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On the Stability of a Stochastic Nonlinear Model of the Heart Beat Rate During a Treadmill Exercise

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Abstract: We investigate the stability properties of a nonlinear stochastic dynamical model of a person's heart beat rate (HBR) during a treadmill exercise. The analysis is based on the Lyapunov direct method and it is valid for systems with either known or unknown parameters. Specifically, we characterize an upper bound on the norm of the cumulative noise that holds in the presence of bounded errors in the model parameters and guarantees p-stability. Numerical simulations are presented that corroborate the theoretical results.

Keywords: Stability analysis, stochastic processes, nonlinear dynamcs, parameter perturbation, system biology.

1. INTRODUCTION

Physical exercise plays an important role in improving the general fitness and quality of life of both healthy individuals and patients with cardiovascular diseases. Further advantages include diabetes control, rehabilitation of spinal cord injury and stroke patients, prediction of cardiac failure from dialysis, developing more efficient weight loss protocols for the obese, physical fitness, individual health programs, etc Mersy (1991); Achten and Jeukendrup (2003); Barbeau et al. (1999); Aronow (2001).

A common and very relevant indicator of the intensity of a physical exercise for a given subject is her heart beat rate (HBR). The HBR can vary as the body's need to absorb oxygen and excrete carbon dioxide changes, such as during sleep, illness and, in particular, when sustaining some physical exercise. Because each individual has a constant blood volume, one of the physiological ways to deliver more oxygen to an organ is to increase the HBR so to make blood pass through the organ more often. These biological facts, together with the ease of use and the low cost of measurement equipment, have made HBR the best-known and most widely used indicator of exercise intensity.

Indeed, HBR monitoring helps physicians manage and control training exercises in order to ensure that the subject is safe during the practice. For further reliability and safety, physicians individualize HBR profiles by taking into account the physiological state of the subject. Such procedures clearly call for appropriate (accurate and reliable) models to simulate HBR. A very flexible, and commonly accepted, simulation model for the HBR during a treadmill exercise was introduced in Cheng et al. (2008). This model describes a specific nonlinear input-output relationship linking the HBR (the *output*) and the treadmill speed (the *input*). One advantage of this approach is that it yields a rather simple characterization consisting of just two states –one representing the deviation of the HBR from the subject's HBR when at rest, and another one modeling any internal peripheral effects.

In a previous work Asheghana and Miguez (2016), we provided a sufficient and necessary (but rather rigid) condition for the stability of the nonlinear model of Cheng et al. (2008). Here, we introduce a stochastic version of the latter model and then proceed to analyse its long-term stability. Stochasticity is introduced in the system by way of an additive Wiener process whose intensity depends on a (possibly nonlinear) transformation of the HBR itself. Compared to its deterministic counterpart, the proposed stochastic model can account for physiological effects that are not explicitly represented, as well as measurement errors. Our main contribution is the stability analysis of this stochastic HBR model. The technique we employ departs from the analysis in Asheghana and Miguez (2016) for the deterministic system. Indeed, we incorporate ideas from Deng et al. (2001) in order to obtain a sufficient condition (on the ranges of the unknown model parameters) for global stability in probability of the stochastic dynamical system.

The rest of the paper is organized as follows. In Section (2) we introduce the model of Cheng et al. (2008) and the basic stability result from Asheghana and Miguez (2016). The new stochastic version of the HBR model is introduced and analyzed in Section (3). Then, in Section (4) we present some illustrative numerical results and finally make some concluding remarks in Section (5).

2. BACKGROUND

The model proposed in Cheng et al. (2008) to simulate the HBR consists of the system of differential equations

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + a_2 x_2(t) + a_2 u^2(t) \\ \dot{x}_2(t) &= -a_3 x_2(t) + \phi(x_1(t)) \end{aligned} \qquad (1)$$

where ϕ is a nonlinearity defined as

$$\phi(y) = \frac{a_4 y}{1 + \exp(-(y - a_5))},$$

$a_i > 0$, $i = 1, ..., 5$, are positive and static model parameters, $x_1(t)$ is proportional to the deviation of the instantaneous HBR from the nominal rate when the subject is at rest (in particular, the instantaneous HBR is $h(t) = 4x_1(t) + 74$) and $x_2(t)$ represents the superposition of various internal, and typically slower, processes that take place in the body during the exercise and affect the HBR. Changes in types and density of hormones, boosted metabolism and the increase of body temperature are some examples of such processes (see Cheng et al. (2008) for additional details and examples). The input signal $u(t)$ is the treadmill speed, which serves as an indicator of the exercise intensity. Note that, according to Eq. (1) and the definition of ϕ, the signals $x_1(t)$, $x_2(t)$ and $u(t)$ are always positive Cheng et al. (2008). In the rest of the paper, we use $v(t) = a_2 u^2(t)$ for simplicity.

In Asheghana and Miguez (2016) we introduced the perturbed model

$$\begin{aligned} \dot{x}_1(t) &= -\tilde{a}_1 x_1(t) + \tilde{a}_2 x_2(t) + v(t) \\ \dot{x}_2(t) &= -\tilde{a}_3 x_2(t) + \tilde{\phi}(x_1(t)), \end{aligned} \qquad (2)$$

where

$$\tilde{\phi}(x_1(t)) = \frac{\tilde{a}_4 x_1(t)}{1 + \exp(-(x_1(t) - \tilde{a}_5))} \qquad (3)$$

and the parameters $\tilde{a}_i$, $i = 1, \ldots, 5$, are unknown but bounded and contained in the open intervals $(\underline{a}_i, \bar{a}_i)$, $i = 1, \ldots, 5$, where $0 < \underline{a}_i < \bar{a}_i < \infty$. It is shown in Asheghana and Miguez (2016) that system (1) is stable if

$$m = \frac{a_1 a_3}{a_2 a_4} > 1, \qquad (4)$$

while the system is unstable with $m < 1$.

3. STOCHASTIC MODEL AND STABILITY ANALYSIS

In this section we extend our results to a "noisy" version of system (2). Consider the stochastic differential equation [1]

$$dx = h(x,t)\,dt + g(x,t)d\omega, \qquad (5)$$

where state variable $x(t)$ is a stochastic process taking values on $\mathbb{R}^n$ with an associated probability measure $\mathbb{P}$, $\omega(t)$ is an independent standard Wiener process, and $h : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ are continuous functions of $x(t)$ that satisfy $h(0,t) = g(0,t) \equiv 0$. The following definitions and Lemma (1) below will be used throughout this section.

Definition 1. (from Khalil and Grizzle (2002)). A real and continuous function $\alpha(r)$, defined for $r \in [0,\infty)$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if $\lim_{r\to\infty} \alpha(r) = \infty$.

Definition 2. (from Deng et al. (2001)). The equilibrium $x \equiv 0$ of system (5) is globally stable in probability if, $\forall\varepsilon > 0$, there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that $\mathbb{P}\{\|x(t)\| < \gamma(\|x(0)\|)\} \geq 1 - \varepsilon$, $\forall t \geq 0$, $x(0) \in \mathbb{R}^n \backslash 0$.

Definition 3. (from Khalil and Grizzle (2002)). We say a real function $V : \mathbb{R}^n \to \mathbb{R}$ belongs to the class $\mathcal{C}^2$ if every partial derivative of V up to order 2 is continuous.

Lemma 1. (from Deng et al. (2001)). Consider system (5) and assume that there exists a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to [0,\infty)$, class $\mathcal{K}_\infty$ functions α_1, α_2 and a continuous non-negative function $W : \mathbb{R}^n \to \mathbb{R}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \qquad (6)$$

and

$$\begin{aligned} \mathcal{L}V(x,t) &= \frac{\partial V}{\partial x} h(x,t) + \frac{1}{2}\mathrm{Tr}\{g^\top(x,t)\frac{\partial^2 V}{\partial x^2} g(x,t)\} \\ &\leq -W(x,t), \end{aligned} \qquad (7)$$

then there is a unique strong solution of (5) for each $x(0) \in \mathbb{R}^n$, the equilibrium $x \equiv 0$ is globally stable in probability and

$$\mathbb{P}\left\{\lim_{t\to\infty} W(x,t) = 0\right\} = 1, \quad \forall x(0) \in \mathbb{R}^n. \quad \diamond \qquad (8)$$

We can recast the perturbed deterministic model of Eq. (2) under the framework in this section if we simply define $x(t) = (x_1(t), x_2(t))^\top$ and

$$f(x,t) = \begin{pmatrix} -\tilde{a}_1 x_1(t) & \tilde{a}_2 x_2(t) \\ \tilde{\phi}(x_1(t)) & -\tilde{a}_3 x_2(t) \end{pmatrix} \qquad (9)$$

where $\tilde{\phi}(x_1(t))$ is defined in (3). Recall here that the parameters $\tilde{a}_i$, $i = 1, ..., 4$, are unknown but bounded, namely $\tilde{a}_i \in (\underline{a}_i, \overline{a}_i)$, for some positive boundaries $0 < \underline{a}_i < \overline{a}_i$. Then the stochastic differential equation

$$dx = f(x,t)\,dt + g(x,t)d\omega \qquad (10)$$

becomes a stochastic version of model (2), whose theoretical properties may vary depending on the form of function $g(x,t)$. If we assume that $g(x,t)$ grows at most sublinearly with respect to x, namely

$$\|g(x,t)\| \leq \eta \|x(t)\|, \quad \forall t > 0, \qquad (11)$$

where $\eta > 0$ is a real constant, then we can provide an explicit condition to guarantee that model (10) is globally and asymptotically stable with probability 1.

[1] In this section we adopt a standard notation for stochastic differential equations (see, e.g., Øksendal (2007)) in order to make the results easily comparable with the literature in the field. The time dependence of x is left implicit.

Theorem 1. Assume that the inequality (4) holds and the constant $\eta > 0$ in (11) satisfies

$$\eta < \sqrt{\underline{a}_1 + \underline{a}_3 - \sqrt{(\underline{a}_1 - \underline{a}_3)^2 + 4\bar{a}_2\bar{a}_4}}. \tag{12}$$

Then, system (10) is globally stable in probability around $x \equiv 0$ and

$$P\left\{\lim_{t\to\infty} \|x(t)\| = 0\right\} = 1, \quad \forall x(0) \in R^n \tag{13}$$

Proof: Define the candidate Lyapunov function

$$V(x(t)) = x^\top(t)Px(t), \quad P = \begin{pmatrix} \tilde{a}_4 & 0 \\ 0 & \tilde{a}_2 \end{pmatrix}. \tag{14}$$

We aim at applying Lemma (1) with function V in (14) in order to prove that model (10) is stable around $x \equiv 0$. Replacing function $h(x,t)$ in (8) by $f(x,t)$ as defined in (9) we obtain

$$\begin{aligned}\mathcal{L}V(x,t) &= x^\top(t)\left(PA + A^\top P\right)x(t) \\ &\quad + g^\top(x(t))Pg(x(t)),\end{aligned} \tag{15}$$

where

$$A = \begin{pmatrix} -\tilde{a}_1 & \tilde{a}_2 \\ \tilde{a}_4\tilde{\Phi}(x_1(t)) & -\tilde{a}_3 \end{pmatrix}$$

and

$$\tilde{\Phi}(x_1(t)) = \frac{1}{1 + \exp\left(-\left(x_1(t) - \tilde{a}_5\right)\right)}. \tag{16}$$

Equations (11) and (15) together lead us to the inequality

$$\mathcal{L}V(x,t) \le x^\top(t)Zx(t), \tag{17}$$

where

$$\begin{aligned} Z &= PA + A^\top P + \frac{\eta^2}{2}\begin{pmatrix} \tilde{a}_4 & 0 \\ 0 & \tilde{a}_2 \end{pmatrix} \\ &= \begin{pmatrix} -\tilde{a}_1\tilde{a}_4 + \tilde{a}_4\dfrac{\eta^2}{2} & \tilde{a}_2\tilde{a}_4\left(\dfrac{1+\tilde{\Phi}(x_1(t))}{2}\right) \\ \tilde{a}_2\tilde{a}_4\left(\dfrac{1+\tilde{\Phi}(x_1(t))}{2}\right) & -\tilde{a}_2\tilde{a}_3 + \tilde{a}_2\dfrac{\eta^2}{2} \end{pmatrix} \end{aligned} \tag{18}$$

In order to use Lemma (1) we first introduce the class $\mathcal{K}_\infty$ functions

$$\begin{aligned} \alpha_1(\|x(t)\|) &= \min\{\underline{a}_2, \underline{a}_4\}\|x(t)\|^2 \\ \alpha_2(\|x(t)\|) &= \max\{\bar{a}_2, \bar{a}_4\}\|x(t)\|^2 \end{aligned} \tag{19}$$

that obviously satisfy (6), and all that remains is to construct a continuous and non-negative function $W(x)$ that satisfies $\mathcal{L}V(x,t) \le -W(x)$, for $\mathcal{L}V(x,t)$ given by (15). From (17), the obvious choice is $W(x) = -x^\top Zx$, as long as we can guarantee that $Z < 0$. Indeed, straightforward calculations (see Appendix A) show that if η is chosen to satisfy

$$\eta < \eta_{max} = \sqrt{\tilde{a}_1 + \tilde{a}_3 - \sqrt{(\tilde{a}_1 - \tilde{a}_3)^2 + 4\tilde{a}_2\tilde{a}_4}}, \tag{20}$$

then both

$$\det(Z) > 0 \quad \text{and} \quad \text{trace}(Z) < 0 \tag{21}$$

which, in turn, imply $Z < 0$. Since $\underline{a}_i \le \tilde{a}_i \le \bar{a}_i$ for every $i = 1, \ldots, 4$, if we select η such that

$$\begin{aligned}\eta &< \min_{\substack{a_i \in (\underline{a}_i, \bar{a}_i) \\ i=1,\ldots,4}} \sqrt{\tilde{a}_1 + \tilde{a}_3 - \sqrt{(\tilde{a}_1 - \tilde{a}_3)^2 + 4\tilde{a}_2\tilde{a}_4}}, \\ &= \sqrt{\underline{a}_1 + \underline{a}_3 - \sqrt{(\underline{a}_1 - \underline{a}_3)^2 + 4\bar{a}_2\bar{a}_4}}\end{aligned} \tag{22}$$

then (21) is guaranteed to hold and hence $Z > 0$ is ensured as well [2]. Therefore, if (22) holds, α_1 and α_2 in (19) together with $W(x) = -x^\top Zx$ satisfy the assumptions of Lemma (1) and, as a consequence, system (10) is globally stable in probability around $x \equiv 0$ and

$$\mathbb{P}\left\{\lim_{t\to\infty}\|W(x,t)\| = 0\right\} = 1, \quad \forall x(0) \in \mathbb{R}^n.$$

Finally, since $Z < 0$ (strictly negative definite), it is apparent that $\lim_{t\to\infty} W(x,t) = 0$ implies $\lim_{t\to\infty}\|x(t)\| = 0$ and, therefore, Eq. (13) holds, which concludes the proof. □

Remark 1. Substitution of the nominal values

$$\begin{aligned} &\hat{a}_1 = 1.84, \quad \hat{a}_2 = 24.32, \\ &\hat{a}_3 = 0.0636, \quad \hat{a}_4 = 0.00321, \quad \hat{a}_5 = 8.32, \end{aligned} \tag{23}$$

(adopted from Cheng et al. (2008)) in (12) reveals that $\eta \le \eta_{max} \simeq 0.2$ is a sufficient condition for this set of parameters. In particular, it is guaranteed that the bound on the right hand side of (22) is real for this range of the parameters. ◇

4. NUMERICAL SIMULATION

In this section, we present computer simulation results that corroborate the validity of upper bound in expression (22) which, in turn, determines how restrictive the constraint $\|g(x,t)\| \le \eta\|x(t)\|$ becomes. We use the nominal parameter values (24), which yield $\eta_{max} \simeq 0.2$, as an upper bound for η. The simulations are based on Eq. (10) with initial values $[x_1(0), x_2(0)]^\top = [2, 0.8]^\top$ and

$$g(x,t) = \eta\begin{pmatrix} \dfrac{x_1(t) - x_2(t)}{\sqrt{2}} \\ \dfrac{x_1(t) + x_2(t)}{\sqrt{2}} \end{pmatrix}. \tag{24}$$

This choice of $g(x,t)$ readily implies that $\|g(x,t)\| = \eta\|x(t)\|$.

Simulations show that system (10) with $g(x,t)$ defined as in (24) is stable with $\eta = 0.15 < \eta_{max} = 0.2$, as expected (see Fig. (1).a). Based on Theorem (1), convergence towards $x \equiv 0$ is not guaranteed for $\eta > \eta_{max}$. This is in agreement with the computer simulations shown in Fig. (1).b and c, where system does not display convergence with $\eta = 0.25$ (Fig. (1).b) and diverges with $\eta = 0.35$ (Fig. (1).c).

We have also studied numerically how the bound η_{max} evolves as the parameters a_i, $i = 1, \ldots, 4$, change. This

[2] Taking derivatives on the right hand side of (20) it is easy to verify that the minimum of η is given by (22), as long as $\tilde{a}_i > 1, i = 1, \ldots, 4$ and (4) is satisfied.

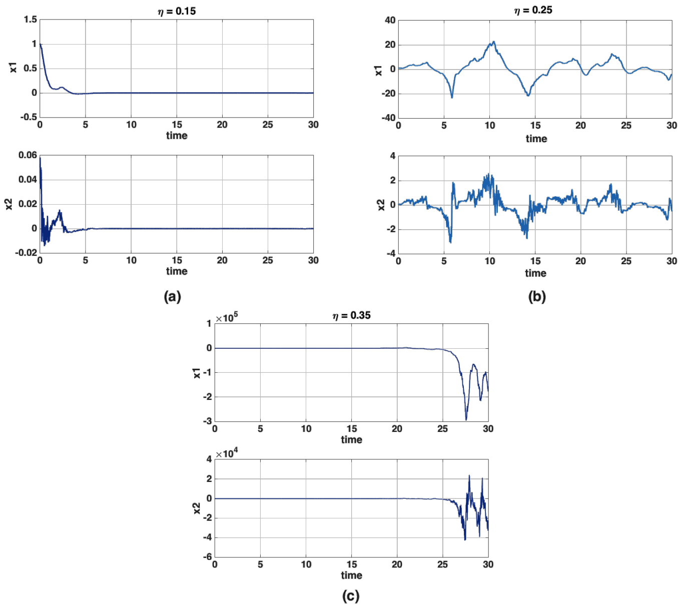

Fig. 1. System behavior with different values of the constant η. (a) For $\eta = 0.15$ we obtain convergence to98y3o2; $x \equiv 0$. (b) For $\eta = 0.25$ the state does not converge. (c) For $\eta = 0.35$ the state diverges.

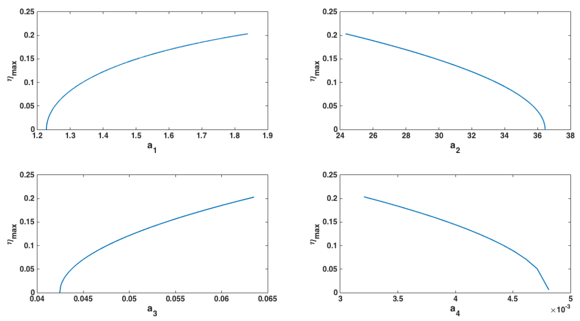

Fig. 2. Evolution of the bound η_{max} in (11) using the nominal values in (24) when we let single parameter run from its nominal to its marginal value. Upper left: a_1 changes, a_2, a_3, a_4 are kept fixed. Upper right: a_2 changes, a_1, a_3, a_4 are kept fixed. Lower left: a_3 changes, a_1, a_2, a_4 are kept fixed. Lower right: a_4 changes, a_1, a_2, a_3 are kept fixed.

is shown in Fig.(2), where each plot represents the bound η_{max} calculated using the nominal values in (24), except for one parameter which varies from its nominal to its marginal value (i.e., the value that makes $m = 1$ in the stability condition of (4)). The figure shows that, for all parameters $a_1, \ldots, a_4$, the bound η_{max} drops when the perturbed non-stochastic system approaches its stability margin. In other words, Fig.(2) shows that the constraint (11) is less restrictive, and the system can tolerate a "larger amount" of noise, when the ratio $m = a_1 a_3 / a_2 a_4$ of (4) is further from 1.

5. CONCLUSIONS

We have tackled the analysis of the stability of a stochastic model of HBR, built upon the one originally introduced in Cheng et al. (2008) in deterministic form. In our study, we have assumed that the system parameters are unknown. Specifically, each parameter a_i is subject to arbitrary perturbations within a fixed interval, while cumulative stochastic noise vanishes when the system state approaches a rest status. In our analysis, we have introduced a sublinear condition, of the form $\|g(x,t)\| \leq \eta \|x(t)\|$, for the function that controls the intensity of the additive Wiener process in the differential equation. Then, we have shown numerically that the constant η (for which we have obtained an explicit upper bound η_{max}) is key to determine the range of parameter values that guarantee system stability.

Appendix A. CALCULATION

The inequality

$$Z = \begin{pmatrix} -\tilde{a}_1\tilde{a}_4 + \tilde{a}_4 \dfrac{\eta^2}{2} & \tilde{a}_2\tilde{a}_4 \left(\dfrac{1 + \tilde{\Phi}(x_1(t))}{2} \right) \\ \tilde{a}_2\tilde{a}_4 \left(\dfrac{1 + \tilde{\Phi}(x_1(t))}{2} \right) & -\tilde{a}_2\tilde{a}_3 + \tilde{a}_2 \dfrac{\eta^2}{2} \end{pmatrix} < 0 \tag{A.1}$$

is satisfied when $\text{trace}(Z) < 0$ and $\det(Z) > 0$. In the sequel, we seek an upper bound for η such that $Z < 0$ is guaranteed. We proceed by constructing intervals for the parameter η that satisfy $\text{trace}(Z) < 0$ and $\det(Z) > 0$ separately and then take their intersection.

Let us start finding conditions on η such that $\text{trace}(Z) < 0$. From the definition of matrix Z in (A.1) we can write

$$\text{trace}(Z) = -\tilde{a}_1\tilde{a}_4 + \tilde{a}_4 \frac{\eta^2}{2} - \tilde{a}_2\tilde{a}_3 + \tilde{a}_2 \frac{\eta^2}{2}. \tag{A.2}$$

The constraint $\text{trace}(Z) < 0$ then implies that

$$\eta < \eta_{tr} = \sqrt{2 \frac{\tilde{a}_1\tilde{a}_4 + \tilde{a}_2\tilde{a}_3}{\tilde{a}_2 + \tilde{a}_4}}. \tag{A.3}$$

where $\eta_{tr} \in \mathbb{R}$ whenever (4) is satisfied.

On the other hand, $\det(Z)$ can be written as

$$\det(Z) = \frac{\tilde{a}_2\tilde{a}_4}{4}\eta^4 - \frac{\tilde{a}_2\tilde{a}_4}{2}(\tilde{a}_1 + \tilde{a}_3)\eta^2 + \tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4 - \tilde{a}_2^2\tilde{a}_4^2 \left(\frac{1 + \tilde{\Phi}(x_1(t))}{2} \right)^2. \tag{A.4}$$

Since

$$\frac{1 + \tilde{\Phi}(x_1(t))}{2} < 1, \quad \forall x_1(t) > 0,$$

it readily follows that

$$\underline{\det}(Z) = \frac{\tilde{a}_2\tilde{a}_4}{4}\eta^4 - \frac{\tilde{a}_2\tilde{a}_4}{2}(\tilde{a}_1 + \tilde{a}_3)\eta^2 + \tilde{a}_1\tilde{a}_2\tilde{a}_3\tilde{a}_4 - \tilde{a}_2^2\tilde{a}_4^2 \tag{A.5}$$

is a lower bound for $\det(Z)$, i.e., $\underline{\det}(Z) < \det(Z)$. The roots of the equation $\underline{\det}(Z) = 0$ are real (assuming (4) holds) and can be easily calculated as

$$\begin{aligned} \eta_1 = -\eta_2 &= \sqrt{\tilde{a}_1 + \tilde{a}_3 - \sqrt{(\tilde{a}_1 - \tilde{a}_3)^2 + 4\tilde{a}_2\tilde{a}_4}} \\ \eta_3 = -\eta_4 &= \sqrt{\tilde{a}_1 + \tilde{a}_3 + \sqrt{(\tilde{a}_1 - \tilde{a}_3)^2 + 4\tilde{a}_2\tilde{a}_4}} \end{aligned}. \tag{A.6}$$

Moreover, a simple inspection reveals that $\underline{\det}(Z) > 0$ when $\eta \in (-\infty, \eta_1) \cup (\eta_3, +\infty)$, and $\underline{\det}(Z) \leq 0$ otherwise. If we substitute η_{tr} into $\underline{\det}(Z)$, we have

$$\underline{\det}(Z)|_{\eta=\eta_{tr}} = -\frac{\tilde{a}_2^2\tilde{a}_4^2}{(\tilde{a}_2+\tilde{a}_4)^2}\left((\tilde{a}_1-\tilde{a}_3)^2+(\tilde{a}_2+\tilde{a}_4)^2\right) < 0, \tag{A.7}$$

hence $\eta_1 < \eta_{tr} < \eta_3$ (by combining (A.3) and (A.6)). Therefore, if $\eta \in (-\infty, \eta_1)$ then

(i) the inequality (A.3) holds and, as a consequence, $\text{trace}(Z) < 0$;
(ii) $\underline{\det}(Z) > 0$ and, as a consequence, $\det(Z) > 0$;
(iii) from (i) and (ii) it follows that $Z < 0$.

Finally note that the bound on the right hand side of (20) is η_1 as defined in Eq. (A.6).

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