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Scaled Group Consensus over Weakly Connected Structurally Balanced Graphs

Kenta Hanada * **Takayuki Wada** * **Izumi Masubuchi** ** **Toru Asai** *** **Yasumasa Fujisaki** *

* *Graduate School of Information Science and Technology, Osaka University, Suita, 565-0871 Japan (e-mail: {k-hanada, t-wada, fujisaki}@ist.osaka-u.ac.jp)*
** *Graduate School of System Informatics, Kobe University, Kobe, 657-8501 Japan (e-mail: msb@harbor.kobe-u.ac.jp)*
*** *Graduate School of Engineering, Nagoya University, Nagoya, 464-8601 Japan (e-mail: asai@nuem.nagoya-u.ac.jp)*

Abstract: A graph Laplacian based distributed protocol that can achieve a group consensus over weighted, signed, directed, and weakly connected graphs is investigated. It is said to achieve the group consensus if the state of agents who belong to the same group converges to a common value, while the one of agents who belong to another group converges to a different value. It is assumed that no agent knows which group she belongs to before the protocol is executed. In this paper, for a given signed graph which contains a directed spanning tree, namely, at least one leader that can affect all of the other agents, a definition of n-structurally balanced is proposed. It is emphasized that this definition is a generalization of the structurally balanced which leads a bipartite consensus. Then, necessary and sufficient conditions are established to guarantee the agents' state reaching the group consensus. The results are illustrated through numerical examples.

Keywords: multi-agent systems, scaled consensus, group consensus, structurally balanced.

1. INTRODUCTION

Consensus algorithms in multi-agent systems have been intensively developed and investigated for last several decades. The objective of the algorithms is to achieve the desired state for all agents in a distributed manner. Hence, the algorithms are expressed as fully local agent interactions and the agents form communication networks. There exist many theoretical convergence analyses such as averaging consensus (Fagnani and Frasca, 2018; Xiao et al., 2007), optimization problems (Nedic and Ozdaglar, 2009; Masubuchi et al., 2016), and modeling opinion dynamics (Friedkin, 2015).

In ordinary consensus algorithms, the state of the agents converges to the same value via attracting among agents, that is, the agents are completely cooperative. However, we have sometimes motivated ourselves to analyze the behavior such that some agents are cooperative, while the other agents are antagonistic or malicious, that is, antagonistic agents repel each other. In the notion of bipartite consensus, the agents divided into a couple of groups. The graph Laplacian based bipartite consensus is achieved autonomously if and only if a given signed graph is equivalently called balanced (Cartwright and Harary, 1956; Harary and Palmer, 1967), cycle balanced (Acharya, 1980), or structurally balanced (Altafini, 2013). Studies based on bipartite consensus have been attracted for a last decade (Altafini and Ceragioli, 2018). On the other hand, scaled consensus (Yu and Shi, 2018; Shang, 2017) can deal with multi partition of the agents. Although many existing studies assume that each agent knows which group she belongs to for all agents, the authors of the present paper revealed the condition of weights over strongly connected signed graph in order to achieve the group consensus even if no agent knows which group she belongs to a priori (Hanada et al., 2019).

In this paper we consider a graph Laplacian based distributed protocol that can achieve a group consensus over weighted, signed, directed, and weakly connected graphs. It is assumed that no agent knows which group she belongs to before the protocol is executed. We should point out that it is not necessary for follower agents to be the same group that a leader agent belongs to. Then, we define n-structurally balanced for signed graphs assuming that it contains a directed spanning tree, namely, at least one leader that can affect all of the other agents. Necessary and sufficient conditions are established to guarantee the agents' state reaching the group consensus. This result is a generalization of the existing studies in Altafini (2013) and Hanada et al. (2019). The results are illustrated through numerical examples.

2. PROBLEM STATEMENTS

Let us consider N agents having the same dynamics

$$x_i[k+1] = x_i[k] + u_i[k], \tag{1}$$

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where $x_i[k] \in \mathbb{R}$ is the state of agent i, $u_i[k] \in \mathbb{R}$ is the input of agent i, and $k \in \mathbb{N}$ is the discrete time. We introduce the following agent interaction

$$u_i[k] = r \sum_{j=1}^{N} a_{ij} \left(w_{ij} x_j[k] - x_i[k] \right), \tag{2}$$

where $r \in \mathbb{R}$ is a communication gain to be determined later, $a_{ij} \in \mathbb{R}$ is a non-negative weight between agent i and j, and $w_{ij} \in \mathbb{R}$ is a non-zero scaling factor between agent i and j if $(i,j) \in \mathcal{E}$ otherwise w_{ij} takes an arbitrary value. The parameter a_{ij} is strictly positive if $(j,i) \in \mathcal{E}$, it is equal to zero if $(j,i) \notin \mathcal{E}$. Furthermore, we suppose that $a_{ij} = a_{ji}$ if both a_{ij} and a_{ji} are strictly positive. We assume that the scaling factors $w_{i1}, w_{i2}, \ldots, w_{iN}$ are known to only agent i for all $i \in \mathcal{V}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is a set of agents.

Let $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}, A)$ be a weighted, signed, and directed graph (sigraph for short), where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of directed edges, and $A = [a_{ij} w_{ij}] \in \mathbb{R}^{N \times N}$ is an adjacency matrix corresponding to the edges. In this paper, we represent the adjacency matrix A as a compact form

$$A = A_0 \circ W, \tag{3}$$

where $A_0 = [a_{ij}] \in \mathbb{R}^{N \times N}$ and $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ are matrices respectively and $\circ$ is the Hadamard product which is elementwise multiplication of matrices. Note that only the matrix A_0 represents connectivity of the agents.

In this paper, we consider the following assumptions.

Assumption 1. All of the following conditions hold.

(1) The sigraph $\mathcal{G}(A)$ contains a directed spanning tree.
(2) We allow self-loop edges. That is, an edge $(i,i) \in \mathcal{E}$ may exists for some $i \in \mathcal{V}$.
(3) We allow bidirected edges. That is, the weight $a_{ij} w_{ij}$ corresponding to the edge (j,i) need not be the same of $a_{ji} w_{ji}$ corresponding to the edge (i,j) if there exist both (i,j) and (j,i).

Remark 1. Assumption 1-(1) ensures that the sigraph $\mathcal{G}(A)$ is weakly connected. From the fact that $a_{ij} = a_{ji}$ if both a_{ij} and a_{ji} are positive, we reword the assumption 1-(3) as follows: the scaling factor w_{ij} need not be the same of w_{ji} if there exist both (i,j) and (j,i).

In this paper, we consider several types of consensus. First, we introduce scaled one.

Definition 1. The system (1) with agent interactions (2) is said to achieve a *scaled* consensus if for any initial state $x[1] \in \mathbb{R}^N$,

$$\lim_{k \to \infty} |x_i[k] - w_{ij} x_j[k]| = 0$$

hold for any i and $j \in \mathcal{V}$ such that $i \neq j$.

We also define a trivial consensus as follows.

Definition 2. The system (1) with agent interactions (2) is said to achieve a *trivial* consensus if for any initial state $x[1] \in \mathbb{R}^N$,

$$\lim_{k \to \infty} x_i[k] = 0$$

hold for any $i \in \mathcal{V}$.

Remark 2. Although the scaled consensus is defined by

$$\lim_{k \to \infty} |c_i x_i[k] - c_j x_j[k]| = 0,$$

hold for any i and $j \in \mathcal{V}$ such that $i \neq j$, where c_i, $c_j \in \mathbb{R}$ are non-zero scalar in several existing studies (Roy, 2015; Hou et al., 2016; Yu and Shi, 2018), it is identical to our definition since we can regard w_{ij} as c_j / c_i.

Remark 3. The trivial consensus is obviously a special case of the scaled consensuses defined by Definition 1. If the system achieves the trivial consensus, the agents form exactly one group.

Next, we consider a partition of the agents. We denote $\mathcal{L} = \{1, 2, \ldots, n\}$ as a set of indices and $\mathcal{V}_\ell \subset \mathcal{V}$ ($\ell \in \mathcal{L}$) as a certain subset (group) of the agents, where $n \in \mathbb{N}$ is the number of groups. Note that we assume that no agent knows which group she belongs to. We now state the following consensus problem:

Definition 3. For a given n, the system (1) with agent interactions (2) is said to achieve an *n-group consensus* if for any initial state $x[1] \in \mathbb{R}^N$, there exist a partition $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n\}$ of a set $\mathcal{V}$ and α_i, $i = 1, 2, \ldots, n$ such that

$$\begin{aligned}
&\lim_{k \to \infty} |x_i[k] - \alpha_\ell| = 0, && \forall i \in \mathcal{V}_\ell,\ \forall \ell \in \mathcal{L}, \\
&\mathcal{V}_\ell \neq \emptyset, && \ell \in \mathcal{L}, \\
&\bigcup_{\ell \in \mathcal{L}} \mathcal{V}_\ell = \mathcal{V},\ \mathcal{V}_{\ell_1} \cap \mathcal{V}_{\ell_2} = \emptyset, && \ell_1 \neq \ell_2,\ \forall \ell_1, \ell_2 \in \mathcal{L}, \\
&\alpha_{\ell_1} \neq \alpha_{\ell_2}, && \ell_1 \neq \ell_2, \forall \ell_1, \ell_2 \in \mathcal{L},
\end{aligned}$$

hold, where $\alpha_\ell \in \mathbb{R}$ is a common consensus value for group $\ell \in \mathcal{L}$.

Remark 4. Definition 3 ensures that each agent can only belong to exactly one group. Suppose that agents can belong to two groups at the same time. Then, $x_i[k] = \alpha_{\ell_1} = \alpha_{\ell_2}$ holds and thus it contradicts Definition 3. That is, the sets $\mathcal{V}_\ell$, $\ell \in \mathcal{L}$ should be the partition of $\mathcal{V}$.

In order to represent (1) and (2) as a compact form, we now define the matrix L as

$$L = D - A,$$

where $D = [d_{ij}] \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that $d_{ij} = \sum_{j=1, j \neq i}^{N} a_{ij}$ if $i = j$ otherwise 0. We also define the diagonal matrix Γ whose diagonal element $\gamma_{ii} = a_{ii}(1 - w_{ii})$. By using the matrices L and Γ, the system (1) and agent interactions (2) can be rewritten as

$$x[k+1] = x[k] + u[k], \tag{4}$$
$$u[k] = -r\Gamma x[k] - rLx[k], \tag{5}$$

where

$$x[k] = [x_1[k]\ x_2[k]\ \cdots\ x_N[k]]^\top \in \mathbb{R}^N,$$
$$u[k] = [u_1[k]\ u_2[k]\ \cdots\ u_N[k]]^\top \in \mathbb{R}^N.$$

Substituting (5) into (4), we have

$$x[k+1] = \left(I_N - r \left(\Gamma + L \right) \right) x[k]. \tag{6}$$

The definition of n-group consensus can also be rewritten as follows.

Definition 4. The system (6) is said to achieve a n-group consensus if for any initial state $x[1] \in \mathbb{R}^N$, there exists a vector $\alpha = [\alpha'_1\ \alpha'_2\ \cdots\ \alpha'_N]^\top \in \mathbb{R}^N$ and the set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that the vector α' consists of n distinct values, $\alpha'_i \in \alpha$, $i = 1, 2, \ldots, N$, and

$$\lim_{k \to \infty} \|x[k] - \alpha'\| = 0$$

holds, where $\|\cdot\|$ is the Euclidean norm.

Remark 5. The definition of the matrix L is different from the graph Laplacian. In fact, the graph Laplacian L_0 is defined as $D - A_0$.

The aim of this paper is to establish the conditions of scaled, n-group, or trivial consensus for the multi-agent system (6) over weakly connected sigraphs even if no agent knows which group she belongs to.

3. CONVERGENCE ANALYSIS OF SCALED AND GROUP CONSENSUS

3.1 Generalization of Structurally Balanced Graphs

In order to investigate the condition of n-group consensus, we introduce the definition of n-structurally balanced graphs for sigraphs. First of all, we recall definitions and notations of paths and cycles in a graph. A directed path $\mathcal{P}_{ij}$ from agent i to j is a concatenation of directed edges of $\mathcal{E}$ as

$$\mathcal{P}_{ij} = \{(i, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p), (i_p, j)\} \subseteq \mathcal{E}$$

in which all edges $(i, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p), (i_p, j)$ are distinct. We denote $\mathcal{C}(A, i, j)$ is a set of all possible paths from agent i to j in the sigraph $\mathcal{G}(A)$. A cycle $\mathcal{P}_{ii} \in \mathcal{C}(A, i, i)$ is a path such that agent i is a beginning and ending one in the sigraph $\mathcal{G}(A)$.

Next, we consider a maximal subgraph $\mathcal{G}(A_m) = (\mathcal{S}, \mathcal{E}_m, A_m)$ of $\mathcal{G}(A)$ such that a root node of the directed spanning tree is in $\mathcal{S}$ and $\mathcal{G}(A_m)$ is strongly connected, where $\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{E}_m \subseteq \mathcal{E}$. The term maximal means that it is largest possible subgraph of $\mathcal{G}(A)$. Note that if there is no strongly connected component in $\mathcal{G}(A)$, the root node is the only element of $\mathcal{S}$ and $\mathcal{E}_m = \emptyset$.

Let $\beta_{ij} \in \mathbb{R}$ be a non-zero value corresponding to agents i and j. Here we define a generalized n-structural balance for the sigraph $\mathcal{G}(A)$.

Definition 5. For given matrices A_0 and W, a sigraph $\mathcal{G}(A)$ is said to be *n-structurally balanced*, where $A = A_0 \circ W$, if there exist scalars β_{ij}, $i \in \mathcal{S}$, $j \in \mathcal{V}$ and a partition $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n\}$ of a set $\mathcal{V}$ such that all of the following conditions hold:

(1) $\mathcal{V}_\ell \neq \emptyset, \forall \ell \in \mathcal{L}$.
(2) $\bigcup_{\ell \in \mathcal{L}} \mathcal{V}_\ell = \mathcal{V}$.
(3) $\mathcal{V}_{\ell_1} \cap \mathcal{V}_{\ell_2} = \emptyset \ (\ell_1 \neq \ell_2), \forall \ell_1, \ell_2 \in \mathcal{L}$.
(4) $\beta_{ij} = 1, \forall i, j \in \mathcal{V}_\ell \ (i \neq j), \forall \ell \in \mathcal{L}$.
(5) $\beta_{ij} = \beta_{ik}, \forall i \in \mathcal{S}, \forall j, k \in \mathcal{V}_\ell \ (i \neq j), \forall \ell \in \mathcal{L}$.
(6) $\beta_{ij} \neq \beta_{ik}, \forall i \in \mathcal{S}, \forall j \in \mathcal{V}_{\ell_1}, \forall k \in \mathcal{V}_{\ell_2} \ (\ell_1 \neq \ell_2) \ \forall \ell_1, \ell_2 \in \mathcal{L}$.
(7) $\beta_{ij} = \prod_{(v_2, v_1) \in \mathcal{P}_{ij}} w_{v_1 v_2}, \forall \mathcal{P}_{ij} \in \mathcal{C}(A, i, j), \forall i \in \mathcal{S}, \forall j \in \mathcal{V}$.

It is said to be *structurally unbalanced* if there does not exist n such that it is n-structurally balanced.

Remark 6. The conditions (1), (2), and (3) are exactly the same of the definition of the set partition. The condition (4) says that the scalar β_{ij} is a unit if agents i and j belong to the same group. The condition (5) claims that the scalar β_{ij} must be the same of β_{ik} if j and k belong to the same group, where departure agent i is in the strongly connected component $\mathcal{S}$. On the other hand, the condition (6) claims that β_{ij} is different from β_{ik} if agent j belongs to another group which agent k belong to, where departure agent i is in $\mathcal{S}$. The condition (7) defines a scalar β_{ij} as the product of scaling factors w_{ij} along paths from agent i to j. Note that β_{ij} must be the same value for any paths $\mathcal{P}_{ij}$.

The following lemma is the existing result (Hanada et al., 2019) for the strongly connected sigraph, that is, the maximal subgraph $\mathcal{G}(A_m)$ is identical to the original sigraph itself.

Lemma 1. (Hanada et al. (2019)). Suppose that a matrix A_0 and scaling factors w_{ij} such that $(j, i) \in \mathcal{E}$ are given, the maximal subgraph $\mathcal{G}(A_m)$ is identical to $\mathcal{G}(A)$, and $\mathcal{G}(A)$ is n-structurally balanced. Then, the following conditions are equivalent.

(1) The sigraph $\mathcal{G}(A)$ is n-structurally balanced.
(2) There exist scaling factors w_{ij} for $(j, i) \notin \mathcal{E}$ such that the bidirected complete sigraph $\mathcal{G}(W)$ is n-structurally balanced.
(3) There exist scaling factors w_{ij} for $(j, i) \notin \mathcal{E}$ such that W satisfies

$$\prod_{(v_2, v_1) \in \mathcal{P}_{ii}} w_{v_1 v_2} = 1, \quad \forall \mathcal{P}_{ii} \in \mathcal{C}(W, i, i), \ \forall i \in \mathcal{V}, \quad (7)$$

where the matrix W has exactly $n-1$ kinds of weights for any rows except zero and one.
(4) There exists a diagonal matrix $C \in \mathbb{R}^{N \times N}$ which consists of exactly n distinct non-zero values such that $CAC^{-1} = A_0$ holds.

We now discuss a uniqueness of the partition.

Lemma 2. For any given $i \in \mathcal{S}$ and $N - 1$ scalars β_{i1}, β_{i2}, $\ldots$, $\beta_{i(i-1)}$, $\beta_{i(i+1)}$, $\ldots$, β_{iN}, the partition of $\mathcal{V}$ is unique if there exist subsets $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, $\mathcal{V}_n$ such that all of β_{ij} and $\mathcal{V}_\ell$ satisfy the condition from (1) to (6) in Definition 5.

Proof. Suppose that there exist subsets $\mathcal{V}_\ell$, $\ell \in \mathcal{L}$, such that all of β_{ij} and $\mathcal{V}_\ell$ satisfy the condition from (1) to (6) in Definition 5. Let us consider that agent $j \in \mathcal{V} \setminus \{i\}$ belongs to $\mathcal{V}_{\ell_1}$ ($\ell_1 \in \mathcal{L}$). Similarly, agent $k \in \mathcal{V} \setminus \{i, j\}$ belongs to $\mathcal{V}_{\ell_2}$ such that $\ell_1 \neq \ell_2$ ($\ell_2 \in \mathcal{L}$).

First, we assume that $\mathcal{V}_{\ell_2} \setminus \{k\} = \emptyset$ holds. Then, we immediately see that it violates the condition (1). Thus, the number n of groups never decreases.

Next, we assume that $\mathcal{V}_{\ell_2} \setminus \{k\} \neq \emptyset$ and $\mathcal{V}_{\ell_1} \cup \{k\}$ hold. Note that this assumption does not violate the conditions (1) through (3). According to the condition (4), $\beta_{ij} = \beta_{ik} = 1$ must be satisfied for any $i \in \mathcal{S}$, which leads the contradiction against the fact that $\beta_{ij} \neq \beta_{ik}$. It ensures that agent k cannot move to another group for any k.

Lastly, we assume that $\mathcal{V}_{\ell_2} \setminus \{k\} \neq \emptyset$ holds and agent k forms an independent group $\mathcal{V}_{n+1} = \{k\}$. Regarding $\mathcal{L}$ as $\{1, 2, \ldots, n, n+1\}$, we see that the conditions (1) through (3) are satisfied. Since $\mathcal{V}_{\ell_2}$ is nonempty, there exists an agent $h \in \mathcal{V}_{\ell_2}$. According to the definition, $\beta_{ih} \neq \beta_{ik}$ must be satisfied for any $i \in \mathcal{S}$, which leads the contradiction against the fact that $\beta_{ih} = \beta_{ik}$. Thus, the number n of groups never increases.

We therefore see that Lemma 2 is derived. ■

Remark 7. The number of scalars β_{ij} in Lemma 2 is $N-1$. It is identical to the number of edges in a directed spanning tree.

Lemma 3. For any given matrices A_0 and W, a partition of $\mathcal{V}$ satisyfing Definition 5 is unique if $\mathcal{G}(A = A_0 \circ W)$ is n-structurally balanced.

Proof. If $\mathcal{G}(A)$ is n-structurally balanced, the scalars β_{ij} are well defined by the condition (7). Furthermore, there exist subsets $\mathcal{V}_\ell$ such that the conditions from (1) to (3) and all scalars β_{ij} saisfy the conditions from (4) to (6). Hence, the statement is true from Lemma 2. ■

Next, we discuss the property of subgraphs whose parent graph is n-structurally balanced. Let us define a subgraph $\mathcal{G}(A') = (\mathcal{V}, \mathcal{E}', A')$ of $\mathcal{G}(A)$ such that $\mathcal{G}(A')$ and $\mathcal{G}(A)$ have a common directed spanning tree, where $\mathcal{E}' \subset \mathcal{E}$, $A' = A'_0 \circ W' = [a'_{ij}w'_{ij}]$, a'_{ij} is a positive arbitrary value if $(j,i) \in \mathcal{E}'$ otherwise 0, $w'_{ij} = w_{ij}$ if $(j,i) \in \mathcal{E}'$ otherwise a non-zero arbitrary value. Conversely, we call the sigraph $\mathcal{G}(A)$ *parent* of $\mathcal{G}(A')$ if $\mathcal{E} \supset \mathcal{E}'$ holds and $w_{ij} = w'_{ij}$ when $(j,i) \in \mathcal{E}'$. Note that a_{ij} in the parent sigraph $\mathcal{G}(A)$ need not to be the same of a'_{ij}. We further introduce the maximal subgraph $\mathcal{G}(A'_m) = (\mathcal{S}', \mathcal{E}'_m, A'_m)$ of $\mathcal{G}(A')$. Note that no agent is removed from the original sigraph $\mathcal{G}(A)$ and $\mathcal{S}' \subseteq \mathcal{S}$ holds. Then, the following lemmas are derived.

Lemma 4. For any given A_0 and W, the subgraph $\mathcal{G}(A' = A'_0 \circ W)$ of $\mathcal{G}(A)$ is n-structurally balanced if $\mathcal{G}(A)$ is n-structurally balanced.

Proof. Suppose that $\mathcal{G}(A)$ is n-structurally balanced for given A_0 and W. Then, subsets $\mathcal{V}_\ell$, $\ell \in \mathcal{L}$, are all unique from Lemma 2. Let us define $\beta'_{ij} = \prod_{(v_2,v_1)\in\mathcal{P}_{ij}} w_{v_1v_2}$ for all $i \in \mathcal{S}'$ and $j \in \mathcal{V} \setminus \{i\}$. From the fact that $\mathcal{C}(A'_0, i, j) \subset \mathcal{C}(A_0, i, j) = \mathcal{C}(A, i, j)$, $\beta'_{ij} = \beta_{ij}$ holds for any $i \in \mathcal{S}'$ and $j \in \mathcal{V}$. That is, β'_{ij} are all well defined if $(i,j) \in \mathcal{E}'$. Selecting exactly the same set $\mathcal{V}_\ell$, $\ell \in \mathcal{L}$ for n-structurally balanced graph $\mathcal{G}(A)$, we see that Lemma 2 is always true for any $i \in \mathcal{S}' \subseteq \mathcal{S}$ and $N-1$ scalars $\beta'_{ij}(= \beta_{ij})$, $j \in \mathcal{V}\backslash\{i\}$. Since there exist scalars β'_{ij}, sets $\mathcal{V}_\ell$ ($\ell \in \mathcal{L}$), and a matrix W such that (3) holds and they satisfy all of the conditions in Definition 5, we conclude that the subgraph $\mathcal{G}(A' = A'_0 \circ W)$ is n-structurally balanced. ■

Lemma 5. For any given A'_0 and W', there exists a matrix W such that the parent sigraph $\mathcal{G}(A = A_0 \circ W)$ of $\mathcal{G}(A' = A'_0 \circ W')$ is n-structurally balanced if the sigraph $\mathcal{G}(A')$ is n-structurally balanced.

Proof. Suppose that the sigraph $\mathcal{G}(A')$ is n-structurally balanced for given A'_0 and W'. Let us define the matrix $W = [w_{ij}]$ such that $w_{ij} = w'_{ij}$ if $(j,i) \in \mathcal{E}'$, $w_{ij} = \beta_{ji}$ if $(j,i) \notin \mathcal{E}'$ and there exists a path from agent j to i in $\mathcal{G}(A')$, otherwise arbitrarily non-zero value. That is, if there is no path from agent j to i, we can design the scaling factor w_{ij}.

Let us consider the case that we add new edge $(j,k) \notin \mathcal{E}'$ to $\mathcal{G}(A)$. Since there is at least one path from agent $i \in \mathcal{S}$ to j, β_{ij} is already well defined. Similarly, β_{ik} is already well defined. Then, we design $w_{ki} = \beta_{ik}/\beta_{ij}$. Applying the same discussion of Theorem 1 ($1 \to 2$) in (Hanada et al., 2019), we see that the statement is derived. ■

Remark 8. Lemma 4 claims that we can immediately obtain n-structurally balanced subgraph if a given sigraph $\mathcal{G}(A_0 \circ W)$ is n-structurally balanced. On the other hand, Lemma 5 insists only the existence of the matrix W such that a parent graph become n-structurally balanced if a given subgraph $\mathcal{G}(A'_0 \circ W')$ is n-structurally balanced. Hence, it is NOT necessary and sufficient condition.

Lemma 6. For given matrices A_0 and W, n-structurally balancedness of the sigraph $\mathcal{G}(A)$ is supposed. Then, $\gamma_{ii} = 0$ for all $i \in \mathcal{V}$.

Proof. Applying the same discussion in the proof of Theorem 1 ($2 \to 3$) in (Hanada et al., 2019), $w_{ii} = 1$ must be satisfied if there exists a self-loop edge (i,i). On the other hand, $a_{ii} = 0$ if there is no self-loop edge (i,i). Thus, $\gamma_{ii} = a_{ii}(1 - w_{ii}) = 0$ holds for any i.

The following theorem is one of the main results of this paper.

Theorem 1. Suppose that sigraph $\mathcal{G}(A)$ is a weakly connected and has no self-loop edge and bidrected edge for given matricx A_0 and scaling factors w_{ij} such that $(j,i) \in \mathcal{E}$. Then, the following conditions are equivalent:

(1) The sigraph $\mathcal{G}(A)$ is n-structurally balanced.
(2) There exists a unique scaling factors w_{ij} for $(j,i) \notin \mathcal{E}$ such that the sigraph $\mathcal{G}((A_0 + A_0^\top) \circ W)$ is n-structurally balanced.
(3) There exists a diagonal matrix $C \in \mathbb{R}^{N\times N}$ which consists of exactly n distinct non-zero values such that $CAC^{-1} = A_0$ holds.

Proof. ($1 \to 2$) Suppose that the given sigraph $\mathcal{G}(A)$ is n-structurally balanced. By applying Lemma 1, we immediately obtain the statement since $\mathcal{G}((A_0 + A_0^\top) \circ W)$ is identical to undirected, that is, strongly connected.

($2 \to 3$) Suppose that there exists a unique matrix W such that the sigraph $\mathcal{G}((A_0 + A_0^\top) \circ W)$ is n-structurally balanced. Since the sigraph $\mathcal{G}((A_0 + A_0^\top) \circ W)$ is strongly connected and n-structurally balanced, there exists a matrix C such that $C((A_0 + A_0^\top) \circ W)C^{-1} = A_0 + A_0^\top$ holds from Lemma 1. Then, the (i,j)-th element of $C((A_0 + A_0^\top) \circ W)C^{-1}$ can be described as

$$a_{ij}w_{ij}c_i/c_j + a_{ji}w_{ij}c_i/c_j = a_{ij} + a_{ji}.$$

Since there is no bidirected edge in A, either $a_{ij} = 0$ or $a_{ji} = 0$ must be satisfied. If $a_{ij} = 0$, $a_{ji}w_{ij}c_i/c_j = a_{ji}$ holds for any i and j ($i \neq j$). On the other hand, if $a_{ji} = 0$, $a_{ij}w_{ij}c_i/c_j = a_{ij}$ holds for any i and j ($i \neq j$). Thus, we see that $CAC^{-1} = A_0$ holds.

($3 \to 1$) Suppose that there exists a diagonal matrix $C \in \mathbb{R}^{N\times N}$ which consists of exactly n distinct non-zero values such that $CAC^{-1} = A_0$ holds. Applying the same discussion of Theorem 1 ($4 \to 1$) in Hanada et al. (2019), we see that the statement is derived. ■

Remark 9. We should note that Lemma 1 which is the existing result can be applicable to only strongly connected graphs, while Theorem 1 can be applied to any weakly connected sigraphs including strongly connected ones.

3.2 Example of 4-structurally balanced graph

Let us consider the sigraph $\mathcal{G}(A^1) = (\mathcal{V}^1, \mathcal{E}^1, A^1)$, where $\mathcal{V}^1 = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{E}^1 = \{(1,2), (2,3), (2,4), (2,6), (3,1), (4,6), (6,5)\}$, and

$$A^1 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 9/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/4 & 3/2 \\ 0 & -1/2 & 1/4 & -1/3 & 0 & 0 \end{bmatrix}.$$

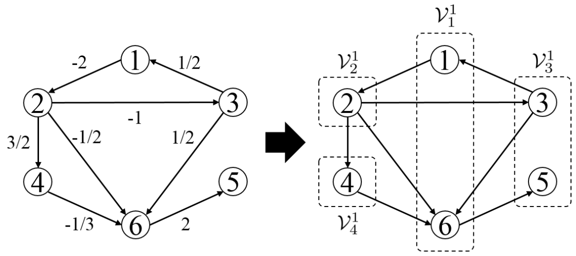

Fig. 1. Topology of example 1: indicated numbers are w_{ij}^1 (not $a_{ij}^1 w_{ij}^1$).

Fig. 1 depicts a topology of $\mathcal{G}(A^1)$. The maximal subgraph $\mathcal{G}(A_m^1)$ of $\mathcal{G}(A^1)$ can be expressed as $(\mathcal{S}^1, \mathcal{E}_m^1, A_m^1)$, where $S^1 = \{1, 2, 3\}$, $\mathcal{E}_m^1 = \{(1, 2), (2, 3), (3, 1)\}$, and $A_m^1 = [[0\ -4\ 0]^\top\ [0\ 0\ -1]^\top\ [2\ 0\ 0]^\top]$.

In this case, we consider 4-structurally balanced. Let us choose $A_0^1 = [a_{ij}^1]$ and $W^1 = [w_{ij}^1]$ such that (3) holds like

$$\begin{aligned} &a_{13}^1 = 4, && a_{21}^1 = 2, && a_{32}^1 = 1, && a_{42}^1 = 3, \\ &a_{56}^1 = 3/4, && a_{62}^1 = 1, && a_{63}^1 = 1/2, && a_{64}^1 = 4, \end{aligned}$$

$a_{ij}^1 = 0$ for all $(i, j) \notin \mathcal{E}^1$,

$$\begin{aligned} &w_{13}^1 = 1/2, && w_{21}^1 = -2, && w_{32}^1 = -1, && w_{42}^1 = 3/2, \\ &w_{56}^1 = 2, && w_{62}^1 = -1/2, && w_{63}^1 = 1/2, && w_{64}^1 = -1/3, \end{aligned}$$

and w_{ij}^1 is an arbitrary non-zero value for all $(i, j) \notin \mathcal{E}^1$. Note that indicated numbers in the left hand side of Fig. 1 are w_{ij}^1 corresponding to edge $(j, i) \in \mathcal{E}^1$. When we select $i = 1 \in \mathcal{S}^1$ for this example, we have $\beta_{12}, \beta_{13}, \ldots \beta_{16}$ as

$$\begin{aligned} &\beta_{12} = w_{21} = -2, \\ &\beta_{13} = w_{21} w_{32} = 2, \\ &\beta_{14} = w_{21} w_{42} = -3, \\ &\beta_{15} = w_{21}^1 w_{32}^1 w_{63}^1 w_{56}^1 = w_{21}^1 w_{42}^1 w_{64}^1 w_{56}^1 = w_{21}^1 w_{62}^1 w_{56}^1 = 2, \\ &\beta_{16} = w_{21} w_{42} w_{64} = w_{21} w_{62} = 1. \end{aligned}$$

These are consistent and satisfy the condition (7). Similarly, when we select $i = 2 \in \mathcal{S}^1$ and $i = 3 \in \mathcal{S}^1$, we have consistent values of $\beta_{21}, \beta_{23}, \ldots, \beta_{26}, \beta_{31}, \beta_{32}, \beta_{34}, \ldots, \beta_{36}$ satisfying the condition (7). Next, let us choose subsets $\mathcal{V}_1^1 = \{1, 6\}$, $\mathcal{V}_2^1 = \{2\}$, $\mathcal{V}_3^1 = \{3, 5\}$, and $\mathcal{V}_4^1 = \{4\}$. Then, we see that scalars β_{ij}, $i, j \in \mathcal{V}^1$, and these subsets satisfy all of the conditions in Definition 5. Hence, this example is actually 4-structurally balanced.

In the last part of this section, we check the statement of Theorem 1. Let us define a matrix C^1 as

$$C^1 = \mathrm{diag}([1\ -1/2\ 1/2\ -1/3\ 1/2\ 1]^\top),$$

where $\mathrm{diag}(a)$ denotes the diagonal matrix whose diagonal elements correspond to the column vector a. Then, we see that the $C^1 A^1 C^{-1} = A_0^1$ holds, that is, Theorem 1 is consistent in this example.

3.3 Convergence Analysis of Consensuses

The following lemma is important to analyze the convergence property of the multi-agent system (6).

Lemma 7. Suppose that the sigraph $\mathcal{G}(A)$ is n-structurally balanced. Then, the matrix L and the graph Laplacian L_0 are similar.

Proof. Employing a transformation to the matrix L by a diagonal matrix C such that $CAC^{-1} = A_0$, we have

$$CLC^{-1} = CDC^{-1} - CAC^{-1} = D - A_0 = L_0.$$

Thus we obtain Lemma 7. ■

Remark 10. Lemma 7 claims that the matrix L has the same eigenvalues of the graph Laplacian L_0. Since we assume that the sigraph $\mathcal{G}(A)$ has a directed spanning tree, the matrix L has eigenvalues $\lambda_i \in \mathbb{C}$ such that $\lambda_1 = 0$, $\lambda_i \neq 0$, $i = 2, 3, \ldots, N$.

We now employ a state coordinate transformation

$$\xi[k] = Cx[k], \qquad x[k] = C^{-1}\xi[k].$$

Applying the transformation to (6), we have

$$\xi[k+1] = \left(I_N - rC\Gamma C^{-1} - rCLC^{-1}\right)\xi[k]. \tag{8}$$

We introduce the weighted average $\bar{x}[k]$ of the states at time k and the deviation $\tilde{x}[k]$ from the average

$$\begin{aligned} \bar{x}[k] &= f^\top Cx[k] = \xi_2[k], \\ \tilde{x}[k] &= x[k] - \mathbf{1}_N \bar{x}[k] \\ &= C\left(I_N - C^{-1}\mathbf{1}_N f^\top C\right)x[k] = C\xi_1[k]. \end{aligned}$$

Selecting the communication gain $0 < r \leq 1/\bar{\sigma}$, where $\bar{\sigma} \in \mathbb{R}$ is the largest singular value of the graph Laplacian L_0, we achieve the second main result of this paper.

Theorem 2. Suppose that matrices A_0 and W are given and the sigraph $\mathcal{G}(A)$ is n-structurally balanced. Let us choose the communication gain r such that $0 < r \leq 1/\bar{\sigma}$ for any k. Then, the following statements hold.

(1) For any initial state $x[1]$, the system (6) achieves the scaled consensus. Then, the state $x[k]$ satisfies
$$\lim_{k\to\infty} x[k] = C^{-1}\mathbf{1}_N f^\top Cx[1]. \tag{9}$$
(2) The system (6) achieves a n-group consensus if $x[1] \in X = \{x[1] | f^\top Cx[1] \neq 0\}$. Then, the state $x[k]$ satisfies (9).
(3) The system (6) achieves a trivial consensus if $x[1] \notin X$ holds.

Proof. Suppose that $\mathcal{G}(A)$ is n-structurally balanced. Then, we have $\Gamma = 0$ from Lemma 6. As a result, the system (8) is a classical graph Laplacian based distributed protocol. Thus, following Fagnani and Frasca (2018), the vector $\xi[k]$ satisfies

$$\lim_{k\to\infty} \|\tilde{\xi}[k]\| = 0, \qquad \lim_{k\to\infty} \xi[k] = \mathbf{1}_N f^\top \xi[1].$$

The norm of the deviation $\tilde{x}[k]$ satisfies

$$\|\tilde{x}[k]\| = \|C^{-1}\tilde{\xi}[k]\| \leq \|C^{-1}\|\|\tilde{\xi}[k]\|.$$

We therefore see that

$$\lim_{k\to\infty} \|\tilde{x}[k]\| = 0, \qquad \lim_{k\to\infty} x[k] = C^{-1}\mathbf{1}_N f^\top Cx[1]$$

hold. Hence, we see that the system (6) achieves the scaled consensus.

If $f^\top Cx[1] = 0$ holds, we have

$$\lim_{k\to\infty} x[k] = C^{-1}\mathbf{1}_N f^\top Cx[1] = 0.$$

As a result, the system (6) achieves the trivial consensus. Thus, we see that the system (6) achieves the n-group consensus if $f^\top Cx[1] \neq 0$ is satisfied. ■

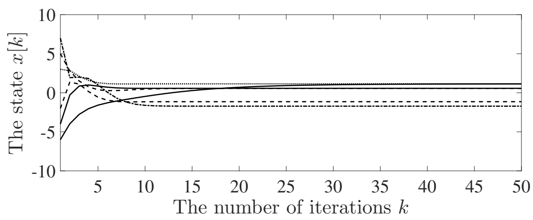

Fig. 2. The state trajectories of scaled and 4-group consensus

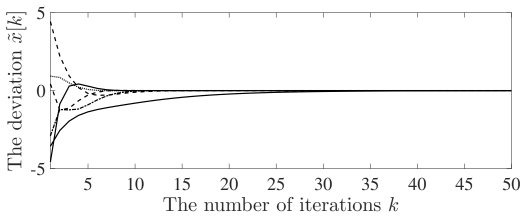

Fig. 3. The deviation of scaled and 4-group consensus

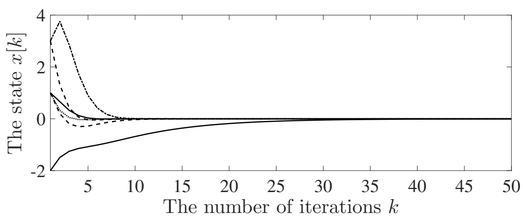

Fig. 4. The state trajectories of scaled and trivial consensus

4. NUMERICAL EXAMPLES

Now, we demonstrate the 4-group consensus with the example in Section 3.2 with $r = 1/\bar{\sigma} = 0.1688$.

4.1 Scaled and 4-group Consensus over 4-structurally Balanced Graph

The initial states were set as

$$x_1[1] = -4, \qquad x_2[1] = -2, \qquad x_3[1] = 3,$$
$$x_4[1] = 7, \qquad x_5[1] = -6, \qquad x_6[1] = 5.$$

Desired consensus points are

$$\lim_{k\to\infty} x_1[k] = 0.5714, \qquad \lim_{k\to\infty} x_2[k] = -1.1429,$$
$$\lim_{k\to\infty} x_3[k] = 1.1429, \qquad \lim_{k\to\infty} x_4[k] = -1.7143,$$
$$\lim_{k\to\infty} x_5[k] = 1.1429, \qquad \lim_{k\to\infty} x_5[k] = 0.5714.$$

Figs. 2 and 3 depict the state trajectory of $x[k]$ and the deviation $\tilde{x}[k]$, respectively. We see that the proposed algorithm achieved the scaled and 4-group consensus.

4.2 Trivial Consensus over 4-structurally Balanced Graph

The initial states were set as

$$x_1[1] = 1, \qquad x_2[1] = 3, \qquad x_3[1] = 1,$$
$$x_4[1] = 3, \qquad x_5[1] = -2, \qquad x_6[1] = 1.$$

Desired consensus points are

$$\lim_{k\to\infty} x_i[k] = 0, \qquad \forall i \in \mathcal{V}^1.$$

Fig. 4 depicts the state trajectory of $x[k]$. We see that the proposed algorithm achieved the trivial consensus even when the sigraph is 4-structurally balanced.

5. CONCLUDING REMARKS

We have considered a group consensus over weighted, signed, directed, and weakly connected graphs. We have proposed a definition of n-structurally balanced for signed graphs assuming that a directed spanning tree is contained. Then, necessary and sufficient conditions has been established to guarantee the agents' state reaching the group consensus.

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