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Towards an MID-based Delayed Design for Arbitrary-order Dynamical Systems with a Mechanical Application

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Abstract: In this study we consider the stabilization of n^{th}-order linear time-invariant (LTI) dynamical systems using *Multiplicity-Induced-Dominancy* (MID)-based controller design in the presence of delays in the input/output channels. A *sufficient condition* is given for the dominancy of a real root with multiplicity $n+1$ using an integral representation of the corresponding characteristic function. Furthermore, this sufficient condition is analyzed in the case when the characteristic function of the open-loop system is real-rooted, and delay intervals are derived for the set of parameters satisfying stabilizability and dominancy conditions. The efficiency of the proposed controller design is shown in the case of a multi-link inverted pendulum.

Keywords: dynamical system, delay, spectral properties, inverted pendulum.

1. INTRODUCTION

Stabilization of unstable equilibria and orbits in the presence of communication delay is an important and challenging task in engineering applications (see, for instance, Michiels et al. (2002); Olgac and Sipahi (2002); Gu et al. (2003); Landry et al. (2005)). In this context, as pointed out by Ma and Chen (2018), computing the control parameters allowing stable operation for large (feedback) delays is not a trivial task. Among the existing approaches, the so-called *parameter-based approach* is largely used in the case of low-order and/or low-complexity controllers (see, e.g., Bhattacharyya et al. (1995); Ackermann (2002); Michiels and Niculescu (2014) and the references therein). In particular, the stability diagrams can be used to visualize stability properties of the closed-loop system in the space of control parameters Bellman and Cooke (1963); Stépán (1989); Hassard (1997); Breda (2012).

It is commonly accepted that the delay in the feedback loop is generally seen as a source of unstable behavior. Furthermore, as the feedback delay gets larger, the stable region in the stability diagrams (charts) gets smaller. In this frame, a challenging task is to find the so-called *critical delay* or *delay margin* [1]. To the best of the authors' knowledge, the idea of exploiting the delay effects in controllers' design was first introduced in Suh and Bien (1979) where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus it can replace the proportional-derivative controller. Furthermore, it was pointed out by Niculescu et al. (2010) that, under appropriate assumptions, the presence of some delay in the control law may induce stability in closed-loop. Finally, in the context of mechanical engineering problems, the effect of time-delay was emphasized in Stépán (1989) where some concrete applications are studied, such as the machine tool vibrations and robotic systems.

It is worth noting that, in the case of stable linear time-delay system, the rightmost root (or spectral abscissa) of the characteristic function is actually the exponential decay rate of its time-domain solution (see, for instance, Mori et al. (1982) for an estimate of the decay rate). Furthermore, to the best of the authors' knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in Hayes (1950). This *Multiplicity-Induced-Dominancy* (MID) property is further explored and analytically derived for scalar delay equations in Boussaada et al. (2016). Next, for second-order systems controlled by a delayed proportional is proposed in Boussaada et al. (2018b, 2017) where its applicability in damping active

[1] i.e. the *maximum delay*, for which the system can still be stabilized by some control law, but for larger delay, the corresponding closed-loop system is unstable.

vibrations for a piezo-actuated beam is proved. Finally, an extension to the delayed proportional-derivative controller case is studied in Boussaada and Niculescu (2018); Boussaada et al. (2018a) where the dominancy property is parametrically characterized and proven by using the argument principle.

In this paper, we analyze a general n^{th}-order linear time-invariant dynamical system with a single delay and show that, under some appropriate conditions, the MID property can be used to assess the delay margin (see, e.g. Boussaada et al. (2015); Boussaada and Niculescu (2018); Boussaada et al. (2018b)). The novelty of the paper lies on the way we are exploiting the root location of the open-loop characteristic polynomial in order to have the multiplicity-induced-dominancy of the overall system, as done in Boussaada et al. (2020) in the particular case of second-order systems.

The remaining paper is organized as follows. The problem statement is presented in Section 2. Section 3 contains some motivating example. Section 4 provides the main ingredients of the dominancy proof, which consists in writing the characteristic function with multiple real roots as an integral operator. The main results are presented in Section 5. An illustrative example is presented in Section 6, where the 5-link inverted pendulum is studied.

2. PROBLEM STATEMENT

Consider the quasipolynomial

$$D(s) = P(s) + e^{-s\tau} Q(s) \,, \tag{1}$$

where the polynomials $P(s)$ and $Q(s)$ are real and have degrees n and $n-1$, respectively:

$$\begin{aligned} P(s) &= a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \,, \\ Q(s) &= b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \ldots + b_1 s + b_0 \,. \end{aligned} \tag{2}$$

Assume that the coefficients a_i of P are fixed and known, and that $a_n > 0$.

The problem we are addressing can be resumed as follows: *computing the values of the delay parameter τ such that the system (1) is stabilizable*[2] *if the control parameters b_i are tuned such that the characteristic function $D(s)$ has a real root s_0 with multiplicity $n+1$?*

3. MOTIVATING EXAMPLE: THE INVERTED PENDULUM

Balancing an inverted pendulum in the presence of feedback delay is a frequently cited example in dynamics and control theory (Atay, 1999; Sieber and Krauskopf, 2004; Li et al., 2017). Different control methods are often implemented in simple inverted pendulum systems (Habib et al., 2017; Xu et al., 2017; Qin et al., 2014; Cieżkowski, 2016). The inverted pendulum is also a basic concept in human balancing models (Maurer and Peterka, 2005; Milton et al., 2016; Milton and Insperger, 2019; Morasso et al., 2019). The equation of motion of an inverted pendulum controlled by a proportional-derivative (PD) controller reads as:

$$\ddot{\varphi}(t) + a_0 \varphi(t) = -b_0 \varphi(t-\tau) - b_1 \dot{\varphi}(t-\tau) \,, \tag{3}$$

with a feedback delay $\tau > 0$ and a system parameter $a_0 < 0$. The characteristic function corresponding to (3) is

$$D(s) = s^2 + a_0 + e^{-s\tau}(b_0 + b_1 s) \,. \tag{4}$$

The open-loop characteristic function $P(s) = s^2 + a_0$ has real roots $\pm\sqrt{-a_0}$ since $a_0 < 0$. This property proves to be useful in Section 5.

The critical delay (delay margin) of the system (3) is well-known from the literature (Schurer, 1948; Stépán, 2009):

$$\tau_{\text{crit}} = \sqrt{-\frac{2}{a_0}} \,, \tag{5}$$

that is, the trivial solution of system (3) can be asymptotically stable if and only if $\tau < \tau_{\text{crit}}$. In the sequel, we will show that the critical delay (5) can be obtained by studying the multiple roots of the characteristic function.

Assume that $D(s)$ has a real root s_0 with algebraic multiplicity at least $\deg P(s) + 1 = 3$. Then $D(s_0) = 0$, $D'(s_0) = 0$ and $D''(s_0) = 0$:

$$\left.\begin{aligned} s_0^2 + a_0 + e^{-s_0\tau}(b_0 + b_1 s_0) = 0 \,, \\ 2s_0 + e^{-s_0\tau}(-\tau(b_0 + b_1 s_0) + b_1) = 0 \,, \\ 2 + e^{-s_0\tau}(\tau^2(b_0 + b_1 s_0) - 2\tau b_1) = 0 \,. \end{aligned}\right\} \tag{6}$$

From (6), one gets:

$$\left.\begin{aligned} b_0 &= e^{s_0\tau}\left(\tau s_0^3 + s_0^2 + a_0 \tau s_0 - a_0\right) , \\ b_1 &= -e^{s_0\tau}\left(\tau s_0^2 + 2s_0 + a_0\tau\right) , \\ s_0 &= \frac{-2 \pm \sqrt{2 - a_0\tau^2}}{\tau} =: s_{0,\pm} \,. \end{aligned}\right\} \tag{7}$$

It can be shown that the triple root $s_{0,+}$ is negative and dominant for every $0 < \tau < \tau_{\text{crit}}$, and therefore the system (3) is asymptotically stable. In particular, at the upper bound $\tau = \tau_{\text{crit}}$ the triple root is $s_{0,+} = 0$ and it is the dominant (rightmost) root of (4) with the control parameters $b_0 = -a_0$ and $b_1 = -a_0\tau_{\text{crit}}$. The dominancy of $s_{0,+}$ may be shown by using the argument principle (see, for instance, Boussaada and Niculescu, 2018; Boussaada et al., 2018a). In the next section, to show the dominancy, we use a method first introduced in Boussaada et al. (2016) and based on an integral representation of the characteristic function (see also Boussaada et al. (2018b)).

4. INTEGRAL REPRESENTATION AND A SUFFICIENT CONDITION FOR DOMINANCY

We have the following result:

Proposition 1. If the quasipolynomial (1) has a real root s_0 with multiplicity at least n, then it can be written as

$$D(s) = (s - s_0)^n \left(a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0; \tau t)}{(n-1)!} \mathrm{d}t \right) , \tag{8}$$

where the family of polynomials $R_k(s;\tau)$ is defined as

$$R_k(s;\tau) = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \; k \in \mathbb{Z}_0^+ \,. \tag{9}$$

Proof. The quasipolynomial $D(s)$ has a root s_0 with algebraic multiplicity at least n if and only if $D^{(k)}(s_0) = 0$, $k = 0, 1, \ldots, n-1$:

[2] i.e. there exist some real parameters b_i ($i = 0, 1, \ldots, n-1$) depending explicitly on $(a_0, a_1, \ldots, a_n, \tau)$ such that the characteristic roots of (1) have negative real part.

$$\left.\begin{aligned} P(s_0)+e^{-s_0\tau}Q(s_0)=0\,,\\ P'(s_0)+e^{-s_0\tau}\left((-\tau)Q(s_0)+Q'(s_0)\right)=0\,,\\ \vdots\\ P^{(k)}(s_0)+e^{-s_0\tau}\sum_{i=0}^{k}\binom{k}{i}Q^{(i)}(s_0)(-\tau)^{k-i}=0\,,\\ \vdots\\ P^{(n-1)}(s_0)+e^{-s_0\tau}\sum_{i=0}^{n-1}\binom{n-1}{i}Q^{(i)}(s_0)(-\tau)^{n-1-i}=0\,. \end{aligned}\right\} \tag{10}$$

Equation (10) gives a linear system of equations for the control parameters. Solving (10) for b_i enables the integral representation of the form (8).

If $D(s)$ has a real root s_0 with multiplicity at least $n+1$ then (8) holds and, in addition, $D^{(n)}(s_0)=0$:

$$\begin{aligned} D^{(n)}(s_0)&=n!\left(a_n+\int_0^1\frac{\tau R_{n-1}(s_0;\tau t)}{(n-1)!}\mathrm{d}t\right)\\ &=R_n(s_0;\tau)=0\,. \end{aligned} \tag{11}$$

Proposition 2. Let s_0 be a real root of $R_n(s_0;\tau)=0$. Assume that the parameters b_i satisfy the conditions (10). If $R_{n-1}(s_0;\tau t)\le 0$, $\forall t$, $0<t\le 1$ then the root s_0 with multiplicity $n+1$ is the dominant root of the characteristic function (8).

Proof. To prove that there exists no root $s_1=\gamma_1+\mathrm{i}\omega_1$ of (8) such that $\gamma_1>s_0$, substitute s_1 into (8). Since $a_n>0$ one can obtain that

$$\begin{aligned} a_n&=\left|\int_0^1 e^{-(s_1-s_0)\tau t}\frac{\tau}{(n-1)!}R_{n-1}(s_0;\tau t)\mathrm{d}t\right|\\ &\le\int_0^1 e^{-(\gamma_1-s_0)\tau t}\frac{\tau}{(n-1)!}\left|R_{n-1}(s_0;\tau t)\right|\mathrm{d}t\,. \end{aligned} \tag{12}$$

Using the condition $R_{n-1}(s_0;\tau t)\le 0$, $\forall t$, $0<t\le 1$ (12) can be written as

$$a_n\le-\int_0^1 e^{-(\gamma_1-s_0)\tau t}\frac{\tau}{(n-1)!}R_{n-1}(s_0;\tau t)\mathrm{d}t=:f(\gamma_1)\,. \tag{13}$$

For $\gamma_1=s_0$ the function f takes the value

$$\begin{aligned} f(s_0)&=-\int_0^1\frac{\tau}{(n-1)!}R_{n-1}(s_0;\tau t)\mathrm{d}t\\ &=-\int_0^1\frac{1}{n!}\frac{\mathrm{d}R_n(s_0;\tau t)}{\mathrm{d}t}\mathrm{d}t\\ &=-\frac{1}{n!}\left(\underbrace{R_n(s_0;\tau)}_{=0}-\underbrace{R_n(s_0;0)}_{=a_n n!}\right)=a_n\,. \end{aligned} \tag{14}$$

For $\gamma_1>s_0$ the value of the integral in (13) is $f(\gamma_1)<a_n$ since $0<e^{-(\gamma_1-s_0)\tau t}<1$ for $\gamma_1>s_0$, $\tau>0$, $0<t\le 1$. Therefore, from (13), we obtain that $a_n<a_n$ which proves the inconsistency of the hypothesis that the characteristic function (8) has a root $s_1=\gamma_1+\mathrm{i}\omega_1$ with $\gamma_1>s_0$.

5. OPEN-LOOP SYSTEMS WITH ONLY REAL ROOTS

In this section, we assume that the polynomial $P(s)$ corresponding to the open-loop system has only *real roots*. In this case, $P(s)$ has the form $P(s)=a_n\prod_{i=1}^{n}(s-s_i)$, $s_i\in\mathbb{R}$, $s_n\le s_{n-1}\le\ldots\le s_1$. To apply the sufficient condition in Proposition 2, first, we need to characterize the algebraic properties of the polynomials $R_k(s;\tau)$. These properties are outlined and discussed in the sequel.

5.1 Interlacing property of polynomials $R_k(s;\tau)$

The two-variable polynomials $R_k(s;\tau)$, $k\in\mathbb{Z}^+$ have the following properties:

$$R_k(s;\tau)=\tau R_{k-1}(s;\tau)+\frac{\partial R_{k-1}(s;\tau)}{\partial s}\,, \tag{15}$$

$$\frac{\partial R_k(s;\tau)}{\partial\tau}=kR_{k-1}(s;\tau)\,. \tag{16}$$

The property (15) allows saying that, for a fixed τ, the polynomials $R_k(s;\tau)$ and $R_{k-1}(s;\tau)$ interlace and $R_k(s;\tau)$ has only real roots for s since $R_0(s;\tau)=P(s)$ has only real roots (Fadeev and Sominski, 1965). Polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ have n distinct real roots for s if $\tau\neq 0$. Let $s_{0,k}$, $k=1,2,...,n$ denote the roots of $R_n(s;\tau)$, $\tau\neq 0$ with $s_{0,n}<s_{0,n-1}<\ldots<s_{0,1}$.

It can also be shown that, for a fixed s, $R_n(s;\tau)$ has only real roots for τ (Pólya and Szegő, 1997). Moreover, $R_k(s;\tau)$, $k=1,2,...,n-1$ has only real roots for τ, and $R_k(s;\tau)$ and $R_{k-1}(s;\tau)$ interlace which are direct consequences of the property (16) above and Rolle's theorem.

It should be mentioned that, in general, the solutions of (15) and (16) can be investigated by using computer algebra techniques (see, e.g., Chyzak et al. (2005)).

5.2 Monotonicity

In the (τ,s)-plane, the algebraic curve $R_n(s;\tau)=0$ has distinct branches, and every branch is strictly increasing since the derivative of the implicit function $R_n(s;\tau)=0$ in a point (s,τ) writes as (using (15) and (16)):

$$\frac{\mathrm{d}s}{\mathrm{d}\tau}=-\frac{\frac{\partial R_n(s;\tau)}{\partial\tau}}{\frac{\partial R_n(s;\tau)}{\partial s}}=-\frac{nR_{n-1}(s;\tau)}{R_{n+1}(s;\tau)}>0\,, \tag{17}$$

where the fraction $\frac{R_{n-1}(s;\tau)}{R_{n+1}(s;\tau)}$ is negative since for a fixed $\tau\neq 0$ at a root s of the polynomial $R_n(s;\tau)$ the function values $R_{n-1}(s;\tau)$ and $R_{n+1}(s;\tau)$ are nonzero and have different signs because of the interlacing property. A similar analysis can be done for the algebraic curve $R_{n-1}(s;\tau)=0$.

5.3 Asymptotic properties

If $\tau\to\infty$ (or $\tau\to-\infty$) then the roots of $R_k(s;\tau)$ for s approach the roots of $P(s)$ (i.e. $s_n\le s_{n-1}\le\ldots\le s_1$). Similarly, if $s\to\infty$ or $s\to-\infty$ then the roots of $R_k(s;\tau)$ for τ approach the roots of $\tau^k=0$ (i.e. 0 with multiplicity k).

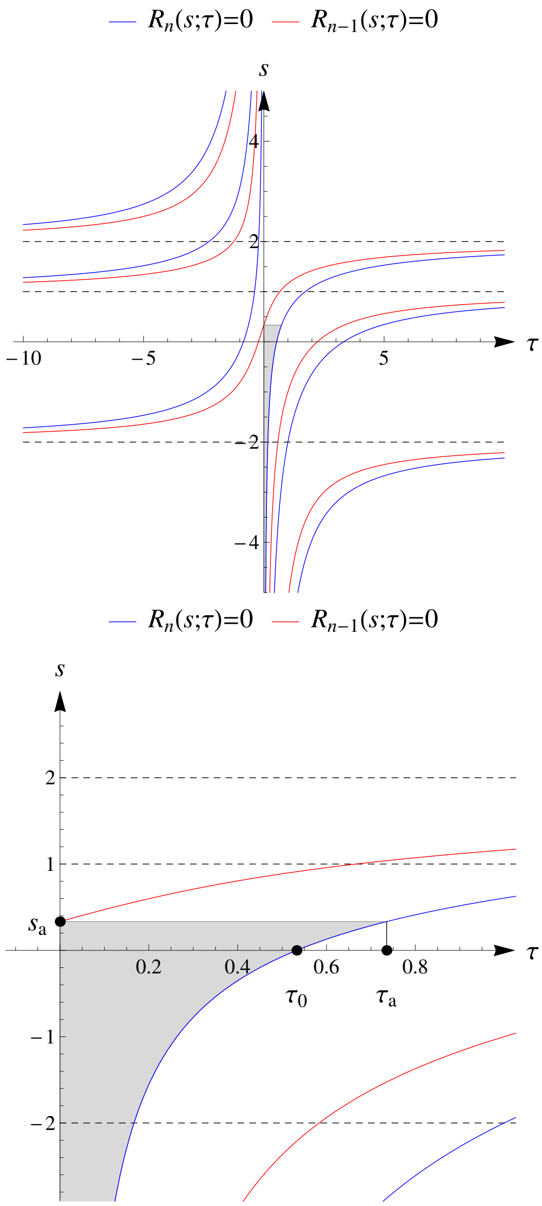

Fig. 1. Roots location of the polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ on the (τ, s) plane for the open-loop characteristic function $P(s) = (s-2)(s-1)(s+2)$. Top: Entire (τ, s) plane. Bottom: Right half-plane near the origin.

5.4 *Roots of* $R_k(s;0)$

If $\tau = 0$ then $R_k(s;\tau) = R_k(s;0) = P^{(k)}(s)$. Therefore, if $k = n$, then $R_n(s;0) = n!a_n$ has no roots for s. If $k = n-1$, then $R_{n-1}(s;0) = \frac{n!}{1!}a_n s + \frac{(n-1)!}{0!}a_{n-1}$ has one root for s:

$$s_{\mathrm{a}} = -\frac{1}{n}\frac{a_{n-1}}{a_n} = \frac{1}{n}\sum_{i=1}^{n} s_i\,, \tag{18}$$

which is the average of the roots of $P(s)$.

5.5 *Sufficient conditions for dominancy and stabilizability if* $P(s)$ *is real-rooted*

Fig. 1 shows the branches of the algebraic curves $R_n(s;\tau) = 0$ and $R_{n-1}(s;\tau) = 0$ corresponding to the interlacing and asymptotic properties on the (τ, s) plane (top), as well as the right half-plane in the neighborhood of the origin. The horizontal asymptotes corresponding to the roots of $P(s)$ are indicated with dashed lines.

Let τ_0 denote the smallest positive root of $R_n(0;\tau) = 0$ for τ. For $\tau > 0$ the first branch of the algebraic curve $R_n(s;\tau) = 0$ corresponds to the greatest s values, and it takes values in the interval $]-\infty, s_1[$. Therefore, if $s_1 > 0$, then τ_0 corresponds to the first branch of $R_n(s;\tau) = 0$. If $s_1 \le 0$,then $R_n(0;\tau) = 0$ has no positive roots. In this case, set $\tau_0 = \infty$.

Moreover, let τ_{a} denote the smallest positive root of $R_n(s_{\mathrm{a}};\tau) = 0$ for τ. If $P(s) \ne a_n(s-s_1)^n$ then τ_{a} corresponds to the first branch of $R_n(s;\tau) = 0$ since $s_n < s_{\mathrm{a}} < s_1$. If $P(s) = a_n(s-s_1)^n$, then $s_{\mathrm{a}} = s_1$ and $R_n(s_{\mathrm{a}};\tau) = 0$ has no roots. In this case, set $\tau_{\mathrm{a}} = \infty$.

The curve $R_n(s;\tau) = 0$ gives a connection between the delay τ and the possible values of the real root s_0 with multiplicity $n+1$, while the curve $R_{n-1}(s;\tau) = 0$ is needed to analyze the sufficient condition given in Proposition 2. It is clear that the condition $R_{n-1}(s_{0,k};\tau t) \le 0$, $\forall t$, $0 < t \le 1$ can be satisfied if and only if $k = 1$ and $0 < \tau \le \tau_{\mathrm{a}}$ (i.e. for the greatest s_0 and in a certain delay interval).

These observations are summarized as follows:

Proposition 3. Consider the case $s_{\mathrm{a}} \ge 0$. Then

1. $\tau_0 \le \tau_{\mathrm{a}}$
2. $s_{0,1}$ is the dominant root of system (8) if $0 < \tau \le \tau_{\mathrm{a}}$, and system (8) is stabilizable if $0 < \tau < \tau_0$.

Proof. If $s_{\mathrm{a}} > 0$ then there is at least one positive root s_1 of $P(s)$, therefore there is a finite τ_0 corresponding to the first branch of $R_n(s;\tau) = 0$. Then the inequality $\tau_0 \le \tau_{\mathrm{a}}$ follows from the monotonicity of the curve $R_n(s;\tau) = 0$. Furthermore, if $s_{\mathrm{a}} = 0$ then $\tau_0 = \tau_{\mathrm{a}}$. Next, the dominancy and stabilizability properties follow from the sufficient condition in Proposition 2 and the fact that $s_{0,1} \ge 0$ if $\tau_0 \le \tau \le \tau_{\mathrm{a}}$.

Proposition 4. Consider the case $s_{\mathrm{a}} < 0$. Then

1. $\tau_0 \ge \tau_{\mathrm{a}}$
2. $s_{0,1}$ is the dominant root of system (8) and system (8) is stabilizable if $0 < \tau \le \tau_{\mathrm{a}}$.

Proof. The proof follows the same lines as the proof of Proposition 3.

6. MULTI-DEGREE-OF-FREEDOM MECHANICAL EXAMPLE: 5-LINK INVERTED PENDULUM

Consider a 5-link inverted pendulum with rods of equal mass m and length l. The control torque is applied at the first (lowest) rod:

$$M = -\sum_{i=1}^{5} p_i\varphi_i(t-\tau) - \sum_{i=1}^{5} d_i{\varphi_i}'(t-\tau)\,. \tag{19}$$

6.1 Derivation of the equation of motion

The equation of motion can be derived by using the Euler–Lagrange equations. The generalized coordinates are chosen to be the pendulum angles φ_i (i.e. angular displacement of the rods from the vertically upward position). The equation of motion linearized around the unstable equilibrium has the form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}\,, \tag{20}$$

where the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ can be written as

$$\mathbf{M} = \frac{1}{6} l^2 m \begin{bmatrix} 26 & 21 & 15 & 9 & 3 \\ 21 & 20 & 15 & 9 & 3 \\ 15 & 15 & 14 & 9 & 3 \\ 9 & 9 & 9 & 8 & 3 \\ 3 & 3 & 3 & 3 & 2 \end{bmatrix}, \mathbf{K} = -\frac{1}{2} glm \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The generalized force $\mathbf{Q}$ reads

$$\mathbf{Q} = -\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{q}(t-\tau) - \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}}(t-\tau).$$

Therefore, the characteristic function has the form (1) where the open-loop characteristic function $P(s)$ reads

$$P(s) = \det\left(s^2\mathbf{M} + \mathbf{K}\right). \tag{21}$$

6.2 Stabilizable delay interval

Since the mass matrix $\mathbf{M}$ is positive-definite and the stiffness matrix $\mathbf{K}$ is negative-definite, then $P(s)$ has only real roots. The average of the roots is $s_{\mathrm{a}} = 0$ since the roots occur in real pairs $\pm s_i$. Therefore, we can apply the results of Proposition 3. More precisely, the system (20) is stabilizable if $0 < \tau < \tau_0$, where τ_0 can be calculated if the system parameters are known. For example, if the parameter $a_0 := \frac{3g}{l} = 1$ then $\tau_0 = 0.3816$. Some numerical simulations are shown in Fig. 2 for $\tau = 0.22$.

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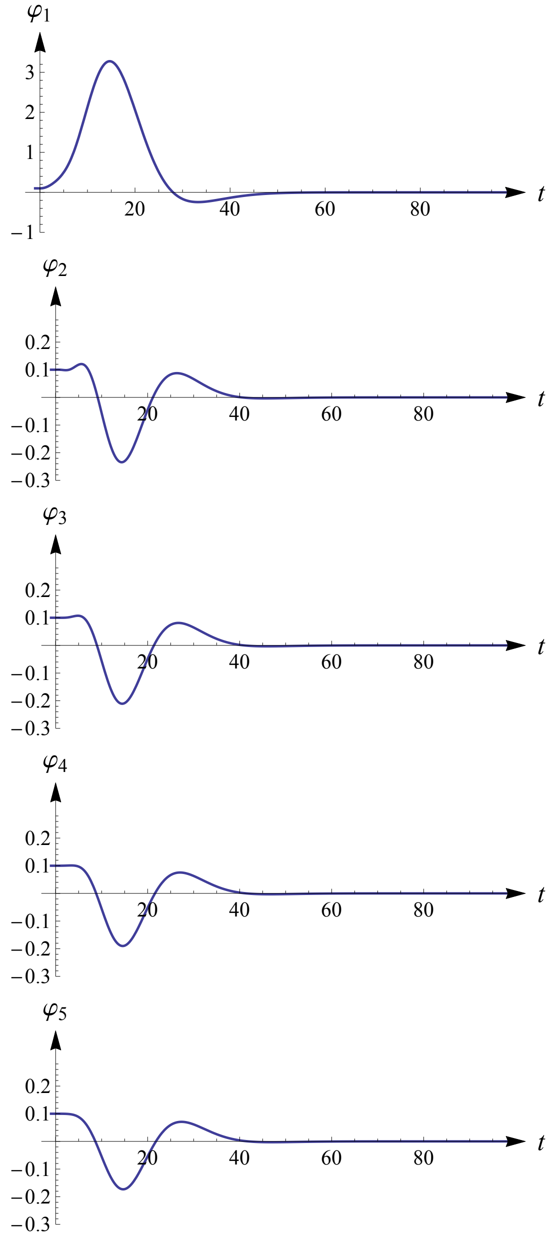

Fig. 2. Time-domain simulation for system (20) with $3g/l = 1$ and $\tau = 0.22 < \tau_0$.

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