Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Robust Analysis and Sensitivity Design of Model Predictive Control

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Abstract: Model Predictive Control (MPC) algorithms have an inherently time domain based design. Design parameters are directly connected to the discrete time domain (sample time, prediction horizon), or impact the discrete time state-space model (weight matrices). We provide an analysis and design method for MPC systems in the frequency domain, including the determination of robustness margins. Pre-designed MPC applications are analyzed for their multi-dimensional gain and phase margin. An adjustment design method to improve unsatisfactory results is given. This approach is shown in a simulation example.

Keywords: Control Design, Optimal Control, Robust Control, Model Predictive Control, Aerospace

1. INTRODUCTION

Model Predictive Control (MPC) is a real-time, optimal control method which notably allows for direct implementation of time domain constraints and disturbance information. Due to these properties, MPC has become increasingly popular in control applications since its introduction.

Robustness is on the forefront of research on MPC. Early approaches included large scale analysis or min-max design, methods which are often difficult to use. Tube-MPC has been proposed as an alternative to these methods, to provide robustness guarantees in MPC design while simultaneously yielding an applicable controller (Bertsekas and Rhodes, 1971). By employing positive invariant sets, it has been shown in (Chisci et al., 2001; Mayne and Langson, 2001) that the uncertainty region around the nominal trajectory shrinks in time, which was then developed further into Tube-MPC applications (Langson et al., 2005). Another approach to control design in face of uncertainty is to include these uncertainties directly in the control design, proposed in Stochastic MPC (Goulart et al., 2006). Both approaches regard the robustness issue from an external disturbance view point.

A more complete analysis with examination of plant parameter variations is often desired in applications. A robust LQR design method is demonstrated in (Ward and Ly, 1996), where additional non-linear parameter optimization is utilized to improve the margins of a closed-loop system. Additionally, in (Søgaard-Andersen et al., 1986) the authors propose a method for parameter adjustment of multi-dimensional controllers for robustness margins, which the present work is based upon. More recent results in (Koduri, 2017) show the applicability of robustness margins to explicit MPC.

The goal of this work is to provide supplemental methods for control design, specifically for linear MPC, that leverage robust analysis with the application of classical control margins. In this direction, we aim to combine robustness analysis of the return difference matrix yielding classical margins with MPC. The purpose of these results is to provide a robust analysis and design method for MPC that can be used in practical applications such as flight control. The robust margins gained in the design method provide quantifiable robustness properties for the plant and can be used in accordance with common certification requirements.

The paper is structured as follows: first, §2 provides background on MPC and frequency domain analysis tools for multi-input, multi-output systems. Next, we present a solution to the singular value decomposition for a closed-loop MPC problem without constraints, and methods to analyze and design for frequency domain properties in §3. The results are exemplified with simulations in §4. A brief conclusion with a look at future work is provided in §5.

Notation

We briefly provide an overview of the notation used. The complex conjugate transpose of the complex-valued matrix M is denoted as M^H. The partial derivative of a matrix inverse with respect to one of its parameters p can be computed as,

$$\frac{\partial M^{-1}(p)}{\partial p} = -M^{-1}\frac{\partial M(p)}{\partial p}M^{-1}.$$

Most subscripts on vectors are used as an abbreviation for (time) parameters of that vector, e.g., $a_\tau = a(\tau)$, unless otherwise noted. Following this notation, the parameter of a vector with a subscript is added to the argument, $a_\tau(k) = a(\tau + k)$.

2. PRELIMINARIES

Applications of MPC in settings such as flight control require a discrete time model. By choosing an appropriate sampling time Δt, the temporal dimension can be discretized as,

$$t = \tau \, \Delta t \ , \quad \tau = 0, 1, 2, \ldots ,$$

where t is the current absolute time and the non-negative integer τ is the discrete time step. A finite-dimensional linear time-invariant system can then be expressed in discrete time as,

$$\begin{aligned} x_{\tau+1} &= A x_\tau + B u_\tau , \\ y_\tau &= C x_\tau , \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $y \in \mathbb{R}^p$ is the output vector, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are system matrices.

2.1 Model Predictive Control

The discrete dynamic system is utilized in the basic MPC setup. In its implicit form, MPC is a real-time optimal control method, where open-loop control trajectories are computed online and the first control element is implemented on the plant. The control loop is closed in feedback form by repeatedly applying this approach at each time step with updated states.

This work is centered around one of the basic MPC settings, a quadratic cost objective, constrained only by linear dynamics. This results in a linear optimal control problem, which can be extended to include additional features, such as state and control constraints. These features are not implemented in the presented work, but future efforts will be focused on analyzing and designing control systems with such features.

The goal of the control problem is to find the optimal solution to the finite horizon cost function, J, with state, control, and discrete time inputs,

$$\begin{aligned} J(x, u, \tau) = \sum_{k=0}^{H_p-1} & x_\tau(k+1)^\top Q x_\tau(k+1) \\ & + u_\tau(k)^\top R u_\tau(k), \end{aligned} \tag{2}$$

where H_p is the prediction horizon, i.e., the number of discrete time steps that are included in the optimization. The matrices Q, assumed to be positive semi-definite, and R, positive definite, are weights on the states and controls, respectively. These can be time-varying weights in general, but are assumed to be time-invariant for the problem at hand.

Often, the terminal weight on the last state element of the trajectory is selected via the Lyapunov analysis, to guarantee certain stability properties (Rawlings et al., 2017). The cost function in (2) is a generalization of this approach and the methods introduced later in this work serve a similar purpose to the role of terminal weights, in giving some measure to the stability of the control system.

Given the cost in (2), the optimization problem V minimizes the cost with respect to the control input, constrained by the state dynamics,

$$\begin{aligned} V(x, \tau) &= \min_u \frac{1}{2} J(x, u, \tau), \\ \text{s.t.} \quad x_\tau(k+1) &= A x_\tau(k) + B u_\tau(k). \end{aligned} \tag{3}$$

With the given cost and constraints, the optimization problem V is a quadratic program. Without additional constraints, the quadratic program can be solved analytically. Let $\mathbf{x}$ and $\mathbf{u}$ be stacked vectors of the state and control trajectories, respectively, over the prediction horizon,

$$\begin{aligned} \mathbf{x} &= \left[x_\tau^\top(1),\, x_\tau^\top(2),\, \ldots,\, x_\tau^\top(H_p) \right]^\top , \\ \mathbf{u} &= \left[u_\tau^\top(0),\, u_\tau^\top(1),\, \ldots,\, u_\tau^\top(H_p-1) \right]^\top . \end{aligned}$$

With these vectors, the state dynamics in (1) can be expressed as,

$$\mathbf{x} = \mathbf{A} x_\tau + \mathbf{B} \mathbf{u}. \tag{4}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are structured forms of the state matrices given by,

$$\mathbf{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{H_p} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{Hp-1}B & A^{Hp-2}B & \cdots & B \end{bmatrix}.$$

Similarly, the cost function J in (2) is restructured to eliminate the sum,

$$\mathbf{J}(\mathbf{x}, \mathbf{u}, \tau) = \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u},$$

where $\mathbf{Q}$ and $\mathbf{R}$ are block-diagonal matrices of the appropriate dimensions with weight matrices Q and R on the diagonal. Now, the state trajectory $\mathbf{x}$ can be replaced by (4) in the optimal control problem V to eliminate the state dynamic constraints and yield the unconstrained quadratic program:

$$\begin{aligned} \mathbf{V}(\hat{x}, \tau) &= \min_{\mathbf{u}} \frac{1}{2} \mathbf{J} \\ &= \min_{\mathbf{u}} \frac{1}{2} \mathbf{u}^\top H \mathbf{u} + c(x_\tau)^\top \mathbf{u}, \end{aligned}$$

with $H = \mathbf{B}^\top \mathbf{Q} \mathbf{B} + \mathbf{R}$, and $c(x_\tau) = \mathbf{B}^\top \mathbf{Q} \mathbf{A} x_\tau$.

The unconstrained quadratic program $\mathbf{V}$ is analytically solvable by setting its partial derivative with respect to the optimization variable $\mathbf{u}$ to zero,

$$\frac{\partial \mathbf{V}}{\partial \mathbf{u}} = H \mathbf{u} + c(x_\tau) = 0.$$

Taking the inverse of the optimization matrix yields the optimal control trajectory,

$$\mathbf{u}^* = -H^{-1} c(x_\tau) = -\mathbf{K} x(\tau).$$

Following the MPC methodology, the first element of this trajectory is applied to the dynamical system as the control input, giving the feedback control law,

$$u(\tau) = -K x(\tau),$$

where K is the feedback matrix consisting of the first m-rows of the optimal control gain $\mathbf{K}$.

2.2 Frequency Domain Analysis

The block diagram in Fig. 1 represents a generic feedback loop, with the plant $P(z)$ and the compensator, or controller, $C(z)$. The variable z describes the complex argument of the plant and controller models in the discrete frequency domain. While an estimator is usually required for state feedback methods, it is assumed in the following discussion that the plant output is equal to the states, i.e., the output matrix C in (1) is equal to identity.

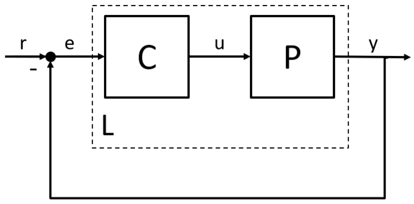

Fig. 1. Closed-loop feedback block diagram: plant P, controller C, loop gain L.

The loop transfer function for this configuration is thus given by,

$$L(z) = P(z)C(z).$$

This product of the plant and controller in series is a key construct in classical control theory. An extension to this expression is the return difference matrix, namely,

$$R(z) = I + L(z).$$

The return difference is often found in the analysis of properties related to the robustness of the closed loop system. In classical single-input, single-output (SISO) analysis, the return difference is complex-valued.

Research in the 1980's led to a better understanding of classical robustness results in the multi-input, multi-output (MIMO) system case, see (Lehtomaki et al., 1981; Safonov et al., 1981; Doyle and Stein, 1981; Shaked, 1986). To employ powerful classical control tools, relevant scalar-valued properties were needed for multi-dimensional systems. A primary result of the research was the discovery of the connection of the system's singular values to robustness properties in the classical sense.

Singular values can be computed as the square root of the respective eigenvalues of the matrix square,

$$\sigma_i(M) \;=\; \sqrt{\lambda_i(MM^H)}.$$

The main approach to determine all singular values of a matrix is by using the singular value decomposition (SVD). Let M be a generic matrix with some parameter argument p. Then the SVD for $M(p)$ is given by:

$$M(p) = V(p)\Sigma(p)Z(p)^H, \quad \Sigma(p) = \mathrm{diag}(\sigma_i),$$

where V and Z are matrices containing the left and right singular vectors, respectively, and Σ is a diagonal matrix, with all singular values on its diagonal. The matrices V, Z satisfy $VV^H = I$ and $ZZ^H = I$.

Robustness for MIMO systems in the classical sense can be determined by singular values of loop transfer matrices. For example, it has been shown that the smallest singular value, $\underline{\sigma}(M)$, of the return difference matrix is closely tied to the robustness of the system at a given loop breaking point (Lehtomaki et al., 1981). Qualitatively, the larger this smallest singular value is for the return difference over a relevant frequency range, the more robust that closed-loop system is over that range. Furthermore, if a lower limit for the smallest singular value over a given frequency range is found, then that limit can be used to determine classic control margins of the MIMO system. As such, let,

$$\inf_{\omega\in\Omega} \underline{\sigma}(I + L(i\omega\Delta t)) = \alpha_0, \tag{5}$$

where ω is the frequency and the set Ω is defined as a limiting frequency range, or bandwidth. In discrete systems, the frequency is given per cycle and needs to be multiplied by the sample time Δt to yield the frequency in Hertz. The following margins then hold for each loop in the MIMO system.

Gain margin:

$$GM = \frac{1}{1 \pm \alpha_0}. \tag{6}$$

Phase margin:

$$PM = \pm\cos^{-1}\left(1 - \frac{\alpha_0^2}{2}\right). \tag{7}$$

From these equations it is evident that an increase of α_0 will yield improved (i.e., larger) margins. The design goal in the next section is therefore to maximize critical points in the minimum singular value plot of the return difference for a system that has been obtained from MPC.

Margins that are determined by SVD are valid for all feedback-loops of the multi-dimensional system in question. But for most separate loops within the system, these values will be conservative. The main benefit of these multi-dimensional margins is evident when cross-feed is considered in MIMO systems. This can occur when inputs from one loop affect the robustness of other, separate loops. In this case, SISO margins will not yield relevant values for the robustness of the system. The margins determined by the singular value analysis of the return difference, on the other hand, include such inter-loop effects and represent a whole picture of the robustness of MIMO systems.

3. SINGULAR VALUE SENSITIVITIES IN MPC

Given the loop properties for robustness analysis presented in the last section, an analysis and design method is developed in this section for time domain based MPC systems in their basic form. For the analysis step in this process, a relevant expression for the minimum singular value in the closed loop MPC system will be provided. The design process following the analysis of the MPC system is built upon the partial derivative of a singular value with respect to some parameter p. This parameter is assumed to be an argument of the original matrix M for which the singular value is determined. This partial derivative is then given by,

$$\frac{\partial\sigma_i(p)}{\partial p} = \mathrm{Re}\left(v_i^H \frac{\partial M(p)}{\partial p} z_i\right). \tag{8}$$

Here, σ_i is the i-th singular value of the matrix M, dependent on some generic parameter argument p.

In order to examine the robustness of MIMO systems, and MPC in particular, the smallest singular value of the return difference is employed. For robustness analysis, let the matrix M in (8) be the return difference matrix of some pre-designed MPC system. It is assumed that Q and R for this system have been designed with relevant time domain requirements (or generic requirements of different nature).

Since the return difference depends on the frequency with which the system is excited, the first step is to choose a relevant excitation bandwidth, with lower and upper frequencies, ω_l and ω_u. The return difference is then analyzed within this bandwidth for the infimum of the smallest singular value. The corresponding frequency at the infimum is designated as the critical frequency, ω_c, which lies within the bandwidth, $\omega_l \leq \omega_c \leq \omega_u$.

As such, the critical frequency is used as the excitation frequency, and the return difference is then,

$$M = I + L(i\omega_c) = I + K\psi(i\omega_c),$$

where $\psi(i\omega_c) = (i\omega_c I - A)^{-1}B$. Here, the output matrix is assumed to be identity. With M being the return difference and following (8), the partial derivative of the smallest singular value of M can be expressed as,

$$\frac{\partial \underline{\sigma}}{\partial p} = \mathrm{Re}\left(\underline{v}^H \frac{\partial (I + L(i\omega_c))}{\partial p}\underline{z}\right).$$

In this work, the primary choice for the parameter p which is to be adjusted for robustness properties are the weight matrices Q and R. These matrices are arguments of the MPC feedback gain K. When p is selected as some element of the weight matrices, i.e., $p = \{q_{ij}; r_{ij}\}$, the partial derivative of the return difference can be reduced to,

$$\frac{\partial (I + L(i\omega_c))}{\partial p} = \frac{\partial K}{\partial p}\psi(i\omega_c).$$

With K being the first m rows of the full quadratic program solution, the derivative is given as,

$$\begin{aligned}\frac{\partial K}{\partial p} &= \left.\frac{\partial\left(H^{-1}\mathbf{B}^\top\mathbf{QA}\right)}{\partial p}\right|_{1:m}\\ &= H^{-1}\frac{\partial\left(\mathbf{B}^\top\mathbf{QA}\right)}{\partial p} + \left.\frac{\partial H^{-1}}{\partial p}\mathbf{B}^\top\mathbf{QA}\right|_{1:m},\end{aligned}$$

for both relevant partial derivative parameters, q_{ij} and r_{ij}, the element for derivation always appears linearly. The dependence of the solution on the first m-rows will subsequently be omitted where appropriate.

There are now two distinct cases for the partial derivative, that depend on the selection of the derivation parameter p. First, let $p = q_{ij}$, which results in the partial derivative,

$$\begin{aligned}\frac{\partial K}{\partial q_{ij}} &= H^{-1}\mathbf{B}^\top\frac{\partial \mathbf{Q}}{\partial q_{ij}}\mathbf{A} + \frac{\partial H^{-1}}{\partial q_{ij}}\mathbf{B}^\top\mathbf{QA},\\ &= H^{-1}\mathbf{B}^\top\frac{\partial \mathbf{Q}}{\partial q_{ij}}\mathbf{A} - H^{-1}\frac{\partial H}{\partial q_{ij}}H^{-1}\mathbf{B}^\top\mathbf{QA},\\ &= H^{-1}\mathbf{B}^\top\frac{\partial \mathbf{Q}}{\partial q_{ij}}\mathbf{A} - H^{-1}\frac{\partial \mathbf{B}^\top\mathbf{QB}+\mathbf{R}}{\partial q_{ij}}\mathbf{K},\\ &= \left.H^{-1}\mathbf{B}^\top\frac{\partial \mathbf{Q}}{\partial q_{ij}}\left(\mathbf{A}-\mathbf{BK}\right)\right|_{1:m}.\end{aligned}$$

The complete partial derivative of the singular value for the q_{ij} case is given as,

$$\begin{aligned}\frac{\partial \underline{\sigma}}{\partial q_{ij}} &= \mathrm{Re}\left(\underline{v}^H\frac{\partial K}{\partial q_{ij}}\psi(i\omega_c)\underline{z}\right) \qquad (9)\\ &= \mathrm{Re}\left(\underline{v}^H\left(H^{-1}\mathbf{B}^\top\frac{\partial \mathbf{Q}}{\partial q_{ij}}\left(\mathbf{A}-\mathbf{BK}\right)\right)_{1:m}\psi(i\omega_c)\underline{z}\right),\end{aligned}$$

where only the first m-rows of the partial derivative of the feedback gain are applied. The derivative of $\mathbf{Q}$ w.r.t. q_{ij} is a sparse matrix with zeroes almost everywhere except for entries of 1 at integer multiples of the entry at ij.

For the second case, let $p = r_{ij}$, to examine the sensitivity of the solution to changes in the input weight matrix R. The derivative is simplified as,

$$\begin{aligned}\frac{\partial K}{\partial r_{ij}} &= \frac{\partial H^{-1}}{\partial r_{ij}}\mathbf{B}^\top\mathbf{QA} = -H^{-1}\frac{\partial H}{\partial r_{ij}}H^{-1}\mathbf{B}^\top\mathbf{QA}\\ &= -H^{-1}\frac{\partial \mathbf{B}^\top\mathbf{QB}+\mathbf{R}}{\partial r_{ij}}\mathbf{K}\\ &= \left.-H^{-1}\frac{\partial \mathbf{R}}{\partial r_{ij}}\mathbf{K}\right|_{1:m}.\end{aligned}$$

Using this result in the full singular value derivative yields,

$$\begin{aligned}\frac{\partial \underline{\sigma}}{\partial r_{ij}} &= \mathrm{Re}\left(\underline{v}^H\frac{\partial K}{\partial r_{ij}}\psi(i\omega_c)\underline{z}\right) \qquad (10)\\ &= \mathrm{Re}\left(-\underline{v}^H\left(H^{-1}\frac{\partial \mathbf{R}}{\partial r_{ij}}\mathbf{K}\right)_{1:m}\psi(i\omega_c)\underline{z}\right).\end{aligned}$$

Next, these results are applied in design improvements to the MPC system.

The relations provided above for the smallest singular value allow for the analysis of a closed-loop MPC system. The robustness of MPC, measured with gain and phase margins, can be determined by applying the control gain solution of the MPC optimal control problem, as given in (3), to the return difference matrix and solving (5) - (7). Should the robustness be deemed unsatisfactory, the next question is how to adjust the MPC design to improve robustness properties of the closed-loop system.

The goal for the design adjustment presented here is to change one or more elements of the MPC weight matrices Q and R, such that the changed closed loop system exhibits improved robustness, with minimal impact on the control performance.

While diagonal elements are the primary choice, with good understanding of the dynamics and state or control interactions, off-diagonal weighting elements can also be selected to improve robustness on performance critical actuators. This can lead to a lesser impact on the control performance compared to a change in the diagonal elements of these variables. It is also possible to adjust multiple weight elements at once by applying the partial derivatives of the singular value in matrix differentiation form.

Once the desired parameters for adjustment have been chosen, they are to be re-selected by changing the original parameters, q_{ij}^O and r_{ij}^O, as,

$$q_{ij} = q_{ij}^O + \epsilon_q\frac{\partial \underline{\sigma}}{\partial q_{ij}}, \qquad (11)$$

$$r_{ij} = r_{ij}^O + \epsilon_r\frac{\partial \underline{\sigma}}{\partial r_{ij}}, \qquad (12)$$

with $\epsilon_{q,r} \geq 0$. This parameter $\epsilon_{q,r}$ may be selected carefully by the designer, thereby introducing an additional tuning parameter within the design process. A more sound approach is to optimally select this parameter through convex optimization algorithms. The optimization problem at hand is given as,

$$\max_{q_{ij}, r_{ij}} \underline{\sigma}\left(I + L(i\omega_c\Delta t, Q, R)\right),$$

with a concave objective function that is smooth almost everywhere (Lewis and Sendov, 2005; Alavian and Rotkowitz, 2016).

The sensitivity based design procedure is summarized as follows, analogous to the procedure given in (Søgaard-Andersen et al., 1986):

i) Design the initial MPC system in (3), tuning matrices Q and R based on control performance requirements.
ii) Determine robustness properties based on singular value analysis, see (5) to (7). Evaluate the robust margins for their suitability. Should the robustness properties be satisfactory, end the procedure here.
iii) Select frequencies at which the margins need to be improved and parameter p for the design adjustment. Calculate the gradients of the smallest singular value with (9) or (10), respectively.
iv) Select updated parameters by applying the preferred method, such as direct tuning or convex optimization. Go to step 2.

This procedure is applied to a pre-designed MPC system in an example in the next section.

4. EXAMPLE

The plant for the simulation example presented in this section is based on data from a wind tunnel test article used for gust load alleviation (GLA) experiments. The dynamics, simulation model, and underlying test article are presented in (Quenzer et al., 2019). The model consists of a half wing aircraft with longitudinal dynamics and a gust generator for vertical gust disturbance.

Four states are used for control design, which are the angle of attack, $x_1 = \alpha$, where the subscript denotes the enumeration of the element in the state vector, the pitch rate, $x_2 = q$, and two states for the first flexible wing bending mode, $x_3 = \eta_1$ and $x_4 = \dot{\eta}_1$. The controllable inputs to the simulation model are the elevator, $u_1 = \delta_E$, and the two ailerons, outboard $u_2 = \delta_{OB}$ and inboard $u_3 = \delta_{IB}$. The output of the wing root strain measurement, μ_{WR}, will be used for visualization purposes.

An MPC controller for GLA is designed for the given model. The time parameters for MPC are selected as 0.04 s for the sampling time, equivalent to 25 Hz, and a prediction horizon of 10 steps. The control performance goal for the pre-designed MPC controller is to minimize wing root strain due to an incoming gust disturbance. With state feedback, this is achieved by penalizing the first wing bending mode. The weighting matrices are then selected as,

$$Q = \text{diag}(0, 0, 20, 0), \quad R = \text{diag}(10^6, 1, 1),$$

where the position in the diagonal of the weighting matrices corresponds to the respective state or control variable.

Figure 2 shows the time history of the wing root strain measurement for a 1-cos shaped gust disturbance. The 1-cos gust is used for discrete gusts in certification documents. It is chosen with a 4° amplitude and 1.5 Hz frequency, which corresponds to the first wing bending frequency. The gust begins at 2 seconds simulation time. In the plot, the dashed line is the open-loop response and the full line is the closed-loop response. The closed-loop controller suppresses wing root strain oscillation.

The minimum singular value of the return difference is given in Figure 3. The sensors, and the system identification based on the sensor measurements, have high cohesion between approximately 0.5 and 2.5 Hz. The upper and lower frequency limits for analysis are therefore chosen based on the sensor range.From Figure 3, the infimum of the singular value is given as $\alpha_0 = 0.52$. This value yields the following robustness margins,

$$GM = \frac{1}{1 \pm \alpha_0} = \{0.66,\ 2.08\},$$

$$PM = \pm \cos^{-1}\left(1 - \frac{\alpha_0^2}{2}\right) = \pm 30.1°.$$

While these parameters are acceptable for robustness margins, in the last part of the example we improve on these margins by utilizing the technique presented in the last section.

The gradients for the diagonal entries in Q and R are shown in Table 1. Reducing the elevator weight and increasing the control action on the elevator will improve the robustness properties of the closed-loop system. The

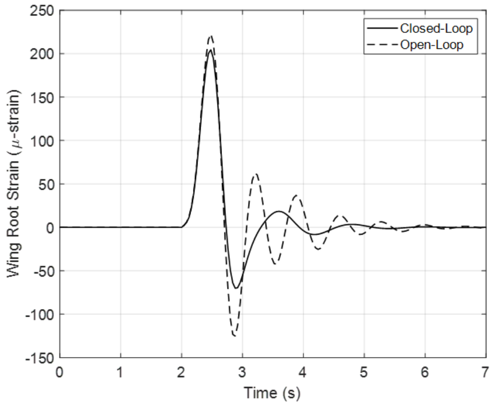

Fig. 2. Wing root strain measurement of the open-loop and initial MPC design.

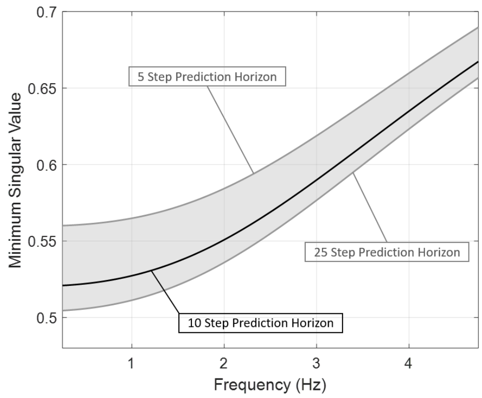

Fig. 3. Robustness measure for the initial MPC design (black line) and range of robustness for varying prediction horizon (shaded gray).

Position	1,1	2,2	3,3	4,4
Q	$-1.1930 \cdot 10^{-5}$	-0.0027	-0.0096	-1.1063
R	$-1.0967 \cdot 10^{-12}$	0.0924	0.0990	-

Table 1. Minimal singular value gradients of the initial MPC design, by parameter. Only diagonal matrix position values are shown.

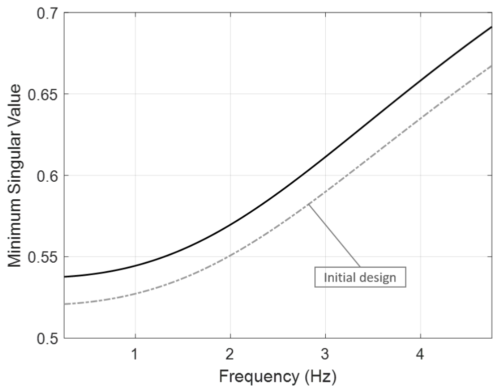

Fig. 4. Robustness measure for the updated MPC system.

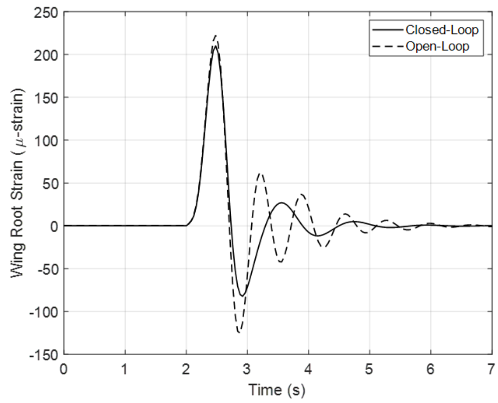

Fig. 5. Wing root strain measurement for the updated MPC system.

graph in Figure 4 shows the minimum singular value plot for an elevator weight of 50 (reduced from 10^6), while all other parameters were kept constant. The minimum singular value at 0.5 Hz increases from 0.52 to 0.54. The new feedback gain matrix K now has non-zero entries in the elevator row.

The wing root strain time history of the updated system is given in Figure 5. Most notably, the peak load alleviation has decreased slightly, which follows a general trend. This trend is visible in the gradients, and shows that the primary way to improve the robustness parameters is to decrease control action on the wing strain. This can be achieved in multiple ways, and the trade-off between control performance in load alleviation and robustness properties has to be carefully evaluated in the design process.

5. CONCLUSION

We have provided an approach to MPC analysis and design, centered around classical robustness properties. The presented method provides robustness margins without requiring conditions on the terminal constraints. Relevant elements for MPC systems, such as constraints and disturbance preview, have not been included, and including these elements is the focus of further work. Extending the explicit MPC analysis for robustness margins in (Koduri, 2017) to involve control synthesis is another promising research direction. Future research also includes experimental validation of the synthesis method and robustness measures in the wind tunnel.

Acknowledgements

The authors acknowledge their many useful discussions with Profs. Kristi Morgansen and Eli Livne from the University of Washington.

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