = 0$, the state variable ε_2 can be seen as a virtual control for equation (13), and from (20)

$$\varepsilon_2 = -\sigma_0(\varepsilon_1). \tag{21}$$

Therefore, the reduced-order SM dynamics can be obtained by introducing (21) in (13), resulting in

$$\begin{aligned}\dot{\varepsilon}_1 = &f_1(\varepsilon_1, -\sigma_0(\varepsilon_1), w) + D_1(\varepsilon_1, -\sigma_0(\varepsilon_1), w)\\ &- \frac{\partial \pi_1(w)}{\partial w}s(w).\end{aligned} \tag{22}$$

The linear representation of the SM dynamics can be obtained by using the previously defined linear matrices

$$\dot{\varepsilon}_1 = (A_{11} - A_{12}C_s)\varepsilon_1 + H_1 w + \phi_1(\varepsilon_1, w), \tag{23}$$

where $C_s = \left[\frac{\partial \sigma_0}{\partial \varepsilon_1}\right]_{(0)}$.

3.3 Proportional-Integral Observer

Since the sliding manifold (20) uses the umnmeasured control error variables, the following observer is proposed to design the system (8) taking $\xi = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w}, \xi_1]^T$:

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\varepsilon}}_1\\ \dot{\hat{\varepsilon}}_2\\ \dot{\hat{w}}\end{bmatrix} = &\begin{bmatrix}f_1(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w}) + D_1(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w})\hat{w} - \frac{\partial \pi_1(\hat{w})}{\partial \hat{w}}s(\hat{w})\\ f_2(\hat{\varepsilon}, \hat{w}) + D_2(\hat{\varepsilon}, \hat{w})\hat{w} - \frac{\partial \pi_2(\hat{w})}{\partial \hat{w}}s(\hat{w}) + B_2(\hat{\varepsilon}, \hat{w})u\\ s(\hat{w})\end{bmatrix}\\ &+ L_1(e - \hat{e}) + L_2\xi_1\end{aligned} \tag{24}$$

$$\dot{\xi}_1 = e - \hat{e} \tag{25}$$

where $\xi_1 = \int(e - \hat{e})dt$; $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$, and $\hat{w}$ are the estimates of ε_1, ε_2 and w, respectively, $\hat{e} = h(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{w}) - q(\hat{w})$, and $L_1 = [L_{11}, L_{12}, L_{13}]^T$, $L_2 = [L_{21}, L_{22}, L_{23}]^T$ are the observer gain matrices.

Using (13)-(14) with its linearization (16) and (24) augmented with (25), the observer error dynamics result in

$$\begin{bmatrix}\dot{\tilde{\varepsilon}}_1\\ \dot{\tilde{\varepsilon}}_2\\ \dot{\tilde{w}}\\ \dot{\xi}_1\end{bmatrix} = R\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{w}\\ \xi_1\end{bmatrix} + \delta_1(\tilde{\varepsilon}, \tilde{w}, t), \tag{26}$$

with $[\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{w}]^T = [\varepsilon_1 - \hat{\varepsilon}_1, \varepsilon_2 - \hat{\varepsilon}_2, w - \hat{w}]^T$, and

$$R = \begin{bmatrix}\bar{A} - L_1\bar{C} & -L_2\\ \bar{C} & 0\end{bmatrix}$$

$$\delta_1(\tilde{\varepsilon}, \tilde{w}, t) = \begin{bmatrix}\phi_1(\tilde{\varepsilon}, \tilde{w}) + L_{11}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_2(\tilde{\varepsilon}, \tilde{w}) + L_{12}\phi_e(\tilde{\varepsilon}, \tilde{w}) + B_2\delta(\varepsilon, w, t)\\ \phi_w(\tilde{\varepsilon}, \tilde{w}) + L_{13}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_e(\tilde{\varepsilon}, \tilde{w})\end{bmatrix}.$$

3.4 Conditions of existence of solution

By setting the sliding manifold with the estimated states

$$\sigma = \hat{\varepsilon}_2 + C_s\hat{\varepsilon}_1, \tag{27}$$

where $C_s \in \Re^{n-m}$ is a control gain matrix, the solvability conditions can be established in the following proposition:

Proposition 1. *Under **A1-A5**, if there exist $C^k (k \geq 2)$ mappings $z_1 = \pi_1(w)$ and $z_2 = \pi_2(w)$, with $\pi_1(0) = 0$ and $\pi_2(0) = 0$, defined in a neighborhood W of the origin, satisfying the following conditions:*

$$f_1(\pi_1(w), \pi_2(w)) + d_1(\pi_1(w), \pi_2(w))w = \frac{\partial \pi_1(w)}{\partial w}s(w) \tag{28}$$

$$h(\pi_1(w), \pi_2(w)) - q(w) = 0 \tag{29}$$

*then, the Error Feedback SM Regulator problem for systems in Regular form (5)-(6), as defined through conditions **C1 - C3**, is solvable.*

Proof. Selecting the discontinuous control in (9) as a common signum function

$$u = -MB_2^{-1}(\hat{\varepsilon}, \hat{w})\,\mathrm{sign}(\sigma(\hat{\varepsilon})), M > 0 \tag{30}$$

it can be seen that for $M > \|B_2(\hat{\varepsilon}, \hat{w})u_{eq}(\hat{\varepsilon}, \hat{w})\|$, where $u_{eq}(\hat{\varepsilon}, \hat{w})$ is calculated from $\dot{\sigma} = 0$, ensures the finite-time convergence of the closed-loop system (13)-(14) with (30) to the sliding manifold $\sigma = 0$, satisfying condition **C1**.

The sliding manifold (27) can be expressed in terms of real states ε and observation error states $\tilde{\varepsilon}$ as

$$\sigma = \varepsilon_2 - \tilde{\varepsilon}_2 + C_s(\varepsilon_1 - \tilde{\varepsilon}_1).$$

Once sliding mode occurs, that is $\sigma = 0$, then

$$\varepsilon_2 = -C_s\varepsilon_1 + \tilde{\varepsilon}_2 + C_s\tilde{\varepsilon}_1,$$

and from (22), the SM dynamics become

$$\begin{aligned}\dot{\varepsilon}_1 = f_1(\varepsilon_1, (-C_s\varepsilon_1 + \tilde{\varepsilon}_2 + C_s\tilde{\varepsilon}_1, w)) - \frac{\partial \pi_1(w)}{\partial w}s(w) \\ + D_1(\varepsilon_1, (-C_s\varepsilon_1 + \tilde{\varepsilon}_2 + C_s\tilde{\varepsilon}_1, w))w.\end{aligned} \tag{31}$$

Thus, the following linear approximation of the SM dynamics is presented

$$\dot{\varepsilon}_1 = (A_{11} - A_{12}C_s)\varepsilon_1 + H_1 w + \tilde{R}_1 + \phi(\varepsilon_1, w), \tag{32}$$

where $\tilde{R}_1 = A_{12}(\tilde{\varepsilon}_2 + C_s\tilde{\varepsilon}_1)$.

Rearranging equations (26), (32), and (3) the closed-loop system motion on the manifold $\sigma = 0$ is described by

$$\dot{\varepsilon}_1 = (A_{11} - A_{12}C_s)\varepsilon_1 + H_1 w - \tilde{R}_1 + \phi_1(\varepsilon_1, w) \tag{33}$$

$$\begin{bmatrix}\dot{\tilde{\varepsilon}}_1\\ \dot{\tilde{\varepsilon}}_2\\ \dot{\tilde{w}}\\ \dot{\xi}_1\end{bmatrix} = R\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{w}\\ \xi_1\end{bmatrix} + \delta_1(\tilde{\varepsilon}, \tilde{w}, t) \tag{34}$$

$$\dot{w} = Sw + \phi_w(w) \tag{35}$$

$$e = h(\varepsilon_1, \varepsilon_2, w) - q(w) \tag{36}$$

In the absence of the perturbation $\Delta(x,t)$, the term $\delta_1(\tilde{\varepsilon}, \tilde{w})$ in (34) vanishes at the origin, and under assumption **A3**, matrices L_1 and L_2 can be chosen in order to make matrix R Hurwitz, thus making the system (34) locally asymptotically stable. Moreover, the term $\tilde{R}_1$ in (33) will vanish as well, and if condition (28) is satisfied, then

$$\begin{aligned}f_1(\pi_1(w), \pi_2(w)) + d_1(\pi_1(w), \pi_2(w))w - \frac{\partial \pi_1(w)}{\partial w}s(w) = \\ (A_{11}\Pi_1 + A_{12}\Pi_2 - \Pi_1 S + D_1)w + \phi_1(0, w) = 0.\end{aligned} \tag{37}$$

Under assumption **A1**, matrix $(A_{11} - A_{12}C_s)$ is Hurwitz by selecting C_s accordingly, and the system (33)-(35) admits a local center manifold given by $z = T(\pi(w))$, thus achieving asymptotic stability of the system (33) origin, fulfilling condition **C2**.

Defining the vector $\chi = [\varepsilon_1\ \tilde{\varepsilon}_1\ \tilde{\varepsilon}_2\ \tilde{w}\ \xi_1]^T$ and considering (37), the SM dynamics (33)-(35) can be represented as

$$\dot{\chi} = \mathcal{A}\chi + \bar{\delta}(\varepsilon_1, w, t) + \Phi(\varepsilon_1, \tilde{\varepsilon}, \tilde{w}) \tag{38}$$

where

$$\mathcal{A} = \begin{bmatrix} A_{11} - A_{12}C_s & \star \\ 0 & R\end{bmatrix}$$

$$\bar{\delta}(\varepsilon_1, w, t) = \begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ B_2\delta(\varepsilon_1, w, t) \\ 0 \\ 0\end{bmatrix}\end{bmatrix}$$

$$\Phi(\varepsilon_1, w, \tilde{\varepsilon}, \tilde{w}) = \begin{bmatrix}\phi_1(\varepsilon_1, w)\\ \phi_1(\tilde{\varepsilon}, \tilde{w}) + L_{11}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_2(\tilde{\varepsilon}, \tilde{w}) + L_{12}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_w(\tilde{\varepsilon}, \tilde{w}) + L_{13}\phi_e(\tilde{\varepsilon}, \tilde{w})\\ \phi_e(\tilde{\varepsilon}, \tilde{w})\end{bmatrix}.$$

From **C3.A**, it follows that there are $\gamma_3 > 0$ and $\gamma_4 > 0$ such that

$$\|\bar{\delta}(\varepsilon_1, w, t)\| \le \gamma_3, \qquad \gamma_3 > 0 \tag{39}$$

$$\|\Phi(\varepsilon_1, w, \tilde{\varepsilon}, \tilde{w})\| \le \gamma_4\|\chi\|, \qquad \gamma_4 > 0 \tag{40}$$

in some admissible region Ω_c.

Since matrices $(A_{11} - A_{12}C_s)$ and R are Hurwitz, matrix $\mathcal{A}$ is Hurwitz as well; thus, the Lyapunov function $V(\chi) = \chi^T P\chi$ can be proposed with $P > 0$ the solution of the Lyapunov equation $\mathcal{A}^T P + P\mathcal{A} = -Q$ for $Q > 0$.

Taking the derivative of $V(\chi)$ along the trajectories of (38) and using (39) - (40), we obtain

$$\begin{aligned}\dot{V}(\chi, t) \le& -(\lambda_{min}(Q) - 2\lambda_{max}(P)\gamma_4)\|\chi\|^2 \\ &+ 2\gamma_3\lambda_{max}(P)\|\chi\|.\end{aligned}$$

Suppose now $\lambda_{min}(Q) - 2\lambda_{max}(P)\gamma_4 = \alpha$, $\alpha > 0$, then

$$\begin{aligned}\dot{V}(\chi, t) &\le -\alpha\|\chi\|^2 + 2\gamma_3\lambda_{max}(P)\|\chi\| \\ &\le -(1-\theta)\alpha\|\chi\|^2 > 0\end{aligned}$$

for $\|\chi\| \ge \frac{2\gamma_3\lambda_{max}(P)}{\alpha\theta}$ with $0 < \theta < 1$. Thus, the solution of (38) is locally uniformly ultimately bounded with an ultimate bound

$$b = \frac{2\gamma_3\lambda_{max}(P)}{\alpha\theta}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$$

fulfilling condition **C3**.

4. EXAMPLE

The Pendubot is an under-actuated electromechanical, non-minimum phase nonlinear system, ideal to evaluate the effectiveness of the proposed method. The mathematical model of this system in regular form is presented as

$$\begin{aligned}\dot{z}_1 &= z_3 - f_{11}(z)\\ \dot{z}_2 &= z_4\\ \dot{z}_3 &= f_{31}(z)\\ \dot{z}_4 &= f_{41}(z) + b_4(z_2)(u + \delta(z,t)) \qquad (41)\\ y &= z_2 \qquad (42)\end{aligned}$$

where the expressions $f_{11}(z) = \frac{D_{22}}{D_{12}(z_2)}z_4$, $f_{31}(z) = b_3(z_2)p_1(z) + \frac{D_{22}}{D_{12}(z_2)}b_4(z_2)p_2(z) + \frac{D_{22}}{D_{12}(z_2)^2}C_3(z_2, z_4)$, and $f_{41}(z) = b_4(z_2)p_2(z)$. The terms $p_1(z)$, $p_2(z)$, $D_{11}(z_2)$, $D_{12}(z_2)$, $C_3(z_2, z_4)$, and D_{22}, depend on operations between plant parameters and elements of the state, alongside with $b_3(z_2) = \frac{D_{22}}{D_{11}(z_2)D_{22} - D_{12}^2(z_2)}$, and $b_4(z_2) = -\frac{D_{12}(z_2)}{D_{11}(z_2)D_{22} - D_{12}^2(z_2)}$.

The reference signal to be tracked, $q(w) = w_2$ will be generated by the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

thus, the tracking error becomes $e = z_2 - w_2$.

Defining the control error as $\varepsilon_i = z_i - \pi_i(w), i = 1, ..., 4,$ the control error dynamics are described by

$$\begin{aligned} \dot{\varepsilon}_1 &= \varepsilon_3 + \pi_3(w) - f_{11}(\varepsilon + \pi(w)) - \frac{\partial \pi_1}{\partial w} s(w) \\ \dot{\varepsilon}_2 &= \varepsilon_4 + \pi_4(w) - \frac{\partial \pi_2}{\partial w} s(w) \\ \dot{\varepsilon}_3 &= f_{31}(\varepsilon + \pi(w)) - \frac{\partial \pi_3}{\partial w} s(w) \\ \dot{\varepsilon}_4 &= f_{41}(\varepsilon + \pi(w)) + b_4(\varepsilon_2, w)(u + \delta(\varepsilon, w, t)) \\ &\quad - \frac{\partial \pi_4}{\partial w} s(w) \end{aligned} \tag{43}$$

where $\pi(w) = [\pi_1(w), \ldots, \pi_4(w)]^T$, and $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_4]^T$.

From Theorem 1 and (43), the corresponding Regulator Equations are given by

$$\frac{\partial \pi_1}{\partial w} s(w) = \pi_3(w) - f_{11}(\pi(w)) \tag{44}$$

$$\frac{\partial \pi_2}{\partial w} s(w) = \pi_4(w) \tag{45}$$

$$\frac{\partial \pi_3}{\partial w} s(w) = f_{31}(\pi(w)) \tag{46}$$

$$0 = \pi_2(w) - w_2. \tag{47}$$

From (47) and (45) it follows that $\pi_2(w) = w_2$, and $\pi_4(w) = -\alpha w_1$. Since the solution of $\pi_1(w)$, and $\pi_3(w)$, involves solving partial differential equations; we take a simpler approach by proposing a polynomial approximation for $\pi_1(w)$ as

$$\pi_1(w) = a_0 + a_1 w_1 + a_2 w_2 + a_3 w_1 w_2 + a_4 w_1^2 + a_5 w_2^2 + a_6 w_1 w_2^2 + a_7 w_1^2 w_2 + a_8 w_1^3 + a_9 w_2^3 + O^4(\|w\|),$$

and from (44),

$$\pi_3(w) = \frac{\partial \pi_1}{\partial w} s(w) - \frac{D_{22}}{D_{12}(w_2)} \alpha w_1. \tag{48}$$

The values for $a_0, \ldots, a_9$ can be found by assigning a value to α and performing a Taylor series expansion in both sides of equation (46), thus leaving a set of algebraic equations to solve.

From (24)-(25), the PI Observer for the Pendubot system has the following form

$$\begin{bmatrix} \dot{\hat{\varepsilon}}_1 \\ \dot{\hat{\varepsilon}}_2 \\ \dot{\hat{\varepsilon}}_3 \\ \dot{\hat{\varepsilon}}_4 \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} \hat{\varepsilon}_3 + \pi_3(\hat{w}) - f_{11}(\hat{\varepsilon} + \pi(\hat{w})) - \frac{\partial \pi_1}{\partial \hat{w}} s(\hat{w}) \\ \hat{\varepsilon}_4 + \pi_4(\hat{w}) + \alpha \hat{w}_1 \\ f_{31}(\hat{\varepsilon} + \pi(\hat{w})) - \frac{\partial \pi_3}{\partial \hat{w}} s(\hat{w}) \\ f_{41}(\hat{\varepsilon} + \pi(\hat{w})) + b_4 u - \frac{\partial \pi_4}{\partial \hat{w}} s(\hat{w}) \\ \alpha \hat{w}_2 \\ -\alpha \hat{w}_1 \end{bmatrix} + L_1(e - \hat{e}) + L_2 \xi$$

$$\dot{\xi} = e - \hat{e},$$

where $e = z_2 - w_2 = \varepsilon_2$, and $\hat{e} = \hat{\varepsilon}_2$.

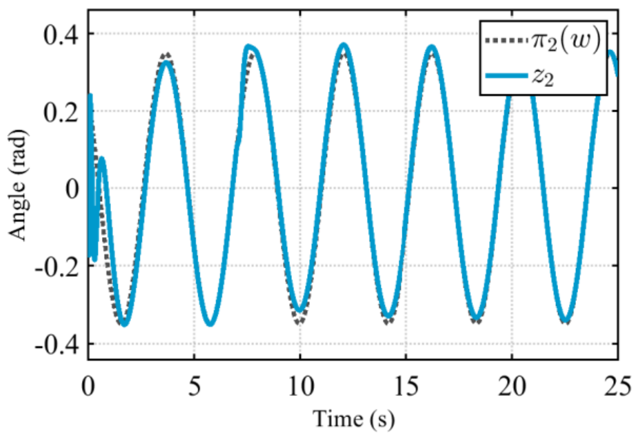

Fig. 1. Output tracking signal z_2 (solid blue), and reference w_2 (dotted gray).

With the estimation of the states defined, the sliding manifold can be designed as

$$\sigma = \hat{\varepsilon}_4 + C_s[\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3]^T, \tag{49}$$

and the discontinuous control is selected as in (30).

4.1 Numerical Evaluation

Using Table 1, the linear matrices A_{11}, A_{12}, are numerically evaluated. The matrix C_s is computed using an LQR algorithm aiming to obtain a feedback law that minimizes a cost functional, thus obtaining $C_s = [49.64, 48.64, 8.0203]^T$. By consequence, the matrix $(A_{11} - A_{12}C_s)$ has the eigenvalues $(-0.99, -6.21 + 0.44i, -6.21 - 0.44i)$.

The PI Observer gains, L_1 and L_2, are computed as in Beale and Shafai (1988), obtaining $L_1 = (1 \times 10^4)[-1.4097, 0.0260, -8.8378, 1.9860, 0.5930, -0.4517]$, $L_2 = (1 \times 10^4)[-0.0386, 0.0100, -1.6190, -0.0186, 0.6034, -0.0168]$, which ensures the matrix R has the eigenvalues $(-150, -70, -18, -13, -7, -2.5, -2)$.

Setting $\alpha = 1.5$, the following values for the coefficients of the regulator equations were found: $a_0 = 1.57$, $a_1 = -0.008$, $a_2 - 0.98$, $a_3 = a_4 = a_5 = 0$, $a_6 = -0.57 \times 10^{-4}$, $a_7 = 0.01$, $a_8 = -2.57 \times 10^{-5}$, $a_9 = -0.007$.

In order to test the efficiency of the proposed method, a numerical simulation was implemented, and the plant parametrs values from Table (1) were used along with the following initial conditions:

$$z_1(0) = 1.57\text{rad}, \qquad z_2(0) = z_3(0) = z_4(0) = 0,$$

$$w_1(0) = w_2(0) = \frac{1}{\sqrt{2}} 0.35.$$

The external perturbation element in (41) is considered as $\delta(z, t) = 0.0001 \sin(0.1t)$ and is present from $t > 7$.

	Link 1	Link 2
Mass (kg)	0.0551	0.0237
Length (m)	0.0825	0.2197
Center of mass distance (m)	0.0523	0.0799
Moment of inertia (kg m^2)	6.272×10^{-5}	1.759×10^{-4}
Friction coefficient (kg $m^2 s^{-1}$)	5.5286×10^{-4}	9.8895×10^{-5}

Table 1. Nominal values for parameters

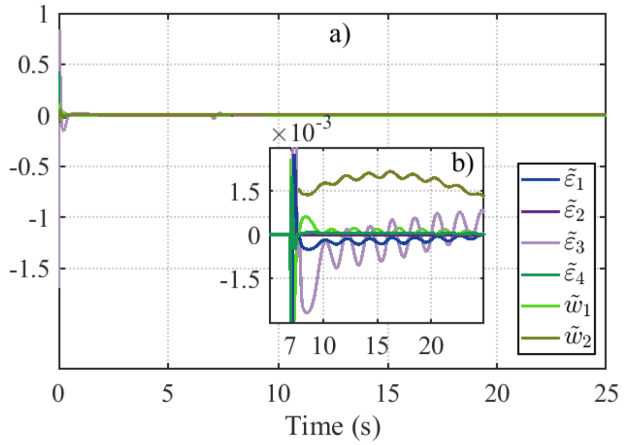

Fig. 2. a) Observation error. b) Zoomed figure

Fig. 1 shows the output of the Pendubot z_2, tracking the desired reference $\pi_2(w) = w_2$. At $t = 7$, a small transient response can be appreciated due to the effect of the perturbation, however, the proposed scheme is robust enough to compensate its effect. In Fig. 2 the boundedness of the observation error is shown, whereas in Fig. 3 the discontinuous control action appears, along with the equivalent control, depicted only for illustration purposes. It can be seen that the equivalent control has a slight offset from the origin line, which can be interpreted as a consequence of the switching control law that enforces a sliding motion on the surface $\sigma(\hat{\varepsilon}) = 0$, and since the observation error is not zero, $\sigma(\hat{\varepsilon}) \neq \sigma(\varepsilon)$.

5. CONCLUSION

The Error Feedback SM Regulator problem for nonlinear systems in regular form with unmodeled external matched perturbations was analyzed, and conditions for existence of solution were derived. It is worth to note that the stability analysis shown the explicit bound of the solution, which is proportional to the bound of the external perturbation. Using a nonlinear PI observer, a sliding manifold was proposed that used the estimated states. The effectiveness of the proposed method is demonstrated by the application of the regulator to the Pendubot system.

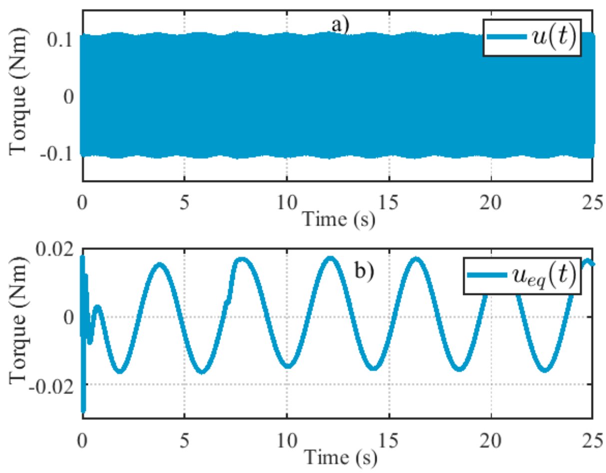

Fig. 3. a) SM control signal $u(t)$. b) Equivalent control $u_{eq}(t)$.

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