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A Novel Probabilistic Fault Detection Scheme with Adjustable Reliability Estimates ⋆

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Abstract: We propose a novel probabilistic fault detection scheme with adjustable reliability estimates. Our scheme consists of two phase, the first is the modelling phase, where a probabilistic fault detection design is devised, while the second is the validation phase, where reliability estimates of the design are adjusted online according to new operation records of the plant and the validated reliability. The modelling phase is based on two methods: residual generation, such as parity space, which is an important tool in fault detection problem, and scenario approach, which is a seminal trick to transfer intractable optimization problem into approximate tractable optimization problem and ensure reliability guarantees. The validation phase leverages the state-of-art posteriori probabilistic bounds of convex scenario programs with validation tests. Such a holistic design-and-validate scheme will can help technicians to make better decision. The efficacy of the proposed approach is illustrated on a simulated case study.

Keywords: Parity space, Fault detection, Scenario Approach, A posteriori Probabilistic Bound

1. INTRODUCTION

Fault Detection (FD) has broken the ground into the modern industry as a seminal tool to identify possible anomalies in time, which is helpful for reducing the risk of huge economic loss due to unexpected breakdown. During the past half century, model-based FD has received remarkable attention and rich results have been achieved. In Ding (2008), model-based FD techniques are divided into observer based methods, parity space-based FD approaches and parameter estimation schemes. The primary challenge of FD design lies in an appropriate trade-off between the sensitivity to faults and the robustness against unknown input. For example, with a higher alert threshold, the false alarm rate (FAR) can be reduced, but at a price of higher missed alarm rate (MAR).

Because FAR and MAR are essentially random probabilities, probability distributions of unknowns must be available. However, this is quite restrictive in practice. On one hand, it is commonly difficult to attain exact knowledge of distributions. On the other hand, even if the distribution is known, one has to compute multiple integrals to evaluate FAR and MAR, which is computationally intensive. To address these issues, the so-called scenario approach or randomized algorithm has been applied in FD design recently (Zhong et al. (2016); Zhou et al. (2018); Ding et al. (2019)). The idea is to take a finite number of constraints into consideration as an approximation based on past samples of uncertainty. In this way, FAR can be desirably controlled with a suitably high confidence level (Campi and Garatti (2008)). Zhong et al. (2016) focused on the problem of FD for a class of nonlinear systems s.t. l_2-norm bounded unknown input. Zhou et al. (2018) proposed an approach using randomized algorithms to design the FD system for ship propulsion systems. Ding et al. (2019) put forward a probabilistic framework for performance assessment and design of observer-based FD systems.

However, the use of scenario approach in probabilistic FD design has several limitations. On one hand, previous works mainly focus on using random algorithms to set the decision threshold only. This is because a plethora of uncertainty samples are needed to make an integrated FD design. On the other hand, due to the randomness of uncertainty sampling, the design made by scenario approach itself is random, and thus it is necessary to evaluate its reliability. Unfortunately, abundant Bernoulli trials as well as validation samples have to be made towards this goal. These aspects heavily compromise the practical use of scenario approach in FD design.

In this paper we propose a novel probabilistic integrated FD scheme to improve the practicability of scenario-based design, which is based on the a posteriori performance guarantee of scenario approach with validation tests (Shang and You (2019)). The entire scheme organically integrates the *design stage* and *validation stage*, where the design matrix and the threshold are simultaneously

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optimized in the design stage. To obtain a reliable estimate of FAR, the structural information underlying the optimal design, which is described by the number decisive support constraints (Campi and Garatti (2018)), is utilized together with the validation data. In this way, an adaptive reliability estimate can be incrementally attained in the validation stage, which allows the decision-maker to adjust his/her belief in an online manner. It turns out that the conservatism of traditional a-priori probability bound (Campi et al. (2008)) can be significantly reduced based on the proposed scheme, especially when a moderate sample size is available.

The organization of this paper is given as follows. Section 2 sets up the system configuration and then introduces the parity space technique, and translates the fault detection into an optimization problem. Section 3 states our novel FD design scheme. Section 4 presents a case study to verify our contributions, followed by final conclusions.

2. PRELIMINARIES

2.1 Parity space based residual generation

First we introduce the system configuration. Consider the following linear time invariant system where system matrices are assumed to be known:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_d d(k) + B_f f(k) \\ y(k) = Cx(k) + Du(k) + D_d d(k) + D_f f(k) \end{cases} \tag{1}$$

where x is system state, u is input, d is disturbance, f is fault and y is output, all with appropriate dimensions.

To detect a fault, we need to collect observations for a period of time before we can make reliable diagnosis, which is based on the so-called temporal redundancy or serial redundancy (Ding (2008)). Suppose at time k, we trace back a time interval of s to the start time of our induction, i.e. $k-s$ (Zhong et al. (2018)):

$$\begin{aligned} y(k) &= Cx(k) + Du(k) + D_d d(k) + D_f f(k) \\ &= C(Ax(k-1) + Bu(k-1) + B_d d(k-1) \\ &\quad + B_f f(k-1)) + Du(k) + D_d d(k) + D_f f(k) \end{aligned} \tag{2}$$

By proceeding in this way, we can arrive at the following succinct expression, which is known as the I/O data model:

$$Y_s(k) = H_{us}U_s(k) + H_{os}x(k-s) + H_{fs}F_s(k) + H_{ds}D_s(k) \tag{3}$$

where

$$H_{us} = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{s-1}B & \cdots & CAB & CB & D \end{bmatrix}, \tag{4}$$

$$H_{os} = \left[C^T (CA)^T \cdots (CA^s)^T \right]^T. \tag{5}$$

H_{ds} and H_{fs} are constructed by replacing (B, D) in the Toeplitz matrix H_{us} with (B_d, D_d) and (B_f, D_f), respectively. $Y_s(k)$ includes output measurements from $k-s$ to k in a batch form, i.e.,

$$Y_s(k) = \left[y(k-s)^T y(k-s+1)^T \cdots y(k)^T \right]^T \tag{6}$$

$U_s(k)$, $F_s(k)$ and $D_s(k)$ can be constructed in a similar manner.

Then, the residual $e(k)$ is defined to measure the extent of the deviation from normality:

$$\begin{aligned} e(k) &= Y_s(k) - H_{us}U_s(k) - H_{os}x(k-s) \\ &= H_{fs}F_s(k) + H_{ds}D_s(k) \end{aligned} \tag{7}$$

$e(k)$ will be ideally zero when there is no fault ($F_s(k) = 0$) and no disturbance ($D_s(k) = 0$) in the system. $e(k)$ is calculated through time series of sensor input $U_s(k)$ and output $Y_s(k)$ and system state $x(k-s)$ at base time $k-s$. However, the existence of $x(k-s)$ poses significant challenges since it cannot be observed from sensors. To overcome this problem, a parity space matrix V is designed to let $VH_{os}x(k-s) = 0$. Multiplying V on both side of the equation $e(k) = Y_s(k) - H_{us}U_s(k) - H_{os}x(k-s)$ yields the residual for fault detection:

$$\begin{aligned} r(k) &= Ve(k) = VY_s(k) - VH_{us}U_s(k) - VH_{os}x(k-s) \\ &= VY_s(k) - VH_{us}U_s(k) \end{aligned} \tag{8}$$

which is obviously unrelated to $x(k-s)$, and thus can be calculated by sensor signals of inputs and outputs given the design matrix V. s is also referred to the order of parity space.

Existence of parity space matrix Parity space matrix V is designed to let $VH_{os}x(k-s) = 0$. Given the uncertainty of base time state $x(k-s)$, to ensure that the equation holds, $VH_{os} = 0$ must be satisfied. Thus V is established through packing the left null vectors of H_{os} in rows.

$$v_1 H_{os} = v_2 H_{os} = ... = 0, \qquad V = [v1; v2; ...] \tag{9}$$

Note that $H_{os} = \left[C^T (CA)^T \cdots (CA^s)^T \right]^T$, $vH_{os} = C\sum_{i=1}^{s} v_i A^i = 0$ implies that v can be deduced from the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$, because it always satisfies $p(A) = 0$ from the Cayley-Hamilton theorem. This ensures the existence of the parity space matrix V provided that the order s is no lower than the system order.

2.2 Translation into an optimization problem

In general, the parity space matrix serves as two different roles in fault detection. First, it serves as a projector: it projects $e(k) = Y_s(k) - H_{us}U_s(k) - H_{os}x(k-s)$ into the left null space of H_{os} (orthogonal complement space of the range space of H_{os}) in order to make sure that $VH_{os}x(k-s) = 0$, thereby diminishing the unobservable system state $x(k-s)$. Second, it serves as a modulator: it modulates the norm of $e(k)$ by stretching or shrinking during the projection $r(k) = Ve(k)$. The latter role is important in the following paragraphs.

In fault detection, we transfer the residual vector $r(k)$ to a scalar form by residual evaluation function. The most ordinary choice is the square of 2-norm of this particular residual vector:

$$J(k) = r(k)^T r(k) \tag{10}$$

If $J(k)$ is less than a pre-defined threshold J_{th}, it will be reasonable to keep calm and no alarms are issued. Otherwise, an alarm is issued to indicate ongoing abnormality.

Note that $J(k)$ is a random variable because of the randomness inherent in disturbance d. So from now on, we will use $J_\delta(k)$ as a function of some random variable δ

to denote its inherent randomness. Ideally, when there is no fault ($F_s(k) = 0$), it is ideal to force $J^0_\delta(k)$ (we adopt the convention of using 0 to represent normality) under J_{th} under any circumstances (i.e. under any realization of random variable δ) in order to suppress false alarm rate (FAR) to zero:

$$J^0_\delta(k) \leq J_{th}, \forall \delta \in \Delta. \tag{11}$$

where δ denotes the randomness, lying in an uncertainty set Δ, and k denotes the current time. Such a robust formulation takes all kinds of possible cases into consideration, which makes the result convincing and reliable but leads to too conservative result as a downside. In other words (assume that extreme zero-measure set does not matter),

$$\mathbb{P}\left\{\delta \in \Delta : J^0_\delta(k) \leq J_{th}\right\} = 1 \tag{12}$$

However, this ideal setting is not practical because of the stringent requirement that FAR will be exactly zero. In fact, there is a desirable trade off between false alarm rate (FAR) and missed alarm rate (MAR), which allows us to sacrifice a certain degree of FAR for a significant improvement of MAR. To concretize such an intuition, the previous robust-style constraint is transformed into a more reasonable chance constraint with an admissible level of FAR ϵ:

$$\mathbb{P}\left\{\delta \in \Delta : J^0_\delta(k) \leq J_{th}\right\} \geq 1 - \epsilon \tag{13}$$

Rewriting the above inequation as $\mathbb{P}\left\{\delta \in \Delta : J^0_\delta(k) > J_{th}\right\} \leq \epsilon$, we find that that the FAR should be less than ϵ. Relaxing the upper bound of FAR form 0 to ϵ will surely make more room for lowering the MAR.

Then we proceed with the objective function. Since the residual evaluation function is the square of 2-norm of $r(k) = Ve(k)$, there are two conflicting requirements in the design of V. First, it should amplify $e(k)$ enough otherwise $J(k) = r(k)^T r(k)$ will stay below threshold J_{th} even with some fault; second, it should not magnify $e(k)$ excessively otherwise $J(k) = r(k)^T r(k)$ will exceed threshold J_{th} even in the absence of faults. In a nutshell, there is an evident trade-off between FAR and MAR. The constraints above ensure that FAR is less than ϵ, and it is reasonable to add a optimization objective function to reduce the MAR to achieve the tradeoff. The amplification of matrix V can be mathematically described in its own norm. What is needed is that the norm is large enough to detect minor fault, thereby reducing the MAR.

Now we arrive at the nascent optimization problem:

$$\begin{aligned} &\max ||V|| \\ &\text{s.t. } \mathbb{P}\left\{\delta \in \Delta : J^0_\delta(k) \leq J_{th}\right\} \geq 1 - \epsilon \;\; (\text{FAR} < \epsilon) \\ &\qquad VH_{os} = 0 \end{aligned} \tag{14}$$

However, the existence of chance constraint $\mathbb{P}\left\{\delta \in \Delta : J^0_\delta(k) - J_{th} \leq 0\right\} \geq 1 - \epsilon$ makes the problem a NP-hard problem from the viewpoint of stochastic programming (Prékopa (2013)), which adds significant difficulty in devising an effective computational treatment.

3. A NOVEL PROBABILISTIC FD SCHEME

Fig. 1 depicts our integrated FD scheme, which consists of two different phases. The first is modelling phase, where a probabilistic fault detection design is devised; the second is validation phase, where reliability estimate of the design is adjusted online with the accumulation of operational data.

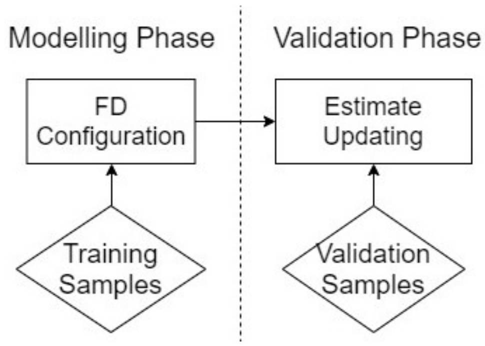

Fig. 1. Two-Phase Fault Detection Scheme

3.1 *The first phase: Modelling phase*

Applying Scenario approach An innovative technology called scenario approach (Campi et al. (2008)) has been created and well accepted to deal with chance constrained programs at a very general level. The main thrust of this technology is that tractability can be obtained through random sampling of constraints, where $\delta^{(1)}, , \ldots, , \delta^{(N)}$ are N independent identically distributed samples from Δ:

$$\begin{aligned} &\max ||V|| \\ &\text{s.t. } J^0_{\delta^{(i)}}(k) - J_{th} \leq 0, \;\; i = 1, \ldots, N \\ &\qquad VH_{os} = 0 \end{aligned} \tag{15}$$

By substituting the residual evaluation function $J^0_{\delta^{(i)}}(k)$, the concrete form of scenario-based optimization problem can be stated as follows:

$$\begin{aligned} &\max ||V|| \\ &\text{s.t. } \;\; {e^0_{\delta^{(i)}}}^T V^T V e^0_{\delta^{(i)}} - J_{th} \leq 0, \;\; i = 1, \ldots, N \\ &\qquad VH_{os} = 0 \end{aligned} \tag{16}$$

For simplicity, we omit the current time k from now on without bringing about confusions. To make sure that the optimization is tractable, the objective function to minimize must be convex while the objective function to maximize must be concave. Unfortunately, the norm is a convex function which is about to be maximised, so it is not a convex problem. The way to resolve this unpleasant issue is to redefine the optimization function. As can be seen from previous explanation, the parity space matrix serves as two roles, namely the projector and the modulator. The "modulator" functions through $V^T V$: $J = r^T r = (Ve)^T (Ve) = e^T V^T V e$. If the Frobenius norm of parity space matrix V is considered, it will be equal to the trace of $V^T V$ which is affine thus concave. So it is reasonable to change objective function from $\max ||V||$ to $\max \text{Trace}(W)$ where $W = V^T V$, and then optimize over W instead of V.

Remember that the role of V, "projector", is reflected in the constrains, which also need to be modify from $VH_{os} = 0$ to $WH_{os} = V^T V H_{os} = 0$. In fact, these two equations are exactly the same. First, any V satisfying $VH_{os} = 0$ must also satisfy $V^T V H_{os} = 0$. Second, let $A = VH_{os}$, thus $V^T V H_{os} = 0$ implies $H_{os}^T V^T V H_{os} = A^T A = 0$. Since the diagonals of A^TA are squares of 2-norm of columns of A, $A^TA = 0$ means every column of A is zero vector, in other words, $A = 0$.

Here arrives our final tractable optimization problem:

$$\begin{aligned} &\max \ \text{Trace}(W) \\ &\text{s.t.} \ \ {e^0_{\delta^{(i)}}}^T W e^0_{\delta^{(i)}} - J_{th} \le 0, \ \ i = 1,, \ldots,, N \\ &\quad\ \ W H_{os} = 0 \end{aligned} \tag{17}$$

To recap, the δ is the unknown random variable modelling the inherent randomness from disturbance, $\delta^{(i)}$ $i = 1 \ldots N$ are samples from this unknown distribution. As for $e^0_{\delta^{(i)}}$, the superscript 0 denotes the normal working condition (no fault occurs), the subscript $\delta^{(i)}$ denotes the ith sample and e denotes the deviation from normality which is a function of random variable δ. The optimal W is thus represented by W_N^*, where the subscript N comes from the sample size in scenario programming.

Violation probability of optimal solution

Definition 1. Violation Probability: The violation probability of a given optimization variable $W \in S_+^{n\times n}$ is defined as the fraction of uncertainty set Δ which makes the constraint violated:

$$V(W) = \mathbb{P}\left\{\delta \in \Delta : {e^0_\delta}^T W e^0_\delta - J_{th} > 0\right\}. \tag{18}$$

Upon deriving the scenario-based FD design, our target is to further estimate the reliability of scenario optimal solution W_N^* by the violation probability $V(W_N^*)$. Note that $V(W_N^*)$ is essentially FAR, since ${e^0_\delta}^T W e^0_\delta$ is exactly the residual generation of possible residual e^0_δ when there is no fault denoted by the superscript 0.

Estimate the prior reliability of optimal solution W_N^* is a random variable because of the dependence of W_N^* on $\delta^{(1)}, \ldots, \delta^{(N)}$. Recall that from the definition, violation probability is in fact FAR, so providing probability bounds to keep it under control is vital for practical usage. The main result of Campi and Garatti (2008) states that the tail probability of $V(W_N^*)$ is upper bounded by the cumulative distribution function of binomial distribution.

Theorem 2. Assume that $\mathbb{P}^N$ is a product probability space due to independence of $\delta^{(1)}, \ldots, \delta^{(N)}$, d is the number of optimization variables. It then holds that:

$$\mathbb{P}^N \left\{V\left(W_N^*\right) > \epsilon\right\} \le \sum_{i=0}^{d-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i}.$$

Note that the above theorem holds independently of the true distribution $\mathbb{P}$. To make N explicit from the equation, we can resort to the following theorem, which simplifies the determination of N given a prespecified confidence level β.

Theorem 3. Given a violation parameter $\epsilon \in (0,1)$ and a confidence parameter $\beta \in (0,1)$. If

$$N \ge \frac{2}{\epsilon}\left(\ln\frac{1}{\beta} + d\right)$$

then, with probability no smaller than $1-\beta$, W_N^* gives an FAR no higher than ϵ, i.e. $\mathbb{P}^N\left\{V\left(W_N^*\right) \le \epsilon\right\} \ge 1-\beta$.

In order to get a acceptable confidence level, such as $1 - 0.00001$, the required smallest sample size is astronomically high, which limits the application of this method to situations where it is expensive to get sample from experiments. For instance, when $d = 55$, $\epsilon = 0.01$ and $\beta = 0.00001$, the required size of sample is at least $\frac{2}{0.01}(\ln\frac{1}{0.00001} + 55) \approx 13303$, which is an unaffordable number to obtain from expensive experiments. We can also see from the result that N is susceptible to d, which will also aggravate the situation.

3.2 The second phase: Validation phase

The scenario-based solution is essentially random since it is calculated based on samples that are i.i.d random variables, so the evaluation of the risk of the randomized solution must be performed before its implementation in real-world situations. A recently promising idea is that there exists affluent information in these a posteriori results, which leads to so-called 'wait-and-judge' scenario approach (Campi and Garatti (2018)) by excavating a particular structure property of the optimal solution—the number of 'decisive' support constraints.

Definition 4. A constraint of the scenario program is a support constraint if its removal changes the solution.

By removing each constraint one by one with others being intact, we can solve a sequence of scenario convex optimization programs and compare the new solution with the original solution. The total number of altered solutions is then equal to the number of support constraints. Given the number of support constraints, we can use bisection method to calculate the a posteriori estimation of probability bound, which will be referred to as BoundS in our paper.

Assumption 5. Existence and uniqueness: For every N and for every sample $\delta^{(i)}$, $i = 1 \ldots N$, equation 17 admits solution, which becomes unique after the application of the tie-break rule.

Assumption 6. Non-degeneracy: For every N, with probability 1 with respect to the sample $\delta^{(i)}$, $i = 1 \ldots N$, the solution to equation 17 with all constraints in place coincides with the solution to the program where only the support constraints are kept.

Theorem 7. (Campi and Garatti (2018)). Assume that s_N^* is the number of support constraints of solution W_N^*, $\beta \in (0,1)$, d is the number of optimization, and N is scenario sample size. For any $k = 0, 1, \ldots, d$, the polynomial equation in the t variable

$$\frac{\beta}{N+1}\sum_{m=k}^{N}\binom{m}{k}t^{m-k} - \binom{N}{k}t^{m-k} - \binom{N}{k}t^{N-k} = 0 \tag{19}$$

has one and only one solution $t(k)$ in the interval $(0,1)$. Letting $\epsilon(k) := 1 - t(k)$, under assumptions 5 and 6, it then holds that

$$\mathbb{P}^N\left\{V\left(W_N^*\right) > \epsilon\left(s_N^*\right)\right\} \le \beta. \tag{20}$$

In the above theorem, the estimate of the upper bound on FAR becomes a function of s_N^*, which allows a flexible adjustment based on the realization of scenario-based FD design. Meanwhile, an alternative yet more popular strategy is to carry out validation test on a collection of new instances of the uncertainty.

Definition 8. Given M validation samples, the violation frequency r_M is the number of validation constraints that are violated by the solution.

By generating more samples from the same probability space and using them to validate the proposed solution, the total number of inconsistent samples is the violation frequency. Given the value of violation frequency, we can readily calculate the posteriori estimation of probability bound, which is essentially based on the hypothesis testing of binomial trials and will be referred to as BoundV.

Theorem 9. r_M is the violation frequency of solution W_N^* on M validation samples. Given a pre-defined confidence level $\beta^* \in (0,1)$, M is validation sample size, it then holds that:

$$\mathbb{P}^M \left\{V(W_N^*) > \eta_M\left(r_M\right)\right\} \leq \beta^*. \tag{21}$$

where the analytic form of $\eta_M(\cdot)$ is expressed as:

$$\eta_M(l) = \begin{cases} \min\limits_{\eta} \left\{\eta : B_M(\eta; l) \leq \beta^*\right\}, & \text{if } l \neq M \\ 1, & \text{if } l = M \end{cases} \tag{22}$$

The violation frequency on validation data can also be used together with support constraints (Campi and Garatti (2018)) to improve our estimates of reliability levels, as proposed by Shang and You (2019). Next we will sketch out the skeleton of how to compute improved estimates with more information available in our hands.

Combine support constraints and violation frequency. Because we have two posteriori information of the scenario program, it is rather reasonable to estimate the probability bound with all the information in hands. It can be proved that the previous two bounds are specific cases of this more general bound. which testifies the correctness and generality of this bound, which will be referred as BoundSV (Shang and You (2019)).

Theorem 10. Assume that M is the validation sample size, r_M is the violation frequency. N is scenario sample size, s_N^* is the number of support constraints, $\beta \in (0,1)$ is the confidence level, and weighting parameters $\{a_m\}$ satisfy

$$a_m \geq 0, m \in \mathbb{N}_{0:N}, \sum_{m=d}^{N-1} a_m > 0, \sum_{m=0}^{N} a_m = 1. \tag{23}$$

$t_{N,M}(k,l)$ is the root of the following polynomial function in t:

$$\begin{aligned} h_{N,M}(t;k,l) = \beta \sum_{m=k}^{N} a_m \binom{m}{k} t^{m-k} \\ - \binom{N}{k} t^{N-k} B_M(1-t;l) \end{aligned} \tag{24}$$

then it holds that $\mathbb{P}^{N+M}\left\{V\left(\mathrm{x}_N^*\right) > \epsilon_{N,M}\left(s_N^*, r_M^*\right)\right\} \leq \beta$ where $\epsilon_{N,M}(k,l) = 1 - t_{N,M}(k,l)$.

In this way, the reliability estimate of the FD design can be adjusted flexibly using sequentially accumulated validation data after the implementation of FD design. Whenever another sample comes, we can test if it violates the constraints to update the upper-bound estimate of the violation frequency. With updated violation sample size and violation frequency, the bounds can be recalculated to incorporate the new finding in reality. From Shang and You (2019), the tendency of the change in our beliefs can be described by the following theorems, which exactly match our intuitions and assure us of the rationale of the estimation.

Theorem 11. BoundV holds that:

$$\eta_M(l) > \eta_{M+1}(l), \qquad \eta_{M+1}(l+1) > \eta_M(l) \tag{25}$$

Now we re-clarify the definition. M denotes the number of validation sample size and l denotes the number of violation frequency. The rationale is really simple: if the next sample does not violate constraints (i.e. $M+1$ and l remains the same), we will have more confidence in the reliability of the result, which leads to a tighter bound of risk; on the other hand, if the next sample does violate constraints (i.e. M and l both add by 1), we will lose some confidence in the reliability of the result, which leads to a looser bound of risk.

Theorem 12. BoundSV satisfies the following monotonicity properties:

$$\epsilon_{N,M}(k,l) > \epsilon_{N,M+1}(k,l), \quad \epsilon_{N,M+1}(k,l+1) > \epsilon_{N,M}(k,l). \tag{26}$$

The interpretation of this theorem is similar to the previous one. In addition to M and l, N denotes the number of scenario sample size and k denotes the number of support constraints, but N,k are not changed during the course of succesive validation.

4. CASE STUDY

4.1 System setup

To testify our findings, a discrete linear time invariant system is constructed, the configuration of which is given as follows:

$$\begin{aligned} &A = \begin{bmatrix} 0.4170 & 0.3023 & 0.1863 \\ 0.7203 & 0.1468 & 0.3456 \\ 0.0001 & 0.0923 & 0.3968 \end{bmatrix}; B = \begin{bmatrix} 0.5388 & 0.2045 \\ 0.4192 & 0.8781 \\ 0.6852 & 0.0274 \end{bmatrix} \\ &C = \begin{bmatrix} 0.6705 & 0.5587 & 0.1981 \\ 0.4173 & 0.1404 & 0.8007 \end{bmatrix}; D = \begin{bmatrix} 0.9683 & 0.6923 \\ 0.3134 & 0.8764 \end{bmatrix}; \\ &B_d = I, D_d = 0 \end{aligned} \tag{27}$$

Because of the controllability of the system, it is sensible to devise a feedback of state to make all the eigenvalues lie in the unit circle on the complex panel, which will make the system stable.

We collect last 3 input and output signals to detect possible fault, in another word the data window is set to 3. For the first batch of run, we accumulate 400 samples. Then in another batch of run, 200 additional samples are collected for usage in further validation step, as shown in Table 1. In order to simulate immanent noise in the system, a zero mean normal noisy with a standard deviation of one is added in the input signal. In the fault detection step, we set the threshold J_{th} to 0.05. The alarm threshold can be modulated in practical usage by technicians in a real plant to achieve an appropriate balance between FAR and MAR. All data samples have to be collected independently. In practice this requirement can be realized by maintaining a large enough sampling gap between two consecutive samples.

Table 1. Parameter Setting of Simulation

Sample Size	Validation Size	Extra Validation Size
400	200	50

4.2 Results and discussions

Input and output series data are fed into the CVX software to solve the convex optimization problem. Upon obtaining the solution, a posteriori information is at hand to attain different tighter bounds. In order to find out which bound is better in reality, by using Monte Carlo simulation, 10^4 more samples are generated from the same probability space for the validation step to get the true violation probability of the scenario solution, namely the TrueBound—0.0215.

Table 2. Three different a-posteriori bounds

BoundS	BoundV	BoundSV	TrueBound
0.0539	0.0689	0.0373	0.0215

After calculation of three bounds which substitute the *bound* in $\mathbb{P}\{V(W_N^*) > bound\} \leq \beta$, the reliability of the solution W_N^* can be stated that: violation probability of W_N^* is less than *bound* with high confidence $(1 - 0.00001)$. Here we compare 3 different posteriori probability bounds with true value. Obviously, the third bound BoundSV which accumulates two kinds of information (i.e. support constraints and validation samples) is less conservative than others.

It is worth mentioning that although we can run inexpensive simulations for thousands of times, in engineering practice the cost of such experiments may be too high to be conducted for the purpose of figuring out the violation probability. In this situation, previously proposed probability bounds can serve as approximate indicators of reliability with small validation sample size.

Online adjustment of posteriori bounds with accumulating validation samples In order to illustrate our findings on the online adjustment of posteriori bounds (i.e. BoundV and BoundSV), we consider the practical case where independent validation scenarios are gradually accumulated after W_N^* is obtained. Once upon fetching a new validation sample, we test whether it is violated or not and then update the posteriori bounds (i.e. BoundV and BoundSV). From the figure 2, we can see see that when the next new validation sample violates the constraints (i.e. 1 in the top gragh), both of the bounds (in the bottom graph) will increase and vice versa, which matched our intuition and the theoretical results in the preceding section.

5. CONCLUSION

In this paper, we have proposed a novel integrated fault detection scheme which is suitable for real-world application. Our design consists of two phase. The first is the modelling phase, where a probabilistic fault detection design is devised, and the second is validation phase, where reliability estimate of the design is adjusted online according to new operation records of the plant. The final remark is that during the whole process, we do not hypothesize any priori information about the distribution of noisy and achieve the goal of distribution free, which ensures that the method proposed in this paper can be applied to wide range of real plants without loss of efficiency.

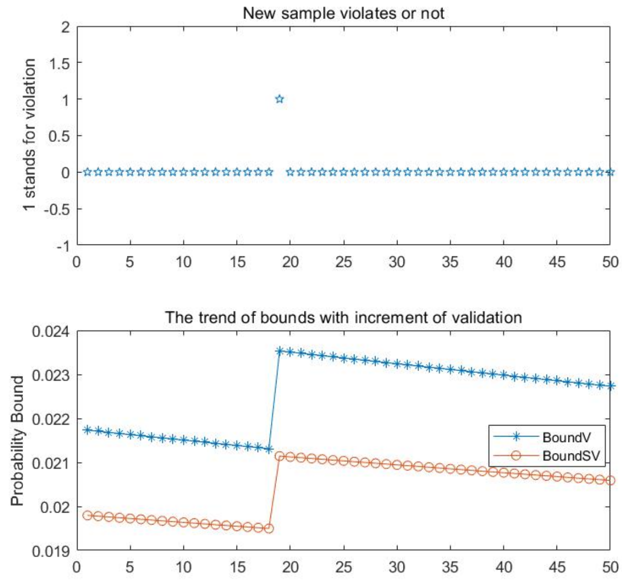

Fig. 2. Updated Bounds Based on Incrementally Arriving Validation Data

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