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Sliding Mode Tracking Control for Nonlinear Sampled-Data Systems with Unmatched Perturbation ⋆

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Abstract: The output tracking problem for a class of sampled-data nonlinear systems exposed in Nonlinear Block Controllable (NBC) form is faced. This paper considers both matched and unmatched perturbations. To formulate a desired sliding manifold on which the impact of unmatched perturbation is attenuated, the Block Control technique combined with the perturbation estimation, is implemented. A discrete-time sliding mode non-switching controller is synthesized such that the system state is driven toward a vicinity of the designed sliding manifold and stays there for all sampled time instants, avoiding chattering and reducing the matched perturbation effect. The effectiveness of the proposed technique is confirmed by simulation.

Keywords: Robust control, robust controller synthesis, sliding mode control, disturbance rejection, nonlinear systems, nonlinear control.

1. INTRODUCTION

Since modern control systems are implemented by computers, the stabilisation and output tracking control design for sampled-data systems remain as important topics of the Sliding Mode (SM) control theory (Utkin et al., 2009).

The main attention has been payed to reaching control design and numerous significant results have been obtained in Furuta (1990); Weibing Gao et al. (1995); Wang et al. (2009). In the proposed SM reaching laws, the switching term was preserved from continuous time SM control to suppress the effect of matched bounded perturbations. However, this term can produce undesired numerical chattering phenomenon in the vicinity of the sliding manifold. This effect was suppressed by implicit Euler discretization of the discontinuous term in Huber et al. (2016) as well by the time variation of the discontinuous term parameter (Chakrabarty and Bandyopadhyay, 2015) or by increasing the relative degree and correspondingly the order of SM (Levant and Livne, 2015; Koch et al., 2016).

To avoid the chattering problem, non-switching reaching controllers has been proposed in Golo and Milosavljevic (2000); Bartoszewicz and Latosinski (2016), including Equivalent-Control-Based (Utkin et al., 2009) and adaptive (Bartolini et al., 1995; Bartoszewicz and Adamiak, 2018) SM controllers. However, as a result of the lack of perturbation for calculating the equivalent control, sliding manifold reaches a boundary layer $\mathcal{O}(\tau)$ with τ as the sample period. In order to mitigate this obstacle, in some researches as Su et al. (2000); Abidi et al. (2007); Sharma and Janardhanan (2019); Zapata-Zuluaga and Loukianov (Dec, 2018), an estimator of perturbation using its previous step has been designed, and an accuracy of $\mathcal{O}(\tau^2)$ in the boundary layer of the sliding manifold has been achieved.

On the other hand, the design of the desired sliding manifold is of great interest. Basely, a classical linear sliding manifold has been synthesized for linear time-invariant (LTI) systems in Utkin et al. (2009); Furuta (1990), and those with matched perturbation in Wang et al. (2009); Huber et al. (2016); Chakrabarty and Bandyopadhyay (2015); Koch et al. (2016). Using multirate ouput feedback technique, a nonlinear sliding manifold has been formulated in Hou and Zhang (2018), again for LTI systems.

In practice, control plants dynamics are nonlinear and can be affected by both matched and unmatched perturbations, and the unmatched part affects the SM dynamics. Therefore, in this case, the central issue is to guarantee the precision bound for the state regulation or smallness of the tracking error. This problem has been analyzed in Abidi et al. (2007), however, again for LTI systems.

This work deals with the aforementioned approach, we consider a SM output tracking problem for a class of nonlinear systems presented in Nonlinear Block Controllable (NBC) form (Loukianov, 2002), with both matched and

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unmatched unmodeled perturbations. The considered class is quite wide and includes, for example, electromechanical systems (robot manipulators, quadrotors, electric motors and etc.). The principal aim is, first, using Block Control (BC) feedback linearization (FL) technique combined with the perturbation estimation, to design a desired nonlinear sliding manifold on which the system motion satisfies a specified transient response, and the effect of unmatched perturbation on the output tracking error is reduced. Then, a discrete-time SM controller, based on the equivalent control combined with the perturbation estimation, is formulated such that the system state is driven into a smaller bounding layer of the designed sliding manifold and stays there for all sampled time instants, avoiding chattering and reducing the matched perturbation effect.

It is worth to note that this work can be considered as a continuation and improvement of the result obtained in the work Zapata-Zuluaga and Loukianov (Dec, 2018) where only part of the nominal dynamics is linearized and, and the rest part is considered as a perturbation. It the present work, the complete nominal dynamics are canceled and linearized; as result, the region of attraction can be incremented. Moreover, in the present work, it is considered a general structure of the NBC form when the dimensions of the blocks are different, i.e., $n_i \leq n_{i-1}$, as in the considered example, permanent magnet synchronous motor control design.

2. PROBLEM FORMULATION

Consider a nonlinear uncertain system

$$\begin{aligned} \dot{x} &= q(x) + G(x)u + g(x,t) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector, $u \in \mathbb{R}^m$ is the control vector, $\text{rank}(G(x)) = m$ for all $x \in \mathbb{R}^n$ and $t \geq 0$, with $p \leq m$; the unknown mapping $g(x,t)$ characterizes external disturbances and parameter variations.

Assume that the system (1) can be presented (possible, under an appropriate nonlinear transformation) in the following NBC form (Loukianov, 2002):

$$\begin{aligned} \dot{x}_i &= q_i(\bar{x}_i) + G_i(\bar{x}_i)\bar{x}_{i+1} + g_i(x,t) \\ \dot{x}_r &= q_r(x) + G_r(x)u + g_r(x,t) \\ y &= x_1, i = 1, \ldots, r-1, \end{aligned} \tag{2}$$

where $x = \left[x_1^\top \ldots, x_r^\top\right]^\top$, $x_j \in \mathbb{R}^{n_j}$, $j = 1, \ldots, r$; $\bar{x}_1 = x_1$, $\bar{x}_i = \left[x_1^\top, \ldots, x_i^\top\right]^\top$, $i = 2, \ldots, r-1$. The indices $n_1, \ldots, n_r$ defines the structure of system (2) and satisfies

$$n_1 \leq n_2 \leq \ldots \leq n_r \leq m, \ \sum_{j=1}^{r} n_j = n, \ p = n_1. \tag{3}$$

The matrix $G_j(\bar{x}_j)$ in each block of (2) has full rank,

$$\text{rank}(G_j(\bar{x}_j)) = n_j \ \forall x \in \mathbb{R}^n, j = 1, ..., r.$$

Now, applying explicit Euler method to the system (2), the sampled-data system becomes

$$\begin{aligned} x_{i,k+1} &= f_i(\bar{x}_{i,k}) + B_i(\bar{x}_{i,k})\bar{x}_{i+1,k} + d_i(x_k,k) \\ x_{r,k+1} &= f_r(x_k) + B_r(x_k)u_k + d_r(x_k,k) \end{aligned} \tag{4}$$

$$y_k = x_{1,k}, \ i = 1, ..., r-1 \tag{5}$$

where $f_i(\bar{x}_{i,k}) = x_{i,k} + \tau q_i(\bar{x}_{i,k})$, $d_i(x_k,k) = \tau g_i(x_k,k)$ and $B_i(\bar{x}_{i,k}) = \tau G_i(\bar{x}_{i,k})$, $i = 1, ..., r$, with $k \in \mathbb{Z}^+ \cup \{0\}$ denotes the discrete time where $\mathbb{Z}^+$ is the set of the positive integers and $x_k, x_{1,k}, \ldots, x_{r,k}$, are the discrete approximation of $x(t), x_1(t), \ldots, x_r(t)$, respectively.

The control objective is to force the output y_k (5) to track a reference signal y_k^{ref}, reducing the effects of unmatched $d_i(\bar{x}_{i,k},k)$, $i = 1, ..., r-1$, and matched $d_r(x_k,k)$ perturbations.

This will be achieved in presence of constraint on the input

$$\|u(t)\| \leq u_{\max}, \qquad u_{\max} > 0. \tag{6}$$

The following assumptions are considered hereinafter.

Assumption 1. All the state variables are available for the measurement.

Assumption 2. The matrix $B_i(\bar{x}_{i,k+1})$ can be decomposed into two parts: the nominal part $\bar{B}_i(\bar{x}_{i+1,k})$ and unknown part $\Delta\bar{B}_i(x_k,k)$, namely

$$\bar{B}_i(\bar{x}_{i,k+1}) = \bar{B}_i(\bar{x}_{i+1,k}) + \Delta\bar{B}_i(x_k,k) \tag{7}$$

with $\bar{B}_i(\bar{x}_{i,k}) = \bar{B}_{i-1}(\bar{x}_{i,k})B_i(\bar{x}_{i,k}), k, \ \ i = 1, \ldots, r-1$.

Remark 1. Assumption 2 means that in the expression

$$\begin{aligned} \bar{B}_i(\bar{x}_{i,k+1}) &= \bar{B}_i(\phi(\bar{x}_{i+1,k} + d_i(x_k,k)), \\ \phi(\bar{x}_{i+1,k}) &= f_i(\bar{x}_{i,k}) + B_i(\bar{x}_{i,k})\bar{x}_{i+1,k} \end{aligned}$$

the nominal part can be distinguished, and the rest part is considered as a disturbance. This is possible, for example, in the case of system parameter variations and additive disturbances.

3. BLOCK TRANSFORMATION

In this section, the concept of Block Control method, adopted for discrete-time setup, is used to transform the original system to a desired form. The relation (3) means $n_i = n_{i+1}$ or $n_i < n_{i+1}$. To include in the design procedure both cases, we consider the following structure:

$$n_1 = n_2 < n_3 < \ldots < n_r = m. \tag{8}$$

Taking in the account the structure (8), the following transformation is introduced:

$$\begin{aligned} z_{1,k} &= x_{1,k} - \alpha_{1,k} := \psi_1(x_{1,k}) \\ z_{2,k} &= B_1(\bar{x}_{1,k})x_{2,k} - \alpha_{2,k} := \psi_2(\bar{x}_{2,k}) \\ z_{i,k} &= \bar{B}_{i-1}(\bar{x}_{i-1,k})x_{i,k} - \alpha_{i,k} := \psi_i(\bar{x}_{i,k}) \\ & i = 3, ..., r \end{aligned} \tag{9}$$

where $z_{i,k}$ is $(n_i \times 1)$ vector, $i = 1, ..., r$;

$$\tilde{B}_i = \begin{bmatrix} \bar{B}_i \\ E_{i,1} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}, \quad \bar{B}_i = (\bar{B}_{i-1}B_i) \in \mathbb{R}^{n_{i-1} \times n_i},$$

$$E_{i,1} = [0 \ I_{n_i - n_{i-1}}] \in \mathbb{R}^{(n_i - n_{i-1}) \times n_i}, \ i = 3, ..., r-1;$$

$$\alpha_{1,k} = y_k^{ref}, \qquad \alpha_{2,k} = K_1 z_{1,k} - f_1(x_{1,k}) - \bar{d}_{1,k-1}$$

$$\alpha_{i+1,k} = \begin{bmatrix} K_i z_{i,k} - \bar{f}_i(\bar{x}_{i,k}) - \bar{d}_{i,k-1} \\ 0 \end{bmatrix}, \ i = 2, ..., r-1.$$

with the design matrices K_i, $i = 1, ..., r-1$.

Theorem 1. Under Assumption 2, using the transformation (9), the system (4) with the structure (3) can be presented in the following desired form:

$$\begin{aligned} z_{1,k+1} &= K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \\ z_{i,k+1} &= K_i z_{i,k} + E_{i,1} z_{i+1,k} + \delta_{i,k}, \ i = 2, ..., r-1 \\ z_{r,k+1} &= \bar{f}_r(z_k) + \bar{B}_r(z_k)u_k + \bar{d}_{r,k} \end{aligned} \tag{10}$$

where $z_k = [z_{1,k}^\top, \ldots, z_{r,k}^\top]^\top$, $\delta_{j,k} = \bar{d}_{j,k} - \bar{d}_{j,k-1}$ for $j = 1, \ldots, r-1$.

The proof is given in Appendix.

It is worth to note that the original systems (4) is represented in the control error system (10) where the nominal part is completely linearized by the transformation (9), and the perturbation unmatched terms $\delta_{i,k}$, $i = 1, ..., r-1$ are of the order $\mathcal{O}(\tau^2)$.

4. DISCRETE-TIME SLIDING MODE CONTROL

To generate sliding mode in (10), the sliding variable is chosen as Loukianov (2002) $\sigma_k = z_{r,k}$ and its dynamics can be obtained from (10) as

$$\sigma_{k+1} = \bar{f}_{r,k} + \bar{B}_{r,k} u_k + \bar{d}_{r,k} \tag{11}$$

with $\bar{f}_{r,k} = \bar{f}_r(z_k)$, $\bar{B}_{r,k} = \bar{B}_r(z_k)$.

To induce chattering-free sliding mode on $\sigma_k = 0$ reducing the matched perturbation $\bar{d}_{r,k}$ effect and considering the control constraint (6), the control u_k is selected of the form

$$u_k = \begin{cases} \tilde{u}_{eq,k} & \text{for } \|\tilde{u}_{eq,k}\| \leq u_{\max} \\ u_{\max} \frac{\tilde{u}_{eq,k}}{\|\tilde{u}_{eq,k}\|} & \text{for } \|\tilde{u}_{eq,k}\| > u_{\max}. \end{cases} \tag{12}$$

with

$$\tilde{u}_{eq,k} = -\bar{B}_{r,k}^\dagger \left(\bar{f}_{r,k} + \bar{d}_{r,k-1}\right) \tag{13}$$

where the perturbation $\bar{d}_{r,k-1}$ can be obtained from (11) of the form $\bar{d}_{r,k-1} = \sigma_k - \bar{f}_{r,k-1} - \bar{B}_{r,k-1} u_{k-1}$ resulting in

$$\begin{aligned} \tilde{u}_{eq,k} &= u_{s,k} + \bar{B}_{r,k}^\dagger \left[-\sigma_k + \bar{f}_{r,k-1} + \bar{B}_{r,k-1}\tilde{u}_{eq,k-1}\right] \\ u_{s,k} &= -\bar{B}_{r,k}^\dagger \bar{f}_{r,k}. \end{aligned} \tag{14}$$

Applying the control (14) to (11), the closed-loop system becomes

$$\sigma_{k+1} = \delta_{r,k}, \;\; \delta_{r,k} = \bar{d}_{r,k} - \bar{d}_{r,k-1}. \tag{15}$$

To analyze stability of closed-loop system (11)-(12) motion over the manifold $\sigma_k = 0$, let us represent the structure of system (11) and control (13), by imposing the term $\sigma_k + \psi_{r,k} = 0$, $\psi_{r,k} = \psi_r(x_k)|_{x_k = \psi_r^{-1}(z_k)}$, into

$$\sigma_{k+1} = \sigma_k + \bar{\psi}_{r,k} + \bar{B}_{r,k} u_k + \bar{d}_{r,k} \tag{16}$$

$$\tilde{u}_{eq,k} = -\bar{B}_{r,k}^\dagger \sigma_k - \bar{B}_{r,k}^\dagger \left(\bar{\psi}_{r,k} + \bar{d}_{r,k-1}\right), \tag{17}$$

where $\bar{\psi}_{r,k} = \bar{f}_{r,k} + \psi_{r,k}$.

For the case $\|\tilde{u}_{eq,k}\| > u_{\max}$, it is assumed that the available control is sufficient to stabilize the system, i.e.

$$u_{\max} > \sup_k [\|\bar{B}_{r,k}^\dagger\| \|\bar{\psi}_{r,k} + \bar{d}_{r,k}\|]. \tag{18}$$

Substituting the control (12) with (17) in (16) yields

$$\begin{aligned} \sigma_{k+1} =& \sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k} - \frac{u_{\max}(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})}{\|\bar{B}_{r,k}^\dagger(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ =& (\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k}) \left(1 - \frac{u_{\max}}{\|\bar{B}_{r,k}^\dagger(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|}\right) \\ &+ \frac{u_{\max}\delta_{r,k}}{\|\bar{B}_{r,k}^\dagger \sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\|}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\sigma_{k+1}\| =& \|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1} + \delta_{r,k}\| \times \\ & \left(1 - \frac{u_{\max}}{\|\bar{B}_{r,k}^\dagger(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|}\right) \\ & + \frac{u_{\max}\delta_{r,k}}{\|\bar{B}_{r,k}^\dagger(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ =& \|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k}\| + \frac{u_{\max}\|\delta_{r,k}\|}{\|\bar{B}_{r,k}^\dagger \sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\|} \\ & - \frac{u_{\max}\|\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1}\| + \|\delta_{r,k}\|}{\|\bar{B}_{r,k}^\dagger(\sigma_k + \bar{\psi}_{r,k} + \bar{d}_{r,k-1})\|} \\ \leq& \|\sigma_k\| + \|\bar{\psi}_{r,k} + \bar{d}_{r,k}\| - \frac{u_{\max}}{\|\bar{B}_{r,k}^\dagger\|} < \|\sigma_k\| \end{aligned} \tag{19}$$

due to (18). Thus, as $\|\sigma_k\|$ decreases monotonically, $\tilde{u}_{eq,k}$ (17) does too, and there will be a time instant $\bar{k}$ such that $\|\tilde{u}_{eq,k}\| \leq u_{\max}$, for $k > \bar{k}$. At this time, the equivalent control $\tilde{u}_{eq,k}$ (13) or (14) is applied, bringing the closed-loop system trajectory in an $\mathcal{O}\left(\tau^2\right)$-neighborhood of the sliding manifold (Su et al., 2000), i.e.

$$\|\sigma_k\|| = \mathcal{O}(\tau^2), \;\; k > \bar{k}$$

achieving quasi-sliding mode.

Remark 2. Stability analysis (19) can be considered as an extension for the perturbed systems of the results obtained in Bartolini et al. (1995) for the nominal case.

5. SLIDING MODE DYNAMICS

Now, SM dynamics will be investigated, i.e. when the closed-loop system motion appears in the $\mathcal{O}(\tau^2)$ vicinity of the manifold $\sigma_k = z_{r,k} = 0$. This motion is governed by the reduced SM equation derived from (10) and (15) as

$$\begin{aligned} z_{1,k+1} &= K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \\ z_{i,k+1} &= K_i z_{i,k} + E_{i,1} z_{i+1,k} + \delta_{i,k}, \qquad i = 2, ..., r-1 \\ z_{r-1,k+1} &= K_{r-1} z_{r-1,k} + E_{r-1,1}\sigma_k + \delta_{r-1,k} \end{aligned}$$

or in compact form

$$\bar{z}_{r-1,k+1} = A_s \bar{z}_{r-1,k} + \delta_k \tag{20}$$

where $\bar{z}_{r-1,k} = \left[z_{1,k}^\top \; z_{2,k}^\top \; \ldots \; z_{r-1,k}^\top\right]^\top$ and

$$\begin{aligned} \mathbb{I}_a &= \text{subdiag}(\mathbb{I}_{n_1}, \ldots, \mathbb{I}_{n_{r-1}}), \\ A_s &= \text{diag}(K_1, \ldots, K_{r-1}) + \mathbb{I}_a, \\ \delta_k &= \left[\delta_{1,k}^\top, \ldots, \delta_{i,k}^\top, \ldots, (\delta_{r-1,k}^\top + \sigma_k)\right]^\top. \end{aligned}$$

Then, a solution of the system (20) is defined by

$$\bar{z}_{r-1,k} = A_s^k \bar{z}_{r-1,0} + \sum_{i=1}^{k-1} A_s^i \delta_{k-i-1}.$$

Since A_s is a Schur matrix, the steady state solution can be estimated by

$$\|\bar{z}_{r-1,k}\| \leq \sum_{i=1}^{k-1} \|A_s^i\| \|\delta_{k-i-1}\|. \tag{21}$$

From $\delta_{i,k} = \mathcal{O}(\tau^2)$, $i = 1, ..., r-1$ and $\|\sigma_k\| = \mathcal{O}(\tau^2)$ it follows $\delta_k = \mathcal{O}(\tau^2)$. Selecting the poles of order $\mathcal{O}(1)$ of the matrix A_s, yields

$$\|\bar{z}_{r-1,k}\| = \mathcal{O}(1)\mathcal{O}(\tau^2) = \mathcal{O}(\tau^2), \tag{22}$$

resulting in the tracking error $z_{1,k}$ (9) ultimate bound

$$\|z_{1,k}\| = \mathcal{O}(\tau^2). \tag{23}$$

The obtained results are formulated in the following theorem.

Theorem 2. Consider the robust tracking problem for the system (4) with the output (5) and the constraint control input (6). Let Assumption 2 with condition (18) are satisfied. Then, a solution of the system (4) closed by the control (12) with (14) is ultimately exponentially bounded by (22), and the tracking error $z_{1,k}$ defined in (9) is ultimately bounded by (23).

6. SIMULATION RESULTS

To show the effectiveness of the proposed approach, the permanent magnet synchronous motor (PMSM) control design problem is considered, and a simulation is carried out. The mathematical model of PMSM in the rotor reference frame is presented as follows (Zhang et al., 2019):

$$\begin{aligned}
\dot{\theta} &= n_p\omega \\
\dot{\omega} &= \frac{1}{J}\left(\frac{3}{2}n_p\lambda_f i_q - B_v\omega - T_L\right) \\
\dot{i}_d &= \frac{1}{L}\left(V_d - Ri_d + Ln_p\omega i_q\right) \\
\dot{i}_q &= \frac{1}{L}\left(V_q - Ri_q - Ln_p\omega i_d - \lambda_f n_p\omega\right),
\end{aligned} \tag{24}$$

where ω is the rotor angular velocity; i_d and i_q are d- and q-axis stator currents, respectively; V_d and V_q are d- and q-axis stator voltages, respectively; T_L is the load torque; n_p is the number of poles-pairs, equals 6; R is the stator resistance, equals 6.9Ω; L is the stator inductance, equals 21mH; λ_f is the magnetic flux linkage, equals 0.3342Wb; J is the moment of inertia, equals 0.002kgm^2; B_v is the frictional coefficient, equals 0.0014N m s/rad. The control objective is to realize tracking of a given desired rotor position reference signal $\alpha_1 = \sin(4\pi t)$rad under an unknown load torque, assumed as

$$T_L = \begin{cases} 0\text{N m} & 0 \le t \le 0.5 \\ 0.5\sin(20\pi t) + 0.5\text{N m} & 0.5 < t \le 1 \\ 0.5\text{N m} & \text{other case,} \end{cases} \tag{25}$$

Applying the explicit Euler's method to discretize the system (24), yields [1]

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + b_1 x_{2,k} \\
x_{2,k+1} &= x_{2,k} + a_2 x_{2,k} + B_2 x_{3,k} + d_{2,k} \\
x_{3,k+1} &= x_{3,k} + \tau f_3(x_k) + \tau B_3 u_k + d_{3,k} \\
y_k &= x_{1,k}
\end{aligned} \tag{26}$$

with $b_1 = \tau n_p$, $d_{2,k} = \Delta a_2 x_{2,k} + \Delta B_2 x_{3,k} - h\frac{T_L}{J}$, $d_{3,k} = \Delta f_{3,k} + \Delta B_3 u_k$ and: where Δa_2, ΔB_2, $\Delta f_{3,k}$ and ΔB_3, are parametric perturbations in a_2, B_2, $f_3(x_k)$ and B_3, respectively. $\tau = 1$ms is the sampling time, $x_{1,k} = \theta_k$, $x_{2,k} = \omega_k$ and $x_{3,k} = [i_{d,k} \quad i_{q,k}]^\top$ are the discrete approximation of $\theta(t)$, $\omega(t)$ and $[i_d(t) \quad i_q(t)]^\top$, respectively. Note that the blocks in this example fulfill (8), i.e. $n_1 = n_2 = 1 < n_3 = m = 2$, where $n_i = \dim(x_i)$, for $i = 1, 2, 3$

[1] Due to space limitations, the expressions a_2, B_2, $f_3(x_k)$ and B_3 are omitted. For more details of those functions see Zhang et al. (2019)

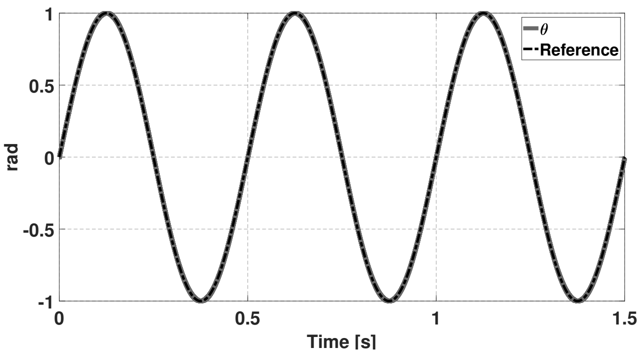

Fig. 1. Rotor position response $\theta(t)$ with proposed controller.

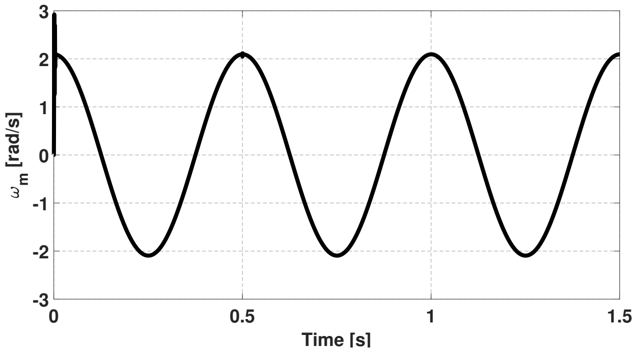

Fig. 2. Rotor angular velocity response $\omega(t)$ with proposed controller.

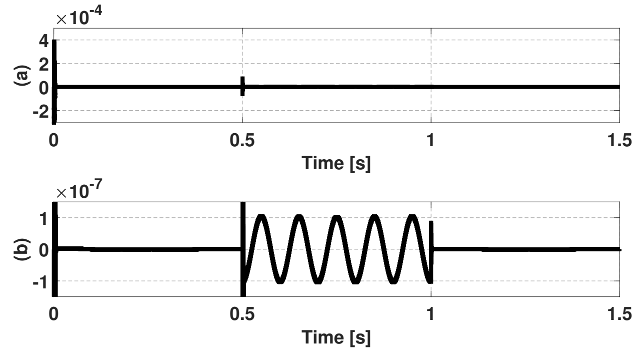

Fig. 3. (a) Sliding variable σ_k, (b) zoom of the same graphic on (a).

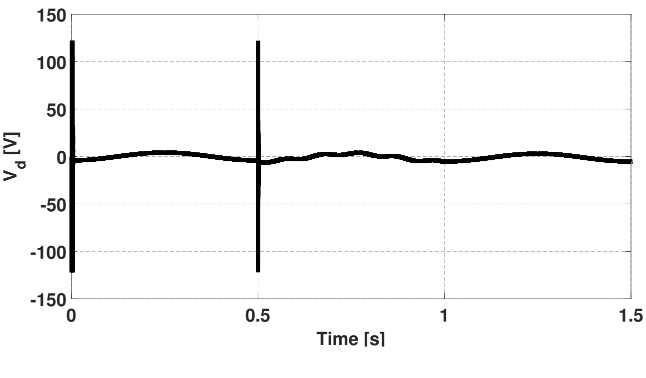

Fig. 4. Control input V_d applied to the system.

Parametric variations are considered in J and L as follows:

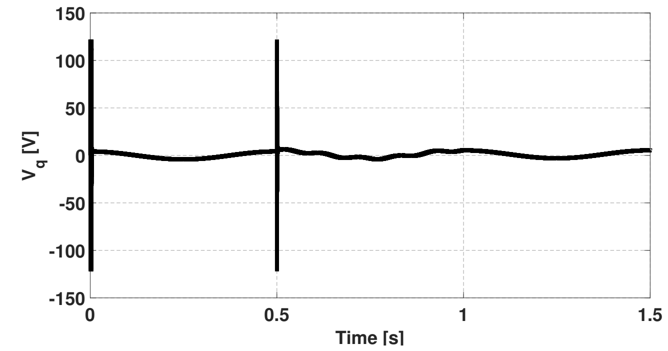

Fig. 5. Control input V_q applied to the system.

$$J = \begin{cases} J_n & t \leq 0.1 \\ -5J_n(t-0.1)+J_n & 0.1 < t < 0.2 \\ 0.9J_n & t \geq 0.2 \end{cases}$$
$$L = \begin{cases} L_n & t \leq 0.3 \\ -L_n(t-0.3)+L_n & 0.3 < t < 0.4 \\ 0.5L_n & t \geq 0.4 \end{cases}, \tag{27}$$

where J_n and L_n are the nominal values of J and L respectively. The control gains (see (20)) are settled as $K_1 = 0.8$ and $K_2 = 0.3$, and the control input is saturated at $u_{\max} = 240$V. Initial conditions of the system are adjusted in $x_0 = [0\ 0\ 0\ 0]^\top$. Fig. 1 shows the rotor position response and the reference signal when the proposed controller is implemented. It can be seen that the controller drives the system output to track the desired value and it remains in such value despite the external perturbation T_L (25) and parametric variations (27). Respectively, velocity response $\omega(t)$ is depicted in Fig. 2. Sliding variable $\sigma_k = z_{3,k}$ response is presented in Fig. 3. It can be seen that it remains in a boundary layer with thickness $\mathcal{O}(\tau^2)$; this result agrees with the theoretical design. Finally, Figs. 4-5 illustrate the control applied to the system, where a continuous signal is obtained, and it stays in the control constraint (6) $\|u_k\| \leq 240$V.

7. CONCLUSIONS

In this paper, based on SM control and perturbance estimation, a robust discrete-time controller was designed for a nonlinear system presented in NBC form with both matched and unmatched perturbations. The proposed controller enables to achieve the $\mathcal{O}(\tau^2)$ order precision of the tracking error. The effectiveness of the proposed control scheme is confirmed by application to PMSM system output tracking control.

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Appendix A. PROOF OF THEOREM 1

To prove the theorem, a constructive step-by-step transformation to the new system (10) will be formulated.

Step 1. On the first step, having the output tracking error $z_{1,k} = x_{1,k} - \alpha_{1,k}$, the first block of (4) is represented as

$$z_{1,k+1} = f_1(x_{1,k}) + B_1(x_{1,k})x_{2,k} + \bar{d}_{1,k} \tag{A.1}$$

where $\bar{d}_{1,k} = d_1(x_k, k) - \alpha_{1,k+1}$.

In this case, $n_1 = n_2$, the $n_2 \times 1$ vector $x_{2,k}$, considered in the system (A.1) as a virtual control, is chosen as

$$x_{2,k} = -B_1^{-1}(x_{1,k})f_1(x_{1,k}) + B_1^{-1}(x_{1,k})[K_1 z_{1,k} + z_{2,k} - \bar{d}_{1,k-1}] \tag{A.2}$$

with K_1 a Schur matrix. From (A.2), the new variable $z_{2,k}$ is expressed as

$$\begin{aligned} z_{2,k} &= \psi_2(\bar{x}_{2,k}), \quad \psi_2(\bar{x}_{2,k}) = B_1(x_{1,k})x_{2,k} - \alpha_{2,k}, \\ \alpha_{2,k} &= K_1 z_{1,k} - f_1(x_{1,k}, k) - \bar{d}_{1,k-1} \end{aligned} \tag{A.3}$$

resulting in first transformed block (10), namely,

$$z_{1,k+1} = K_1 z_{1,k} + z_{2,k} + \delta_{1,k}, \;\; \delta_{1,k} = \bar{d}_{1,k} - \bar{d}_{1,k-1} \tag{A.4}$$

The term $\bar{d}_{1,k-1}$ is obtained from (A.1) as

$$\bar{d}_{1,k-1} = z_{1,k} - f_1(x_{1,k-1}) - B_1(x_{1,k-1})x_{2,k-1}. \tag{A.5}$$

Step 2. At this step, under Assumption 2, namely (7),

$$B_1(x_{1,k+1}) = \bar{B}_1(\bar{x}_{2,k}) + \Delta B_1(x_k, k) \tag{A.6}$$

we define

$$\bar{B}_2(\bar{x}_{2,k}) = \bar{B}_1(\bar{x}_{2,k})B_2(\bar{x}_{2,k}). \tag{A.7}$$

Using (A.3), (A.6) and (A.7), the second block of (4) is transformed into

$$z_{2,k+1} = \bar{f}_2(\bar{x}_{2,k}) + \bar{B}_2(\bar{x}_{2,k})x_{3,k} + \bar{d}_{2,k} \tag{A.8}$$

where $\bar{f}_2(\bar{x}_{2,k}) = \bar{B}_1(\bar{x}_{2,k})f_2(\bar{x}_{2,k})$ and $\bar{d}_{2,k} = -\alpha_{2,k+1} + \bar{B}_1(\bar{x}_{2,k})d_2(x_k, k) + \Delta B_1(x_k, k)x_{2,k+1}$.

Taking into account the structure $n_2 < n_3$, the virtual $n_3 \times 1$ control $x_{3,k}$ in (A.8) is chosen of the form

$$\begin{aligned} x_{3,k} =& \bar{B}_2^{\dagger}(x_{1,k})[K_2 z_{2,k} + E_{3,1}z_{3,k} - \bar{d}_{2,k-1}] \\ & - \bar{B}_2^{\dagger}(x_{1,k})\bar{f}_2(\bar{x}_{2,k}), \end{aligned} \tag{A.9}$$

with pseudo inverse matrix $\bar{B}_2^{\dagger}(x_{1,k})$ and $E_{3,1} = [I_{n_2} \; 0] \in \mathbb{R}^{n_2 \times n_3}$ where I_{n_2} is identity matrix.

Now, the transformation (A.9) is extended as

$$\begin{aligned} \bar{B}_2(\bar{x}_{2,k})x_{3,k} =& - \bar{f}_2(\bar{x}_{2,k}) + K_2 z_{2,k} + E_{3,1}z_{3,k} - \bar{d}_{2,k-1} \\ M_2(\bar{x}_{2,k})x_{3,k} =& E_{3,2}z_{3,k} \end{aligned} \tag{A.10}$$

where the matrix $M_2(\bar{x}_{2,k})$ is selected such that the matrix $\tilde{B}_2(\bar{x}_{2,k}) = \begin{bmatrix} \bar{B}_2(\bar{x}_{2,k}) \\ M_2(\bar{x}_{2,k}) \end{bmatrix} \in \mathbb{R}^{n_3 \times n_3}$ has rank n_3; $E_{3,2} = [0 \; I_{n_3 - n_2}] \in \mathbb{R}^{(n_3-n_2)\times n_3}$ and $I_{n_3} = \begin{bmatrix} E_{3,1} \\ E_{3,2} \end{bmatrix} \in \mathbb{R}^{n_3 \times n_3}$.

From (A.10), the new variable $z_{3,k}$ is obtained as

$$\begin{aligned} z_{3,k} &= \psi_3(\bar{x}_{3,k}), \quad \psi_3(\bar{x}_{3,k}) = \tilde{B}_2(\bar{x}_{2,k})x_{3,k} - \alpha_{3,k}, \\ \alpha_{3,k} &= \begin{bmatrix} -\bar{f}_2(\bar{x}_{2,k}) + K_2 z_{2,k} + E_{3,1}z_{3,k} - \bar{d}_{2,k-1} \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, the second transformed block (10) becomes

$$z_{2,k+1} = K_2 z_{2,k} + E_{3,1}z_{3,k} + \delta_{2,k} \tag{A.11}$$

$$\delta_{2,k} = \bar{d}_{2,k} - \bar{d}_{2,k-1} \tag{A.12}$$

where $\bar{d}_{2,k-1}$ is obtained from (A.8) as

$$\bar{d}_{2,k-1} = z_{2,k} - \bar{f}_2(\bar{x}_{2,k-1}) - \bar{B}_2(\bar{x}_{2,k-1})x_{3,k-1}. \tag{A.13}$$

Step i. At this stage, after $(i-1)$ steps, we have $(i-1)$ transformed blocks of the system (10):

$$\begin{aligned} z_{1,k+1} &= K_1 z_{1,k} + z_{2,k} + \delta_{1,k} \\ &\vdots \\ z_{i-1,k+1} &= K_{i-1}z_{i-1,k} + E_{i-1,1}z_{i,k} + \delta_{i-1,k}, \end{aligned} \tag{A.14}$$

with $z_{i,k} = \psi_i(\bar{x}_{i,k}), \; \psi_i(\bar{x}_{i,k}) = \tilde{B}_{i-1}(\bar{x}_{i-1,k})x_{i,k} - \alpha_{i,k}$.

Proceeding in the same way, under Assumption 2, namely

$$\tilde{B}_{i-1}(\bar{x}_{i-1,k+1}) = \bar{B}_{i-1}(\bar{x}_{i,k}) + \Delta B_{i-1}(x_k, k), \tag{A.15}$$

we define

$$\bar{B}_i(\bar{x}_{i,k}) = \bar{B}_{i-1}(\bar{x}_{i,k})B_i(\bar{x}_{i,k}). \tag{A.16}$$

Using then (A.14)-(A.16), the (i)-th block of (4) is transformed into

$$z_{i,k+1} = \bar{f}_i(\bar{x}_{i,k}) + \bar{B}_i(x_{i,k})x_{i+1,k} + \bar{d}_{i,k} \tag{A.17}$$

where $\bar{f}_i(\bar{x}_{i,k}) = \bar{B}_{i-1}(x_{i,k})f_i(\bar{x}_{i,k})$ and $\bar{d}_{i,k} = -\alpha_{i,k+1} + \bar{B}_{i-1}(\bar{x}_{i,k})d_i(x_k, k) + \Delta B_i(x_k, k)x_{i+1,k}$.

With the structure $n_i < n_{i+1}$, the virtual $(i+1)\times 1$ control $x_{i+1,k}$ in (A.17) is formulated as

$$\begin{aligned} x_{i+1,k} =& - \bar{B}_i^{\dagger}(x_{1,k})\bar{f}_i(\bar{x}_{i,k}) \\ & + \bar{B}_i^{\dagger}(x_{1,k})[K_i z_{i,k} + E_{i+1,1}z_{i+1,k} - \bar{d}_{i,k-1}], \end{aligned} \tag{A.18}$$

with pseudo inverse matrix $\bar{B}_i^{\dagger}(x_{1,k})$ and $E_{i+1,1} = [I_{n_i} \; 0] \in \mathbb{R}^{n_i \times n_{i+1}}$, I_{n_i} is identity matrix.

Now, the transformation (A.18) is extended as

$$\begin{aligned} \bar{B}_i(x_{i,k})x_{i+1,k} =& - \bar{f}_i(\bar{x}_{i,k}) + K_i z_{i,k} + E_{i+1,1}z_{i+1,k} - \bar{d}_{i,k-1} \\ M_i(x_{i,k})x_{i+1,k} =& E_{i+1,2}z_{i+1,k} \end{aligned} \tag{A.19}$$

where the matrix $M_i(x_{1,k})$ is selected such that the matrix $\tilde{B}_i(x_{1,k}) = \begin{bmatrix} \bar{B}_i(x_{1,k}) \\ M_i(x_{1,k}) \end{bmatrix} \in \mathbb{R}^{n_{i+1} \times n_{i+1}}$ has rank n_{i+1}; $E_{i+1,2} = [0 \; I_{n_{i+1}-n_i}]^{\top} \in \mathbb{R}^{(n_{i+1}-n_i)\times n_{i+1}}$ and $I_{n_{i+1}} = \begin{bmatrix} E_{i+1,1} \\ E_{i+1,2} \end{bmatrix} \in \mathbb{R}^{n_{i+1} \times n_{i+1}}$ is identity matrix.

From (A.19), it follows: $z_{i+1,k} = \tilde{B}_i(\bar{x}_{1,k})x_{i,k} - \alpha_{i+1,k}$

$$\alpha_{i+1,k} = \begin{bmatrix} -\bar{f}_i(\bar{x}_{i,k}) + K_i z_{i,k} + E_{i+1,1}z_{i+1,k} - \bar{d}_{i,k-1} \\ 0 \end{bmatrix}.$$

Substituting (A.18) into (A.17), results in

$$z_{i,k+1} = K_i z_{i,k} + E_{i+1,1}z_{i+1,k} + \delta_{i,k}, \quad \delta_{i,k} = \bar{d}_{i,k} - \bar{d}_{i,k-1},$$

$$\bar{d}_{i,k-1} = z_{i,k} - \bar{f}_i(\bar{x}_{i,k-1}) - \bar{B}_i(x_{1,k-1})x_{i,k-1}.$$

Step r. Finally, the variable $z_{r,k} = \bar{B}_{r-1}(\bar{x}_{r-1,k})x_k - \alpha_{r,k}$ is introduced with dynamics

$$z_{r,k+1} = \bar{f}_r(x_k) + \bar{B}_r(x_k)u_k + \bar{d}_{r,k}$$

where $\bar{f}_r(x_k) = \bar{B}_{r-1}(x_k)f_r(x_k)$, $\bar{B}_r(x_k) = \bar{B}_{r-1}(x_k)B_r(x_k)$ and $\bar{d}_{r,k} = B_{r-1}(x_k)d_r(x_k, k) - \alpha_{r,k+1} + \Delta B_{r-1}(x_k, k)\, x_{r,k+1}$.