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On the Discretization of Robust Exact Filtering Differentiators

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Abstract: The calculation of time derivatives has a state-space representation of the form of a perturbed linear system. This description enables the application of the so-called robust filtering differentiator, i.e. a non-linear state observer with extended state and homogeneous input injection terms. Therefore, given the practical importance of having an accurate discretization of such differentiator, this paper presents the design of two discrete-time implementation schemes. The first discrete-time realization is explicit, while the second one is implicit. The implicit allows reducing the numerical chattering phenomenon caused by the explicit discretization of discontinuous terms. Numerical comparisons between the presented scheme and an existing discrete-time representation show that the performance of the proposed explicit implementation scheme is similar to the most recent results from the literature. Finally, the proposed implicit discrete-time realization presents better accuracy, especially when considering large sampling periods.

Keywords: Nonlinear observers and filter design, Sliding mode control, Observer design.

1. INTRODUCTION

The problems of filtering a noisy signal and differentiation in real-time are crucial issues due to their practical interest in signal processing and control engineering. These problems have been addressed using various methods: Kalman filter (Kalman, 1960), algebraic methods (Mboup et al., 2009), non-linear observers (Chitour, 2002; Davila et al., 2005) to name a few. Between those non-linear approaches, sliding mode techniques are convenient to design observers due to their exceptional robustness properties (Utkin, 2013). For example, high-order sliding mode homogeneous differentiators (Levant, 2003; Levant and Livne, 2011) and robust filtering differentiators (Levant and Livne, 2019) can estimate the n time derivatives of a signal, showing good robustness properties in the presence of noise and exact finite-time convergence in the absence of noise. However, one of the main disadvantages of these techniques is its numerical implementation, which can cause numerical chattering, a decreased robustness and other undesired effects.

In practice, observation algorithms are usually discretized to be implemented in a digital environment. However, the discrete-time approximations of the continuous-time algorithms are far from being straightforward. Indeed, for high-gain and sliding mode differentiators, an inadequate discrete-time version of the algorithms may lead to numerical chattering (Drakunov and Utkin, 1989; Utkin, 1994), i.e., high oscillations only due to the numerical methods used in the discretization scheme. Several algorithms have been proposed for the implementation of discrete-time sliding mode controllers (Drakunov and Utkin, 1989; Su et al., 2000; Nguyen et al., 2017; Abidi et al., 2007). Concerning the homogeneous differentiator, some explicit discretization algorithms have been derived in (Livne and Levant, 2014; Koch and Reichhartinger, 2018; Koch et al., 2020; Barbot et al., 2020; Levant and Livne, 2019) to preserve the estimation accuracy properties. In (Livne and Levant, 2014), a discrete-time realization of the homogeneous differentiator, which preserves the computational simplicity of the one-step Euler scheme, has been introduced. In (Koch and Reichhartinger, 2018), the proposed discrete algorithm is less sensitive to gain overestimation. A discrete-time differentiator, which includes nonlinear higher-order terms, has been derived in (Koch et al., 2020) to preserve the asymptotic accuracy properties known from the continuous-time differentiator despite the presence of noise. The work in (Barbot et al., 2020) extends the results from (Livne and Levant, 2014) while also considering non-homogeneous hybrid differentiators.

Recently, some implicit discretization algorithms have been investigated to ensure a smooth stabilization of the sliding surface in discrete-time for the case without disturbance (Acary et al., 2012; Brogliato et al., 2020; Huber et al., 2016). Such algorithms remove the numerical chattering effects due to the time discretization and allow the use of large sampling periods without reducing too much the performances. However, implicit methods were only applied to first-order sliding mode controllers (Acary et al., 2012), twisting controllers (Huber et al., 2016) and super-twisting controllers (Brogliato et al., 2020). Nevertheless, an implicit discretization algorithm has been recently proposed in (Carvajal-Rubio et al., 2019) for the homogeneous differentiator.

This paper proposes two discretization algorithms, based on the recent results presented in (Carvajal-Rubio et al., 2019), for the robust filtering differentiator given in (Levant and Livne, 2019). The first one is an explicit exact discrete-time version of the filtering differentiator, while the second one is an implicit discretization algorithm that removes the numerical chattering effects. Some simulations are given to compare the discrete-time algorithm presented in (Levant and Livne, 2019) with the proposed ones (explicit and implicit methods). It will be shown that the proposed scheme provides estimates of the derivatives of a given signal with good accuracy and robustness properties even when a large sampling period is considered.

The rest of the paper is as follows. Section 2 introduces the problem and recalls some preliminaries on the exact filtering differentiator. In Section 3, two discretization algorithms for the robust filtering differentiator are given (i.e., explicit and implicit discrete-time algorithms). At last, in Section 4, some simulations are done to highlight the interest of the proposed scheme when a significant sampling period is considered.

Notation. For $x \in \mathbb{R}$, the absolute value of x, represented by $|x|$, is given as $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. $\operatorname{sign}(x)$ is a set-valued function defined as $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$, and $\operatorname{sign}(x) \in [-1, 1]$ if $x = 0$. For $c \geq 0$, the signed power c of x is defined as $\lfloor x \rceil^c = |x|^c \operatorname{sign}(x)$.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1 Problem statement

The objective of a differentiator is to obtain online the first n derivatives of a function even if there is noise in the measurement. Here, it is represented as $f_0(t)$, $f_0 : \mathbb{R} \to \mathbb{R}$. Moreover, $f_0(t)$ is assumed a function at least $(n+1)-th$ differentiable and its $n+1$ derivative is bounded by a known real number $L > 0$, i.e., $|f_0^{(n+1)}(t)| \leq L$. Furthermore, $f(t) = f_0(t) + \Delta(t)$, where $f(t)$ is the input of the differentiator and $\Delta(t)$ is the noise in the measurement. It is also assumed that $\Delta(t)$ is a Lebesgue-measurable bounded noise with $|\Delta(t)| \leq \delta$ for a real number $\delta > 0$, which can be unknown.

In order to compute the derivatives $f_0^{(1)}(t)$, $f_0^{(2)}(t)$, $\cdots$, $f_0^{(n)}(t)$, a state space representation is used. The state variables are defined as $x_i(t) = f_0^{(i)}(t)$ and $\boldsymbol{x} = [\, x_0 \; x_1 \; x_2 \; \cdots \; x_n \,]^T \in \mathbb{R}^{n+1}$. Therefore, one can obtain the following representation for the differentiation problem in the state space:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{e}_{n+1} f_0^{(n+1)}(t) \\ y_o(t) &= \boldsymbol{e}_1^T \boldsymbol{x}(t) + \Delta(t) \end{aligned} \tag{1}$$

with the canonical vectors $\boldsymbol{e}_1 = [\, 1 \; 0 \; \cdots \; 0 \; 0 \,]^T$, $\boldsymbol{e}_{n+1} = [\, 0 \; 0 \; \cdots \; 0 \; 1 \,]^T$ and $\boldsymbol{A} = [\boldsymbol{0}_{(n+1)\times 1} \; \boldsymbol{e}_1 \; \boldsymbol{e}_2 \; \cdots \; \boldsymbol{e}_n]$, which is a nilpotent matrix of appropriate dimensions. Notice that the successive time derivatives of $f_0(t)$ can be obtained through the design of a state observer.

2.2 Homogeneous high-order differentiator

In order to obtain the first n derivatives of $f_0(t)$, a continuous-time observer has been proposed in (Levant, 2003). For $\Delta(t) = 0$, it can be represented in the non-recursive form:

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}(\sigma_0) \tag{2}$$

where $\boldsymbol{u}(\sigma_0) = [\Psi_{0,n}(\sigma_0) \; \Psi_{1,n}(\sigma_0) \; \cdots \; \Psi_{n,n}(\sigma_0)]^T$, $\Psi_{i,n}(\cdot) = -\lambda_{n-i} L^{\frac{i+1}{n+1}} \lfloor \cdot \rceil^{\frac{n-i}{n+1}}$, $\boldsymbol{B}$ is the identity matrix of appropriate dimensions, $\sigma_0 = z_0 - x_0$ and $\boldsymbol{z} = [\, z_0 \; z_1 \; z_2 \; \ldots \; z_n \,]^T$ is the finite-time estimate of the state vector $\boldsymbol{x}$ using adequate parameters $\lambda_i > 0$ (see (Reichhartinger et al., 2017; Levant, 2018) for instance). Since the function $\lfloor z_0 - f(t) \rceil^0$ is discontinuous at $z_0 = f$, the solutions of system (2) are understood in the Filippov sense (Filippov, 2013).

2.3 Finite-time-exact robust filtering differentiator (FTER)

Although, differentiator (2) offers good performance when there exists a Lebesgue-measurable bounded noise $\Delta(t)$ such that $|\Delta(t)| \leq \delta$ with small in average δ, its performance becomes significantly reduced when δ is large. Due to this reason, in (Levant, 2018), a new finite-time exact robust filtering differentiator has been proposed, with the following structure:

$$\begin{aligned} \dot{\omega}_{i_f} &= -\lambda_{m+1-i_f} L^{\frac{i_f}{m+1}} \lfloor \omega_1 \rceil^{\frac{m+1-i_f}{m+1}} + \omega_{i_f+1} \\ \dot{\omega}_{n_f} &= -\lambda_{n+1} L^{\frac{n_f}{m+1}} \lfloor \omega_1 \rceil^{\frac{n+1}{m+1}} + z_0 - g(t) \\ \dot{z}_{i_d} &= -\lambda_{n-i_d} L^{\frac{n_f+1+i_d}{m+1}} \lfloor \omega_1 \rceil^{\frac{n-i_d}{m+1}} + z_{i_d+1} \\ \dot{z}_n &= -\lambda_0 L \lfloor \omega_1 \rceil^0 \\ & i_f = 1, 2, \cdots, n_f - 1. \quad i_d = 0, 1, 2, \cdots, n-1. \end{aligned} \tag{3}$$

where $m = n + n_f$, $n_f \geq 0$, n_f is the filtering order and the parameters λ_i are selected as in (2). Moreover, $g(t) = f_0(t) + \upsilon(t)$, where $\upsilon(t)$ is comprised of $n_f + 1$ components, $\upsilon(t) = \upsilon_0(t) + \upsilon_1(t) + \cdots + \upsilon_{n_f}(t)$, $\upsilon_j(t)$ is a signal of the global filtering order j and integral magnitude $\epsilon_j \geq 0$ with $j = 0, 1, \cdots, n_f$. More details can be founded in (Levant and Livne, 2019). In (Levant and Livne, 2019), it is shown that differentiator (3) offers the following accuracy:

$$\begin{aligned} & |z_i - f_0^{(i)}(t)| \leq \mu_i L \rho^{n+1-i}, \;\; \mu_i > 0, \;\; i = 0, 1, 2, \cdots, n. \\ & \rho = \max\left[\left(\frac{\epsilon_0}{L}\right)^{\frac{1}{n+1}}, \left(\frac{\epsilon_1}{L}\right)^{\frac{1}{n+2}}, \cdots, \left(\frac{\epsilon_{n_f}}{L}\right)^{\frac{1}{m+1}} \right] \end{aligned}$$

2.4 Discretization (FTER-D)

In practice, the differentiation algorithms are usually discretized in order to be implemented in a digital environment. In (Levant, 2018), a discrete-time filtering differentiator is presented as follows:

$$\begin{aligned}
\omega_{i_f,k+1} &= \omega_{i_f,k} + \tau\left(\omega_{i_f+1} + \Psi_{i_f-1,m}\left(\omega_{1,k}\right)\right)\\
\omega_{n_f,k+1} &= \omega_{n_f,k} + \tau\left(z_0 - g_k + \Psi_{n_f-1,m}\left(\omega_{1,k}\right)\right)\\
z_{i_d,k+1} &= z_{i_d,k} + \tau\left(z_{i_d+1,k} + \Psi_{n_f+i_d,m}\left(\omega_{1,k}\right)\right)\\
z_{n,k+1} &= z_{n,k} + \tau\Psi_{m,m}\left(\omega_{1,k}\right)\\
i_f &= 1,2,\cdots,n_f-1. \quad i_d = 0,1,2,\cdots,n-1.
\end{aligned} \tag{4}$$

with $\tau = t_{k+1} - t_k$, $w_{i,k} = w_i(\tau k)$, $z_{i,k} = z_i(\tau k)$, $g_{0,k} = g_0(\tau k)$, $k = 0, 1, 2, \cdots$. For differentiator (4), $f_{0,k} = f_0(\tau k)$ is assumed to be as in the continuous differentiator (3), $\upsilon_k = \upsilon(\tau k)$ is comprised of $n_f + 1$ components, $\upsilon_k = \upsilon_{0,k} + \upsilon_{1,k} + \ldots + \upsilon_{n_f,k}$, where $\upsilon_{i,k}$ are of the global sampling filtering order j and integral magnitude ϵ_j with $j = 0, 1, \cdots, n_f$ (Levant and Livne, 2019). Furthermore, it is assumed that the set of admissible sampling-time sequences contains sequences for any $\tau > 0$. According to (Levant and Livne, 2019), the discrete differentiator (4) provides the following accuracy:

$$\begin{aligned}
&|\sigma_{i,k}| \leq \mu_i L\rho^{n+1-i}, \quad \mu_i > 0, \quad \sigma_{i,k} = z_{i,k} - x_{i,k},\\
&\rho = \max\left[\tau, \left(\frac{\epsilon_0}{L}\right)^{\frac{1}{n+1}}, \left(\frac{\epsilon_1}{L}\right)^{\frac{1}{n+2}}, \cdots, \left(\frac{\epsilon_{n_f}}{L}\right)^{\frac{1}{m+1}}\right],\\
&i = 0,1,2,\cdots,n.
\end{aligned}$$

3. DISCRETIZATION OF ROBUST EXACT FILTERING DIFFERENTIATOR

In this Section, two discrete-time realizations of the filtering differentiator are proposed. The first one is an explicit one, which is based on an exact discretization, while the second one is an implicit algorithm.

3.1 Explicit Discretization of the robust exact filtering differentiator

Applying the procedure presented in (Carvajal-Rubio et al., 2019) to system (3), the following discrete-time realization of the differentiator is obtained:

$$\begin{aligned}
\omega_{i_f,k+1} =& -\frac{\tau^{(n_f-i_f+1)}}{(n_f-i_f+1)!}g_k + \sum_{l=i_f}^{n_f}\frac{\tau^{(l-i_f)}}{(l-i_f)!}w_{l,k}\cdots\\
&+\sum_{l=0}^{n}\frac{\tau^{(n_f+l-i_f+1)}}{(n_f+l-i_f+1)!}z_{l,k}\cdots\\
&+\sum_{l=i_f}^{m+1}\frac{\tau^{(l-i_f+1)}}{(l-i_f+1)!}\Psi_{l-1,m}\left(\omega_{1,k}\right)\\
z_{i_d,k+1} =& \sum_{l=i_d}^{n}\frac{\tau^{(l-i_d)}}{(l-i_d)!}z_{l,k} + \frac{\tau^{(l-i_d+1)}}{(l-i_d+1)!}\Psi_{n_f+l,m}\left(\omega_{1,k}\right)\\
i_f =& 1,2,\cdots,n_f. \quad i_d = 0,1,2,\cdots,n.
\end{aligned} \tag{5}$$

Using Taylor series expansion with Lagrange's remainders (see (Firey, 1960)) on system (1) the following discrete-time system is obtained:

$$\begin{aligned}
x_{i_d,k+1} &= \frac{\tau^{n-i_d+1}}{(n-i_d+1)!}f_0^{(n+1)}(\rho_{i_d}) + \sum_{l=i_d}^{n}\frac{\tau^{(l-i_d)}}{(l-i_d)!}x_{l,k}\\
i_d &= 0,1,2,\cdots,n.
\end{aligned} \tag{6}$$

$\rho_{i_d} \in (t_k, t_{k+1})$, $x_{i_d,k} = x_{i_d}(\tau k)$, and $|f_0^{(n+1)}(\rho_{i_d})| \leq L$. Then, the discrete form of the error system is given as:

$$\begin{aligned}
\omega_{i_f,k+1} =& \frac{\tau^{(n_f-i_f+1)}}{(n_f-i_f+1)!}(\sigma_{0,k} - \upsilon_k) + \sum_{l=i_f}^{n_f}\frac{\tau^{(l-i_f)}}{(l-i_f)!}w_{l,k}\cdots\\
&+\sum_{l=1}^{n}\frac{\tau^{(n_f+l-i_f+1)}}{(n_f+l-i_f+1)!}(\sigma_{l,k} + x_{l,k})\ldots\\
&+\sum_{l=i_f}^{m+1}\frac{\tau^{(l-i_f+1)}}{(l-i_f+1)!}\Psi_{l-1,m}\left(\omega_{1,k}\right)\\
\sigma_{i_d,k+1} =& \frac{\tau^{n-i_d+1}}{(n-i_d+1)!}f_0^{(n+1)}(\rho_{i_d}) + \sum_{l=i_d}^{n}\frac{\tau^{(l-i_d)}}{(l-i_d)!}\sigma_{l,k}\cdots\\
&+\sum_{l=i_d}^{n}\frac{\tau^{(l-i_d+1)}}{(l-i_d+1)!}\Psi_{n_f+l,m}\left(\omega_{1,k}\right)\\
i_f =& 1,2,\cdots,n_f. \quad i_d = 0,1,2,\cdots,n.
\end{aligned} \tag{7}$$

with $\sigma_{i,k} = \sigma_i(\tau k)$. Due to the terms $x_{i,k}$, differentiator (5) does not guarantee convergence for functions with unbounded first n derivatives. Therefore, in order to avoid these terms on the error system (7), the following discretization is proposed based on the structure of (5):

$$\begin{aligned}
\omega_{i_f,k+1} =& \frac{\tau^{(n_f-i_f+1)}}{(n_f-i_f+1)!}(z_{0,k} - g_k) + \sum_{l=i_f}^{n_f}\frac{\tau^{(l-i_f)}}{(l-i_f)!}w_{l,k}\cdots\\
&+\sum_{l=i_f}^{m+1}\frac{\tau^{(l-i_f+1)}}{(l-i_f+1)!}\Psi_{l-1,m}\left(\omega_{1,k}\right)\\
z_{i_d,k+1} =& \sum_{l=i_d}^{n}\frac{\tau^{(l-i_d)}}{(l-i_d)!}z_{l,k} + \frac{\tau^{(l-i_d+1)}}{(l-i_d+1)!}\Psi_{n_f+l,m}\left(\omega_{1,k}\right)\\
i_f =& 1,2,\cdots,n_f. \quad i_d = 0,1,2,\cdots,n.
\end{aligned} \tag{8}$$

Henceforth, the differentiator (8) is referenced as FTER-E.

3.2 Implicit Discretization of the robust exact filtering differentiator

Now, consider the implicit discrete-time algorithm of the robust filtering differentiator. By usign FTER-E with $\omega_{1,k+1}$ instead of $\omega_{1,k}$, an implicit scheme is obtained. However, to implement this discrete-time differentiator $\omega_{1,k+1}$ needs to be calculated at time $t = t_k$. Using the difference equation for $\omega_{1,k+1}$, the following inclusion is obtained:

$$\begin{aligned}
w_{1,k+1} &+ a_m\left\lfloor\omega_{1,k+1}\right\rceil^{\frac{m}{m+1}} + \cdots + a_1\left\lfloor\omega_{1,k+1}\right\rceil^{\frac{1}{m+1}}\cdots\\
&+ b_k \in -a_0\mathrm{sign}\left(\omega_{1,k+1}\right)
\end{aligned} \tag{9}$$

where $b_k = -\frac{\tau^{n_f}}{n_f!}(z_{0,k} - g_k) - \sum_{l=1}^{n_f}\frac{\tau^{(l-1)}}{(l-1)!}w_{l,k}$ and $a_l = \frac{\tau^{m-l+1}}{(m-l+1)!}\lambda_l L^{\frac{m-l+1}{m+1}}$, with $a_i \in \mathbb{R}^+$ and $b_k \in \mathbb{R}$. As in (Carvajal-Rubio et al., 2019; Brogliato et al., 2020), a new support variable is introduced as $\xi_{k+1} \in \mathrm{sign}\left(\omega_{0,k+1}\right)$. Using a similar scheme that the one presented in (Carvajal-Rubio et al., 2019), $\omega_{1,k+1}$ and ξ_{k+1} are given as follows:

- **Case 1**: $b_k > a_0$. $\xi_{k+1} = -1$ and $\omega_{1,k+1} = -\left(r_0\right)^{m+1}$, where r_0 is the unique positive root of the polynomial:

$$p(r) = r^{m+1} + a_m r^m + \cdots + a_1 r + (-b_k + a_0) \quad (10)$$

- **Case 2**: $b_k \in [-a_0, a_0]$. $\omega_{1,k,+1} = 0$ and $\xi_{k+1} = -\frac{b_k}{a_0}$.
- **Case 3**: $b_k < -a_0$. $\xi_{k+1} = 1$ and $\omega_{1,k+1} = r_0^{m+1}$, where r_0 is the positive root of the polynomial:

$$p(r) = r^{m+1} + a_m r^m + \cdots + a_1 r + (b_k + a_0) \quad (11)$$

Furthermore, the pair $\omega_{1,k+1} \in \mathbb{R}$ and $\xi_{k+1} \in [-1, 1]$ is unique for each set of values of a_l and b_k. With the new variable ξ_{k+1} the implicit discrete differentiator is implemented as follows:

$$\begin{aligned}
\omega_{i_f,k+1} =& \frac{\tau^{(n_f - i_f + 1)}}{(n_f - i_f + 1)!}(z_{0,k} - g_k) + \sum_{l=i_f}^{n_f} \frac{\tau^{(l-i_f)}}{(l - i_f)!} w_{l,k} \dots \\
&+ \sum_{l=i_f}^{m+1} \frac{\tau^{(l-i_f+1)}}{(l - i_f + 1)!} \widetilde{\Psi}_{l-1,m}(\omega_{1,k+1}) \\
z_{i_d,k+1} =& \sum_{l=i_d}^{n} \frac{\tau^{(l-i_d)}}{(l - i_d)!} z_{l,k} + \frac{\tau^{(l-i_d+1)}}{(l - i_d + 1)!} \widetilde{\Psi}_{n_f+l,m}(\omega_{1,k+1}) \\
i_f =& 1, 2, \cdots, n_f. \quad i_d = 0, 1, 2, \cdots, n. \\
\widetilde{\Psi}_{i,m}(\omega_{1,k+1}) &= -\lambda_{m-i} L^{\frac{i+1}{m+1}} |\omega_{1,k+1}|^{\frac{m-i}{m+1}} \xi_{k+1}.
\end{aligned} \quad (12)$$

Hereafter, the differentiator (12) is referenced as FTER-I.

Remark 1. The variable ξ_{k+1} is defined for any value of $\omega_{1,k+1}$ and $\text{sign}(0) \in [-1, 1]$, $\xi_{1,k+1}$ is smoother than the function $\text{sign}(\omega_{1,k+1})$.

Remark 2. To implement differentiator (12), r_0 needs to be computed when $b_k \notin [-a_0, a_0]$. Hence, a root finding method is needed. Here, the Halley's method is used (Scavo and Thoo, 1995).

4. SIMULATION RESULTS

In order to analyze and compare the performance of differentiators (4), (8) and (12), two indexes are used: the mean square error of z_i in the time interval $[t_{min}, t_{max}]$ (denoted M_i) and Y_i, which is defined as $Y_i = \max\{|\sigma_{i,k}| \in \mathbb{R} \mid 10s \le t_k \le t_{max}\}$. In the following simulations, the filtering differentiator uses the following parameters $n = 3$, $n_f = 2$, $\lambda_0 = 1.1$, $\lambda_1 = 6, 75$, $\lambda_2 = 20.26$, $\lambda_3 = 32.24$, $\lambda_4 = 23.72$ and $\lambda_5 = 7$. Notice that the parameters λ_i are chosen as in (Levant, 2018). Finally, the initial condition for the differentiators is $\omega_{i,0} = 0$ and $z_{i,0} = 0$. Then FTER-I has the following structure:

$$\begin{aligned}
\omega_{1,k+1} =& \omega_{1,k} + \tau\omega_{2,k} + \frac{\tau^2}{2!}(z_{0,k} - g_k) \dots \\
&+ \sum_{l=1}^{6} \frac{\tau^l}{l!} \widetilde{\Psi}_{l-1,5}(\omega_{1,k+1}) \\
\omega_{2,k+1} =& \omega_{2,k} + \tau(z_{0,k} - g_k) + \sum_{l=2}^{6} \frac{\tau^{l-1}}{(l-1)!} \widetilde{\Psi}_{l-1,5}(\omega_{1,k+1}) \\
z_{0,k+1} =& z_{0,k} + \tau z_{1,k} + \frac{\tau^2}{2!} z_{2,k} + \frac{\tau^3}{3!} z_{3,k} \dots \\
&+ \sum_{l=0}^{3} \frac{\tau^{l+1}}{(l+1)!} \widetilde{\Psi}_{l+2,5}(\omega_{1,k+1}) \\
z_{1,k+1} =& z_{1,k} + \tau z_{2,k} + \frac{\tau^2}{2!} z_{3,k} + \sum_{l=1}^{3} \frac{\tau^l}{l!} \widetilde{\Psi}_{l+2,5}(\omega_{1,k+1}) \\
z_{2,k+1} =& z_{2,k} + \tau z_{3,k} + \sum_{l=2}^{3} \frac{\tau^{(l-1)}}{(l-1)!} \widetilde{\Psi}_{l+2,5}(\omega_{1,k+1}) \\
z_{3,k+1} =& z_{3,k} + \tau \widetilde{\Psi}_{5,5}(\omega_{1,k+1})
\end{aligned} \quad (13)$$

For the first scenario, $f_0(t) = t^4 + \sin(t)$, $L = 25$, $\tau = 0.1s$, $t_{min} = 10s$, $t_{max} = 25s$, $v(t) = 0$, $\sigma_{0,0} = 0$, $\sigma_{1,0} = -1$, $\sigma_{2,0} = 0$ and $\sigma_{3,0} = 1$. Figure 1 shows the corresponding estimation errors using the three differentiators. Variables Y_i and the M_i are summarised in Table 1.

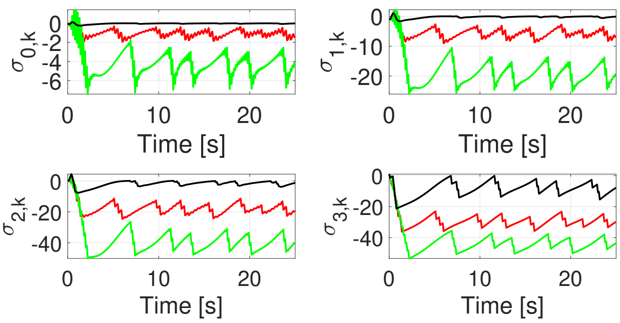

Fig. 1. Estimation error for the first 3 derivatives of $f_0(t)$, FTER-D in red, FTER-E in blue and FTER-I in black.

For this scenario, the three differentiators converge in finite-time in spite of the unbounded functions toward $f_0(t)$, $f_0^{(1)}(t)$, $f_0^{(2)}(t)$ and $f_0^{(3)}(t)$. Moreover using the differentiator FTER-I, one obtains the best results as it can been noticed in Table 1 and Figure 1.

	FTER-D	FTER-E	FTER-I
Y_0	1.8347	6.9573	0.0736
Y_1	8.7351	25.1874	0.5835
Y_2	24.18	47.741	3.8547
Y_3	35.7077	50.3692	15.4041
M_0	1.1476	4.4312	0.0256
M_1	5.9919	19.3807	0.212
M_2	18.4511	39.9016	1.7431
M_3	29.7708	43.9609	8.4592

Table 1. Y_i and M_i for Scenario I.

In the second scenario, $g(t) = \sin(3t)+\cos(2t)-\sin(t)+\upsilon(t)$, with $\upsilon(t) \sim \text{iid}\mathcal{N}(0, 0.1^2)$, $L = 98$, $\tau = 0.1s$, $t_{min} = 10s$, $t_{max} = 25s$, $\sigma_{0,0} = -1$, $\sigma_{1,0} = -2$, $\sigma_{2,0} = 4$ and $\sigma_{3,0} = 26$. Figures 2-5 show the corresponding estimation errors using the three differentiators. Variables Y_i and the M_i are summarised in Table 2.

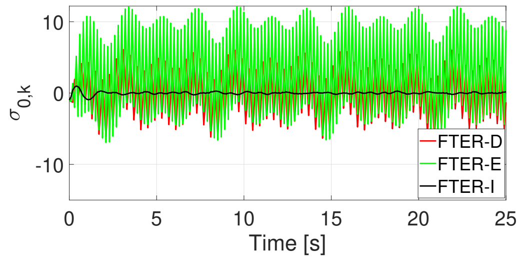

Fig. 2. Estimation error for $f_0(t)$.

For this scenario, the best result for the first two derivatives has been obtained using the proposed implicit differentiator, i.e., FTER-I. For the last derivative, the explicit differentiators, i.e., FTER-D and FTER-E, present better indexes Y_3 and M_3 than the implicit one.

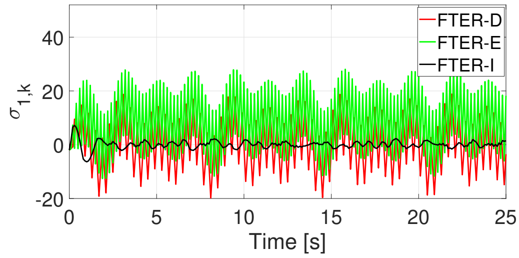

Fig. 3. Estimation error for the first derivative of $f_0(t)$.

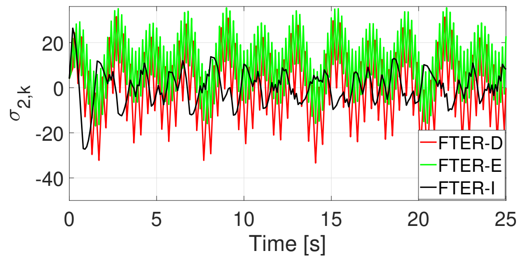

Fig. 4. Estimation error for the second derivative of $f_0(t)$.

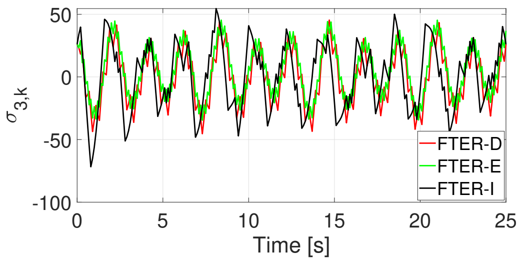

Fig. 5. Estimation error for the third derivative of $f_0(t)$.

	FTER-D	FTER-E	FTER-I
Y_0	6.322853	12.145781	0.274985
Y_1	19.63017	28.059032	2.268494
Y_2	33.374147	35.453408	12.964999
Y_3	45.347208	45.279109	50.014388
M_0	3.146707	7.543292	0.097569
M_1	9.009185	15.711948	0.882423
M_2	14.033387	17.492077	5.676374
M_3	20.517656	20.648051	23.703817

Table 2. Y_i and M_i for Scenario II.

For the last scenario, in order to test the differentiator under noise and different sampling times, the indexes Y_i are given for different constant sampling times in the interval $\tau \in [0.0001s, 1s]$ with a step of $0.0001s$. Furthermore, $f_0(t) = \sin(3t) + \cos(2t) - \sin(t)$, $L = 98$, $t_{min} = 10s$, $t_{max} = 100s$ and the noise is selected as in (Levant and Livne, 2019), $\upsilon(t) = \cos(10000t + 0.7791) + \varepsilon_t$, with $\varepsilon_t \sim \text{iid}\mathcal{N}(0, 0.5^2)$. The results are summarised in Figures 6-8.

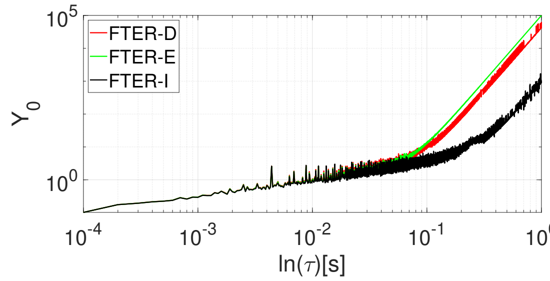

Fig. 6. Y_0 for $\tau \in [0.0001s, 1s]$.

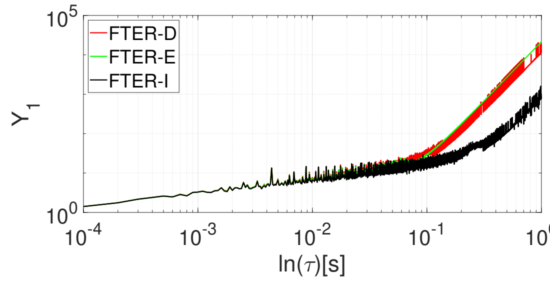

Fig. 7. Y_1 for $\tau \in [0.0001s, 1s]$.

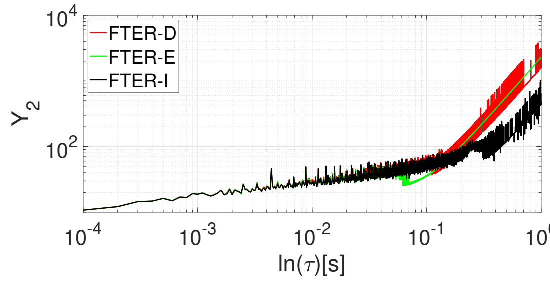

Fig. 8. Y_2 for $\tau \in [0.0001s, 1s]$.

From Figures 6-8, one can see that the differentiator FTER-I gives a better performance for the estimation of $f_0(t)$, $f_0^{(1)}(t)$ and $f_0^{(2)}(t)$ or at least similar for the different sampling times. Although, these figures could indicate that for low frequencies, the estimation of the second and third derivatives of the signal is better for the FTER-D and FTER-E compared with FTER-I.

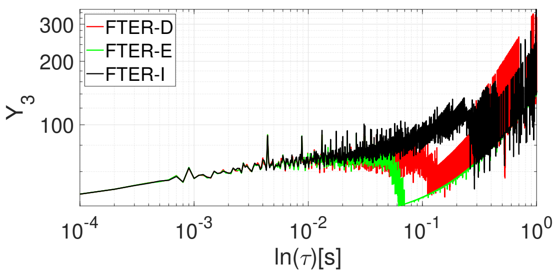

Fig. 9. Y_3 for $\tau \in [0.0001s, 1s]$.

5. CONCLUSION

Two novel discretization algorithms have been presented for the robust filtering differentiator. The first one, which is based on an exact discretization of the continuous differentiator, is explicit. In contrast, the second one is an implicit algorithm which enables to remove the numerical chattering phenomenon and to preserve the estimation accuracy properties. Both algorithms have shown a competitive performance in simulations for free-noise input and when the first n derivatives are unbounded. It is also shown a better performance for the current proposal when compared to the discrete version given in (Levant and Livne, 2019). Moreover, in simulations and under noise, FTER-I presents a better estimation for $f_0(t)$, $f_0^{(1)}(t)$, and $f_0^{(2)}(t)$ than the obtained results using FTER-D and FTER-E. Future works will address convergence and robustness proofs for the proposed discretization schemes.

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