Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On a Network SIS Model with Opinion Dynamics

Weihao Xuan, Ruijie Ren, Philip E. Paré
Mengbin Ye, Sebastian Ruf, Ji Liu

Abstract: This paper proposes a network continuous-time susceptible-infected-susceptible (SIS) model coupled with individual opinion dynamics, where the opinion dynamics models an individual's perceived severity of illness or perceived susceptibility. The effects of opinion dynamics on the network SIS model are studied by analyzing the limiting behaviors of the system model, equilibria of the system and their stability.

Keywords: SIS model, opinion dynamics, health belief model

1. INTRODUCTION

Epidemiological models have been studied for a long time for the purpose of understanding the spread of infectious diseases through large-population societies (Kermack and McKendrick, 1927). Notable among them are SIS and SIR (susceptible-infected-recovered) models. Originally these models only considered a well-mixed population, in which each agent had an equal probability of interaction with any other agent. The first extension to a network SIS model was proposed by Lajmanovich and Yorke (1976) for the study of gonorrhea in which the interactions among the individuals is described by a graph. Network models have lately attracted considerable attention in various fields including the control community (Fall et al., 2007; Mieghem et al., 2009; Khanafer et al., 2016) as the network SIS model and its variants can model the spread of many types of objects, such as malware in computer networks and attacks in cyber-physical systems (Eshghi et al., 2017). For recent development in analysis and control of network epidemics models, see surveys by Mei et al. (2017); Nowzari et al. (2016).

It is intuitive that individuals' perceived severity of the illness or perceived susceptibility affect the spread of infectious diseases. For instance, an increase in perceived susceptibility can lead to behavioral changes including how frequently individuals interact with others, how much medical treatment they receive, whether they want to travel or not, and so on. Recently, there has been interest in the literature on combining disease spread models with human awareness (Funk et al., 2009; Paarporn et al., 2017; Liu et al., 2017) and human behavior (Rizzo et al., 2014; Funk et al., 2010). Those models with human awareness capture transitions in the individuals' awareness of the disease, moving from completely unaware to alerted of the disease. However, these models lack an explicit dynamics to model how individuals' perceived severity of the disease forms and changes over time. Compared with the awareness models, an opinion dynamics model is able to capture a wider range of perceived susceptibilities. In our proposed model, the opinion of the severity of the illness can additively boost the healing rate and simultaneously reduce the susceptibility rate of an individual on a sliding scale. Additionally, opinion dynamics are particularly important in modern epidemiology, considering the rapid development of communication technology and fast-growing online social networking services, which have greatly changed the nature of the network of interactions. With these ideas in mind, we aim to develop a network model which couples epidemic spreading and opinion dynamics, and which allows us to understand the effects of the two dynamics on each other.

There are very few existing mathematical models combining epidemic spreading and opinion dynamics. The work of Ni et al. (2011) only considers a single well-mixed population, thus without any network. Our paper is motivated by recent work of Ruf et al. (2017, 2019) in which linear consensus-type opinion models are coupled with product adoption. Although the model proposed here shares some similarity with the model by Ruf et al. (2017, 2019), our model is driven by different social psychological phenomena, namely the health belief model, and possesses quite different limiting behavior.

The health belief model was proposed as a social psychological theory to explain and predict how health-related behaviors change (Glanz et al., 2008). It was developed by social psychologists from the U.S. Public Health Service in the 1950s and is still one of the best known and most widely used theories in health behavior research (Glanz et al., 2008; Rosenstock, 1974). The health belief model proposes that people's engagement, or lack thereof, in health-promoting behavior can be explained by their beliefs about health problems, perceived benefits of action, and perceived barriers to action. Therefore, their beliefs [1] in their perceived susceptibility and/or in their perceived

⋆ W. Xuan and R. Ren are with the School of Mechanical Engineering at University of Leeds, UK ({mn16whx, cn16rr}@leeds.ac.uk). P. E. Paré is with the Division of Decision and Control Systems at KTH Royal Institute of Technology, Stockholm, Sweden (philipar@kth.se). M. Ye is with the Faculty of Science and Engineering, ENTEG, University of Groningen, Netherlands (e-mail: m.ye@rug.nl). S. Ruf is with the Center for Complex Networks Research and the Department of Psychology at Northeastern University, USA (s.ruf@northeastern.edu). J. Liu is with the Department of Electrical and Computer Engineering at Stony Brook University, USA (ji.liu@stonybrook.edu).

[1] In this paper, beliefs and opinions are used interchangeably.

severity of the illness affects how susceptible they are and/or how effective they will be at healing from these diseases. Furthermore, there should be something that provokes the health-promoting behavior. In this paper, we capture this behavior by modeling the agent's belief of the severeness of the disease using opinion dynamics over an information network. We couple these opinion dynamics with SIS spread dynamics over a human contact network capturing both parts of the health belief model. To the best of our knowledge, this is the first mathematical model of the health belief model of this kind.

There exist both continuous- and discrete-time network SIS models, as well as both deterministic and probabilistic ones (Nowzari et al., 2016). It turns out that the most popular deterministic network model, first proposed by Lajmanovich and Yorke (1976), can be obtained by mean-field approximations of a counterpart probabilistic model (Mieghem et al., 2009). We thus adopt this most popular model and modify it to take opinion dynamics into account, as a first step along this new line of research. This paper validates the modification with a social psychological explanation and explains how individuals' opinions evolve over time and influence their curing and infection rates. It turns out that the proposed deterministic continuous-time network model, with a simple coupling with opinion dynamics, is already more complicated to analyze compared with the popular network SIS model. We thus leave other possible models, such as discrete-time and probabilistic ones which are probably even more challenging in analysis, as future directions.

The contributions of this paper are two-fold. First, we propose a novel network SIS model which couples opinion dynamics for the first time. Second, we analyze the model by characterizing its limiting behavior, equilibria, and their stability, which reveals the effects of opinion dynamics on the network SIS spread processes.

Notation: For any positive integer n, we use $[n]$ to denote the index set $\{1, 2, \ldots, n\}$. A nonnegative $n \times n$ matrix is called a stochastic matrix (or row-stochastic matrix) if its row sums are all equal to 1. We use $\mathbf{0}$ and $\mathbf{1}$ to denote the vectors whose entries all equal to 0 and 1, respectively, and I to denote the identity matrix, while the dimensions of the vectors and matrices are to be understood from the context. Sometimes we also use I_n and $0_{n\times n}$ to denote the identity and zero matrix of size $n \times n$, respectively. For a real square matrix M, we use $\rho(M)$ to denote its spectral radius. For any real number x, we use $|x|$ to denote the absolute value of x. For any two real vectors $a, b \in \mathbb{R}^n$, we write $a \geq b$ if $a_i \geq b_i$ for all $i \in [n]$, $a > b$ if $a \geq b$ and $a \neq b$, and $a \gg b$ if $a_i > b_i$ for all $i \in [n]$. For any two sets $\mathcal{A}$ and $\mathcal{B}$, we use $\mathcal{A}\backslash\mathcal{B}$ to denote the set of elements in $\mathcal{A}$ but not in $\mathcal{B}$. A real square matrix is called Metzler if its off-diagonal entries are all nonnegative (Berman and Plemmons, 1979). For a matrix A, $\sigma(A)$ is the set of eigenvalues of A and $s(A) := \max\{\Re(\lambda) : \lambda \in \sigma(A)\}$. We will use the terms "individual" and "agent" interchangeably.

2. THE MODEL

In this section, we propose a model for describing a network SIS model coupled with an opinion dynamics in a social network.

Consider a social network of $n > 1$ agents, labeled 1 through n. Each agent i can only learn, and be influenced by, the opinions of certain other agents called the neighbors of agent i. Neighbor relationships among the n agents are described by a directed graph $\mathbb{G}$, called the neighbor graph. Agent j is a neighbor of agent i whenever (j, i) is an arc in $\mathbb{G}$. Thus, the directions of arcs indicate the directions of information flow and infection. Each agent i has an opinion z_i that represents agent i's opinion about the severity of the epidemic virus spreading over the network. The opinion z_i is a real-valued quantity that evolves as a function of the opinions of its network neighbors and its probability of infection, with the precise dynamics being defined in the sequel.

We consider a virus modeled by an SIS process. Let $x_i \in [0, 1]$ denote the probability of agent i being infected by the virus with dynamics as follows:

$$\dot{x}_i(t) = -\delta_i x_i(t) + (1 - x_i(t)) \sum_{j\in\mathcal{N}_i} \beta_{ij} x_j(t),$$

where δ_i is the curing rate of agent i, $\mathcal{N}_i$ is the set of the neighbors of agent i, β_{ij} is the infection rate from agent j to agent i. The item $-\delta_i x_i(t)$ represents how each agent cures itself, and $(1 - x_i(t)) \sum_{j\in\mathcal{N}_i} \beta_{ij} x_j(t)$ represents how each agent can be infected by its neighbors. The above dynamics was first proposed by Lajmanovich and Yorke (1976). It has been shown by Mieghem et al. (2009) that the model is also the same as one derived from a mean-field approximation of a networked Markov chain SIS model.

An individual's opinion of how severe the virus is will affect their behavior. Without loss of generality, we assume that each z_i takes values between 0 and 1. A value of $z_i = 0$ means agent i believes the virus does not any threat, and $z_i = 1$ means agent i believes the virus poses a maximal threat. It is natural to assume that the larger z_i is, the less agent i will interact with its neighbors and the more likely it will seek treatment if it becomes infected. With this behavior in mind, we modify the traditional SIS dynamics presented above as follows:

$$\begin{aligned}\dot{x}_i(t) = &- \left[\delta_{\min} + (\delta_i - \delta_{\min}) z_i(t)\right] x_i(t) \\ &+ (1 - x_i(t)) \sum_{j\in\mathcal{N}_i} \left[\beta_{ij} - (\beta_{ij} - \beta_{\min}) z_i(t)\right] x_j(t),\end{aligned} \tag{1}$$

where $\delta_{\min}$ and $\beta_{\min}$ denote the minimum curing rate and infection rate, respectively. In the case when $z_i = 0$, which implies that agent i does not consider the virus a threat, it will take no action to protect itself and thus is maximally exposed to the infection. In the case when $z_i = 1$, which implies that agent i believes the virus is extremely serious, it will interact with others as little as possible and seek out all the medical treatment options possible. Therefore the model allows an agent's opinion to affect how susceptible they are and how effectively they heal from the virus, capturing the health belief model (Glanz et al., 2008; Rosenstock, 1974).

Now we model how each agent's opinion evolves. We adopt the canonical Abelson model, which in the 1960s helped lay the foundation for the study of opinion dynamics and which has also been studied in the controls community as the linear consensus protocol in continuous time. We propose the following modified Abelson dynamics:

$$\dot{z}_i(t) = (x_i(t) - z_i(t)) + \sum_{j\in\mathcal{N}_i}(z_j(t) - z_i(t)). \qquad (2)$$

If $x_i(t) - z_i(t) = 0$, then (2) simplifies to the Abelson model, which will lead all agents' opinions to a consensus. The term $(x_i(t)-z_i(t))$ captures how an agent's probability of infection impacts its opinion: the sicker an agent is the higher their perceived susceptibility. Consider the case where $\sum_{j\in\mathcal{N}_i}(z_j(t)-z_i(t)) = 0$. If z_i is small but the agent is infected, z_i will increase. If z_i is large but the agent is healthy, z_i will decrease. This behavior is sensible since an agent's sickness level should affect their belief of how severe the virus is, consistent with the health belief model (Glanz et al., 2008; Rosenstock, 1974). Meanwhile, agents share their opinions on the severity of the disease, captured in the second summand on the right of (2).

We impose the following natural restrictions on the parameters of the model throughout the paper.

Assumption 1. *For all $i \in [n]$, there hold $x_i(0), z_i(0) \in [0,1]$, $\delta_i \geq \delta_{\min} > 0$, and $\beta_{ij} \geq \beta_{\min} > 0$ for all $j \in \mathcal{N}_i$. The neighbor graph $\mathbb{G}$ is strongly connected.*

It is worth noting that a strongly connected $\mathbb{G}$ is equivalent to matrix $B = [\beta_{ij}]_{n\times n}$ being irreducible.

This paper aims to analyze the networked system consisting of dynamics (1) and (2) under Assumption 1.

3. MAIN RESULTS

We can write the dynamics for the network as follows:

$$\dot{s}(t) = \begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix}W & 0_{n\times n}\\ I_n & -(L+I_n)\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}, \qquad (3)$$

where

$$W = -\Big[\delta_{\min}I_n + (D-\delta_{\min}I_n)Z(t)\Big] + (I_n - X(t))\Big[B - Z(t)(B-\beta_{\min}A)\Big] \qquad (4)$$

and $Z = \mathrm{diag}(z_1,\ldots,z_n)$, $X = \mathrm{diag}(x_1,\ldots,x_n)$, $D = \mathrm{diag}(\delta_1,\ldots,\delta_n)$, $B = [\beta_{ij}]_{n\times n}$, A is the unweighted (entries either 0 or 1) adjacency matrix of neighbor graph $\mathbb{G}$, and L is the Laplacian matrix of $\mathbb{G}$. We remark that W is a Metzler matrix. It is also known that when $\mathbb{G}$ is strongly connected, L has a simple eigenvalue at zero, and the remaining eigenvalues all have negative real parts (Horn and Johnson, 2012). In fact, L is an irreducible singular M-matrix (Berman and Plemmons, 1979). Moreover $L+A$ is a nonsingular M-matrix for any nonnegative diagonal A having at least one positive diagonal entry (Qu, 2009, Theorem 4.31). This implies that $L+I$ has all eigenvalues with positive real part.

The following lemma shows that the system is well-posed.

Lemma 1. *If $x_i(0), z_i(0) \in [0,1]$ for all $i \in [n]$, then $x_i(t), z_i(t) \in [0,1]$ for all $i \in [n]$ and $t \geq 0$.*

Proof: Observe that the model is a system of polynomial ordinary differential equations (ODEs) over the compact space $[0,1]^{2n}$. This implies that the system of ODEs in the model is Lipschitz on $[0,1]^{2n}$ and as such the solutions $s_i(t)$ are continuous for all $i \in [2n]$. Suppose to the contrary that the lemma is not true. Then there is an index $i \in [2n]$ such that $s_i(t)$ is the first state to go outside $[0,1]$. Consider the case where $i \in [n]$, i.e. the $x_i(t)$ variable leaves $[0,1]$. If $x_i(t)$ becomes negative then there exists a time $t_0 \geq 0$ such that $x_i(t_0) = 0$, $\dot{x}(t_0) < 0$, $s_j(t) \in [0,1]\,\forall t \in [0,t_0]$ and $\forall j \neq i$. However, by Eq. (1),

$$\dot{x}_i(t_0) = \sum_{j\in\mathcal{N}_i}[\beta_{ij} - (\beta_{ij}-\beta_{\min})z_i(t)]\,x_j(t_0) \geq 0$$

giving a contradiction. To show $x_i(t)$ cannot exceed one, we apply similar arguments and observe that

$$\dot{x}_i(t_0) = -[\delta_{\min} + (\delta_i - \delta_{\min})z_i(t_0)] \leq 0.$$

This equality would contradict $\dot{x}_i(t_0) > 0$, which is required for $x_i(t)$ to exceed one.

Consider the case where $i \in \{n+1,\cdots,2n\}$, i.e. the $z_i(t)$ variable leaves $[0,1]$. If $z_i(t)$ becomes negative then there exists a time $t_0 \geq 0$ such that $z_i(t_0) = 0$, $\dot{z}(t_0) < 0$, $s_j(t) \in [0,1]\,\forall t \in [0,t_0]$ and $\forall j \neq i$. However, by Eq. (2),

$$\dot{z}_i(t_0) = x_i(t_0) + \sum_{j\in\mathcal{N}_i} z_j(t_0) \geq 0$$

giving a contradiction. To show $z_i(t)$ cannot exceed one, we apply similar arguments and observe that

$$\dot{z}_i(t_0) = (x_i(t_0) - 1) + \sum_{j\in\mathcal{N}_i}(z_j(t_0) - 1) \leq 0.$$

This equality would contradict $\dot{z}_i(t_0) > 0$, which is required for $z_i(t)$ to exceed one. ■

It is easy to see that $x = z = \mathbf{0}$ is an equilibrium of the system given by (3). Note that $x = \mathbf{0}$ corresponds to the case when no individual is infected, which implies that $z = \mathbf{0}$ is the only equilibrium of (2) since $-(L+I_n)$ is an nonsingular matrix. We thus call this trivial equilibrium the healthy state.

In the sequel, we define $D_{\min} = \delta_{\min}I_n$ for simplicity. Note that $D_{\min}^{-1}B = \delta_{\min}^{-1}B$.

Proposition 1. *If $\rho(D_{\min}^{-1}B) \leq 1$, then the healthy state is the unique equilibrium of the system.*

To prove the proposition, we need the following lemmas. First recall the following property for Metzler matrices from (Berman and Plemmons, 1979).

Lemma 2. *Suppose that M is an irreducible Metzler matrix. Then, $s(M)$ is a simple eigenvalue of M and there exists a unique (up to scalar multiple) vector $x \gg \mathbf{0}$ such that $Mx = s(M)x$. Let $z > \mathbf{0}$ be a vector in $\mathbb{R}^n$. If $Mz < \lambda z$, then $s(M) < \lambda$. If $Mz = \lambda z$, then $s(M) = \lambda$. If $Mz > \lambda z$, then $s(M) > \lambda$.*

Lemma 3. [Proposition 1, Liu et al. (2019)] *Suppose that Λ is a negative diagonal matrix in $\mathbb{R}^{n\times n}$ and N is an irreducible nonnegative matrix in $\mathbb{R}^{n\times n}$. Let $M = \Lambda + N$. Then, $s(M) < 0$ if and only if $\rho(-\Lambda^{-1}N) < 1$, $s(M) = 0$ if and only if $\rho(-\Lambda^{-1}N) = 0$, and $s(M) > 0$ if and only if $\rho(-\Lambda^{-1}N) > 1$.*

For any two nonnegative vectors a and b in $\mathbb{R}^n$, we say that a and b have the same sign pattern if they have zero entries and positive entries in the same places, i.e., $a_i = 0$ if and only if $b_i = 0$, and $a_i > 0$ if and only if $b_i > 0$ for all $i \in [n]$.

Lemma 4. [Lemma 1, Liu et al. (2019)] *Suppose that $Mx = y$ where $M \in \mathbb{R}^{n\times n}$ is an irreducible nonnegative* *matrix and $x, y > \mathbf{0}$ are in $\mathbb{R}^n$. If x has at least one zero entry, then x and y cannot have the same sign pattern. In particular, there exists an index $i \in [n]$ such that $x_i = 0$ and $y_i > 0$.*

Lemma 5. *If $s \geq \mathbf{0}$ is an equilibrium of the system, then*

$$\begin{aligned} x = [&I + Z(D_{\min}^{-1}D - I) \\ &+ \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A))x)]^{-1} \\ &\times [(I - Z)D_{\min}^{-1}B + ZD_{\min}^{-1}\beta_{\min}A]x \end{aligned}$$

where $Z = \mathrm{diag}((L+I)^{-1}x)$.

Proof: From Eq. (2), the equilibrium z satisfies $x = (L + I)z$. Recalling that $(L + I)$ is nonsingular (see below (4)), then

$$z(t) = (L + I)^{-1}x(t). \tag{5}$$

From Eq. (1), the equilibrium x satisfies

$$\begin{aligned} Bx(t) = &Z(t)\,(B - \beta_{\min}A)\,x(t) + D_{\min}x(t) \\ &+ Z(t)(D - D_{\min})x(t) \\ &+ X(t)(B - Z(t)(B - \beta_{\min}A))x(t). \end{aligned}$$

From Assumption 1, $D_{\min}$ is a positive diagonal matrix, thus, $D_{\min}$ is nonsingular and $D_{\min}^{-1}$ is also a positive nonsingular matrix. It follows that,

$$\begin{aligned} D_{\min}^{-1}Bx &= D_{\min}^{-1}Z\,(B - \beta_{\min}A)\,x + x + D_{\min}^{-1}Z(D - D_{\min})x \\ &\quad + D_{\min}^{-1}X(B - Z(B - \beta_{\min}A))x \\ &= ZD_{\min}^{-1}\,(B - \beta_{\min}A)\,x + x + ZD_{\min}^{-1}(D - D_{\min})x \\ &\quad + XD_{\min}^{-1}(B - Z(B - \beta_{\min}A))x \end{aligned}$$

then

$$\begin{aligned} &(I - Z)D_{\min}^{-1}Bx + ZD_{\min}^{-1}\beta_{\min}Ax \\ &= x + Z(D_{\min}^{-1}D - I)x + XD_{\min}^{-1}(B - Z(B - \beta_{\min}A))x \\ &= (I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A))x))x. \end{aligned}$$

Since $s \geq 0$, and $D_{\min}^{-1}D - I$, $B - Z(B - \beta_{\min}A)$, $D_{\min}^{-1}$, D are nonnegative matrix, it follows that $I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A))x)$ is a positive diagonal matrix which is nonsingular. Therefore,

$$\begin{aligned} x = &(I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A)) \\ &x))^{-1}((I - Z)D_{\min}^{-1}B + ZD_{\min}^{-1}\beta_{\min}A)x, \end{aligned}$$

which completes the proof. ∎

Lemma 6. *If s is a nonzero equilibrium of the system, then $\mathbf{0} \ll s \ll \mathbf{1}$.*

Proof: Suppose that s is nonzero equilibrium of the system. Then, by Lemma 1, it must be true that $s \geq \mathbf{0}$. By Lemma 3,

$$\begin{aligned} x = &(I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A)) \\ &x))^{-1}((I - Z)D_{\min}^{-1}B + ZD_{\min}^{-1}\beta_{\min}A)x \end{aligned}$$

From Assumption 1, it is straightforward to verify that $(I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A))x))^{-1}((I - Z)D_{\min}^{-1}B + ZD_{\min}^{-1}\beta_{\min}A)$ is an irreducible nonnegative matrix. If x has at least one zero entry, then x and $(I + Z(D_{\min}^{-1}D - I) + \mathrm{diag}(D_{\min}^{-1}(B - Z(B - \beta_{\min}A))x))^{-1}((I - Z)D_{\min}^{-1}B + ZD_{\min}^{-1}\beta_{\min}A)x$ must have different zero sign pattern by Lemma 4, which is impossible. Thus, $x \gg \mathbf{0}$.

From Eq (5), we get $z = (-(L + I))^{-1}(-x)$. since $-(L + I)$ is a nonsingular irreducible Hurwitz Metzler matrix, $(-(L + I))^{-1}$ is a strictly negative matrix. Since $x \gg \mathbf{0}$, $z \gg \mathbf{0}$. Thus $s \gg \mathbf{0}$.

Next we show that $x \ll \mathbf{1}$. Since s is an equilibrium of the system, it follows from (1) that, for all $i \in [n]$, one has

$$x_i = \frac{\sum_{j \in \mathcal{N}_i} (\beta_{ij} - (\beta_{ij} - \beta_{\min})\,z_i)\,x_j}{(\delta_{\min} + (\delta_i - \delta_{\min})\,z_i) + \sum_{j \in \mathcal{N}_i} (\beta_{ij} - (\beta_{ij} - \beta_{\min})\,z_i)\,x_j}.$$

The fact that $\delta_i > 0$, $\beta_{ij} > 0$ for $j \in \mathcal{N}_i$ by Assumption 1, and $0 < z_i \leq 1$ from Lemma 1, one obtains that $(\delta_{\min} + (\delta_i - \delta_{\min})\,z_i) > 0$ and also that $\sum_{j \in \mathcal{N}_i} (\beta_{ij} - (\beta_{ij} - \beta_{\min})\,z_i)\,x_j > 0$. It follows that $x_i < 1$ for all $i \in [n]$.

By Eq. (2), we have $(N_i + 1)z_i = x_i + N_i z_j$, where N_i is the number of neighbors of agent i ($N_i = |\mathcal{N}_i|$). Thus $z_i = \frac{x_i + N_i z_j}{1 + N_i}$. Since $0 < x_i < 1$ and $0 < N_i z_j \leq N_i$, it follows that $z_i < 1$ for all $i \in [n]$. Thus $s \ll \mathbf{1}$. ∎

We are now in a position to prove Proposition 1.

Proof of Proposition 1: Suppose to the contrary, that there is a nonzero equilibrium s of the system. By Lemma 6, it must be true that $s \gg \mathbf{0}$. Since x is an equilibrium, it follows that

$$\begin{aligned} (-D_{\min}^{-1} + B)x = &Z(D - D_{\min})x + XBx \\ &+ (I - X)(Z(B - \beta_{\min}A))x. \end{aligned}$$

Since by Assumption 1, B is nonnegative and irreducible, by Lemma 1, $s_i \in [0, 1]$ it follows $Bx \gg \mathbf{0}$, so $XBx \gg \mathbf{0}$. Since $D - D_{\min}$ is nonegative, one has $(D - D_{\min})x \geq \mathbf{0}$ and $(B - \beta_{\min}A)x \geq \mathbf{0}$. This yields $Z(D - D_{\min})x \geq \mathbf{0}$ and $(I - X)(Z(B - \beta_{\min}A))x \geq \mathbf{0}$, which implies that $Z(D - D_{\min})x + XBx + (I - X)(Z(B - \beta_{\min}A))x \gg \mathbf{0}$. Thus, $(-D_{\min}^{-1} + B)x \gg \mathbf{0}$.

Since $(-D_{\min}^{-1} + B)$ is an irreducible Metzler matrix, by Lemma 2, $s(-D_{\min}^{-1} + B) > 0$. Thus, $\rho(D_{\min}^{-1}B) > 1$ by Lemma 3, which contradicts the assumption of the lemma that $\rho(D_{\min}^{-1}B) \leq 1$. Therefore, $x \gg \mathbf{0}$ cannot be an equilibrium of the system if $\rho(D_{\min}^{-1}B) \leq 1$. Since $z = (L + I)^{-1}x$, and $(L + I)^{-1}$ exists, $s = \mathbf{0}$ is the unique equilibrium of the system if $\rho(D_{\min}^{-1}B) \leq 1$. ∎

We next characterize the stability of the healthy state.

Note that the Jacobian of system (3) evaluated at (x, z) is

$$df_{x,z} = \begin{bmatrix} W - \tilde{V}(x, z) & -(D - \delta_{\min}I_n)X - (I_n - X)\tilde{B} \\ I_n & -(L + I_n) \end{bmatrix},$$

where $\tilde{V}(x, z)$ and $\tilde{B}$ are diagonal matrices with the ith diagonal entry being the ith entry of the vectors $(B - (B - \beta_{\min}A)Z)x$ and $(B - \beta_{\min}A)x$, respectively. It can be verified that the Jacobian evaluated at $s = \mathbf{0}$ is

$$df_{\mathbf{0},\mathbf{0}} = \begin{bmatrix} -D_{\min} + B & 0_{n \times n} \\ I_n & -(L + I_n) \end{bmatrix}.$$

Since $-(L + I_n)$ is a stable matrix when $\mathbb{G}$ is strongly connected (see proof of Theorem 1), it is clear that the stability of the healthy state depends on the matrix $-D_{\min} + B$. From Lemma 3, $s(-D_{\min} + B) < 0$ if and only if $\rho(D_{\min}^{-1}B) < 1$, and $s(-D_{\min} + B) > 0$ if and only if $\rho(D_{\min}^{-1}B) > 1$. Thus, we are led to the following results.

Proposition 2. *If $\rho(D_{\min}^{-1}B) < 1$, the healthy state is locally exponentially stable. If $\rho(D_{\min}^{-1}B) > 1$, the healthy state is unstable.*

Corollary 1. *If $\delta_{\min} > \sum_{j\in\mathcal{N}_i}\beta_{ij}$ for all $i \in [n]$, then $\mathbf{0}$ is asymptotically stable.*

Proof: Since $D_{\min}^{-1}B = \delta_{\min}^{-1}B$ is a nonnegative irreducible matrix, (Varga, 2009, Lemma 2.8) yields

$$\rho(\delta_{\min}^{-1}B) < \max_{i\in[n]} \sum_{j=1}^{n} \delta_{\min}^{-1}\beta_{ij}. \tag{6}$$

The inequality $\delta_{\min} > \sum_{j\in\mathcal{N}_i}\beta_{ij}$ for all $i \in [n]$ implies that $\delta_{\min}^{-1}\sum_{j\in\mathcal{N}_i}\beta_{ij} < 1$ for all $i \in [n]$, which combined with Eq. (6) delivers that if $\delta_{\min} > \sum_{j\in\mathcal{N}_i}\beta_{ij}$ for all $i \in [n]$, then $\rho(\delta_{\min}^{-1}B) < 1$. ■

The following theorem shows that in the case when $\rho(D_{\min}^{-1}B) \le 1$, the healthy state is globally stable.

Theorem 1. *If $\rho(D_{\min}^{-1}B) \le 1$, then the healthy state is asymptotically stable for all initial conditions. If $\rho(D_{\min}^{-1}B) < 1$, then it is exponentially stable.*

Proof: First note that

$$\begin{aligned}\dot{x}(t) &= -[D_{\min} + (D - D_{\min})Z(t)]x(t)\\ &\quad + (I - X(t)[B - Z(t)(B - \beta_{\min}A)]x(t)\\ &= -D_{\min}x(t) + (I - X(t))Bx(t) - (D - D_{\min})Z(t)x(t)\\ &\quad - (I - X(t))Z(t)(B - \beta_{\min}A)x(t)\\ &\le -D_{\min}x(t) + (I - X(t))Bx(t). \end{aligned}\tag{7}$$

The inequality implies further that

$$\dot{x} \le \dot{y} = -D_{\min}y(t) + By(t)$$

because $I_n - X(t)$ is a nonnegative diagonal matrix with entries between zero and one. Now, from Lemma 3, $\rho(\delta_{\min}^{-1}B) \le 1$ implies $s(-D_{\min} + B) \le 0$, i.e. $-D_{\min} + B$ is Hurwitz. Initialise $y(0) = x(0)$, and from the fact that $\dot{y} = -\delta_{\min}y(t) + By(t)$ converges to $y = \mathbf{0}$ exponentially, we conclude that $\dot{x} = \mathbf{0}$. Since $x = \mathbf{0}$ is the unique equilibrium of Eq. (3), we conclude that $\lim_{t\to\infty} x(t) = \mathbf{0}$ exponentially fast. If $s(-D_{\min} + B) = 0$, since $(-D_{\min} + B)$ is an irreducible Metzler matrix, by Lemma A.1 in Khanafer et al. (2016), there exists a positive diagonal matrix P such that $(-D_{\min} + B)^\top P + P(-D_{\min} + B)$ is negative definite. Consider the Lyapunov function $V(x(t)) = x(t)^\top P x(t)$. From Eq. (1) and (7), when $x(t) \neq 0$, we have

$$\begin{aligned}\dot{V}(x(t)) &= 2x(t)^\top P\dot{x}(t)\\ &\le 2x(t)^\top P(-D_{\min} + B - X(t)B)x(t).\end{aligned}$$

Using the proof of Proposition 2 in Liu et al. (2019), one can show that $x(t) = \mathbf{0}$ is asymptotically stable with the domain of attraction $[0, 1]^n$.

The convergence of $\dot{z}(t) = -(L + I)z(t)$ to $\mathbf{0}$ exponentially fast follows immediately from the fact that $-(L + I)$ is Hurwitz. We are interested in the convergence of the system

$$\dot{z}(t) = -(L + I)z(t) + x(t). \tag{8}$$

Now, Eq. (8) is input-to-state stable because $\dot{z}(t) = -(L + I)z(t)$ has a globally exponentially stable equilibrium at $z = \mathbf{0}$ (Khalil, 2002, Lemma 4.6). Thus, with input $x(t)$ vanishing to zero asymptotically (or exponentially fast), $z = \mathbf{0}$ is asymptotically (or exponentially) stable for (8) with domain of attraction $[0, 1]^n$. ■

Compared with many existing mathematical epidemiology models, the value of $\rho(D_{\min}^{-1}B)$ can be regarded as an effective reproduction number $\mathcal{R}_0$, the precise definition being dependent on the particular model, in the sense that the disease is eradicated if $\mathcal{R}_0 \le 1$ and persists if $\mathcal{R}_0 > 1$. Extensive simulations show that this is also the case here: if $\rho(D_{\min}^{-1}B) > 1$, then the system (3) will asymptotically converge to a unique nonzero state. We call a nonzero equilibrium of the system, if it exists, an endemic equilibrium. From Lemma 6, the limiting endemic state must be strictly positive. However, complete analysis of its uniqueness and stability, both local and global, has so far eluded us.

We now present some analysis to identify a condition under which at least one nonzero equilibrium exists. To do so, we first need some additional preliminaries. Since $-D + \beta_{\min}A$ is an irreducible Metzler matrix, $\phi \triangleq s(-D + \beta_{\min}A)$ is an eigenvalue of $-D + \beta_{\min}A$ with an associated right eigenvector $\boldsymbol{y} \gg \mathbf{0}$. Without loss of generality, assume $\max_i y_i = 1$. Now, define for any $\epsilon \in [0, 1)$ a convex and compact subset of $[0, 1]^{2n}$ as

$$\Xi_\epsilon \triangleq \{s \in [0, 1]^{2n} : x_i \ge \epsilon y_i \, \forall\, i \in [n]\}. \tag{9}$$

Note that $\Xi_0 = [0, 1]^{2n}$ and for any $\epsilon > 0$, $\Xi_\epsilon \subset [0, 1]^{2n}$.

To proceed, we let $\partial[0, 1]^{2n}$ and $\mathrm{Int}([0, 1]^{2n})$ denote the boundary and the interior of the cube $[0, 1]^{2n}$, respectively.

Lemma 7. *Consider the system (3). If $s(t) \in \partial[0, 1]^{2n} \setminus \mathbf{0}$, then $s(t + \tau) \in \mathrm{Int}([0, 1]^{2n})$ for some finite τ.*

The proof of Lemma 7 is straightforward and thus omitted due to space limitations.

Theorem 2. *Suppose that $s(-D + \beta_{\min}A) > 0$. Then, there exists a sufficiently small $\bar{\epsilon}$ such that Ξ_ϵ defined in Eq. (9) for every $\epsilon \in (0, \bar{\epsilon}]$ is a positive invariant set for the system in Eq. (3). Moreover, Eq. (3) has at least one nonzero equilibrium in $\mathrm{Int}([0, 1]^{2n})$.*

Proof: Given the result of Lemma 7, it follows that the positive invariance of Ξ_ϵ is established if we can prove that for all $i \in [n]$, there holds $\dot{x}_i > 0$ whenever $x_i = \epsilon y_i$ and $x_j \in [\epsilon y_j, 1]$ for $j \neq i$. Toward that end, observe from Eq. (1) that

$$\begin{aligned}\dot{x}_i &= -(\delta_{\min}(1 - z_i) + \delta_i z_i)\epsilon y_i + (1 - \epsilon y_i)\\ &\quad \times \sum_{j\in\mathcal{N}_i} (\beta_{ij}(1 - z_i) + \beta_{\min}z_i)(x_j - \epsilon y_j + \epsilon y_j), \end{aligned}\tag{10}$$

and note that we have dropped the argument t for brevity. Since $x_j - \epsilon y_j \ge 0$ and $\epsilon y_i < 1$ by hypothesis,

$$(1 - \epsilon y_i)\Big(\sum_{j\in\mathcal{N}_i} \beta_{ij}(1 - z_i) + \beta_{\min}z_i)(x_j - \epsilon y_j) \ge 0.$$

This implies that Eq. (10) obeys the following inequality:

$$\begin{aligned}\dot{x}_i &\ge -(\delta_{\min}(1 - z_i) + \delta_i z_i)\epsilon y_i\\ &\quad + \sum_{j\in\mathcal{N}_i} (\beta_{ij}(1 - z_i) + \beta_{\min}z_i)\epsilon y_j\\ &\quad - \epsilon^2 y_i \sum_{j\in\mathcal{N}_i} (\beta_{ij}(1 - z_i) + \beta_{\min}z_i)y_j. \end{aligned}\tag{11}$$

Verify that $\phi y = (-D + \beta_{\min}A)y$ implies that $\delta_i y_i + \sum_{j\in\mathcal{N}_i} \beta_{\min}y_j = \phi_i y_i$. Recalling that $\delta_i \ge \delta_{\min}$ and $\beta_{ij} \ge \beta_{\min}$ for $j \in \mathcal{N}_i$, we obtain

$$-\delta_{\min} y_i + \sum_{j\in\mathcal{N}_i} \beta_{ij} y_j \geq -\delta_i y_i + \sum_{j\in\mathcal{N}_i} \beta_{\min} y_j = \phi_i y_i. \quad (12)$$

Since $z_i \in [0,1]$, it follows from Eq. (12)

$$z_i(-\delta_i y_i + \sum_{j\in\mathcal{N}_i} \beta_{\min} y_j) + (1-z_i)(-\delta_{\min} y_i + \sum_{j\in\mathcal{N}_i} \beta_{ij} y_j) \geq \phi_i y_i > 0. \quad (13)$$

Using Eq. (13), the right-hand side of Eq. (11) can then be further bounded as

$$\dot{x}_i \geq \epsilon\phi y_i - \epsilon^2 y_i \sum_{j\in\mathcal{N}_i} (\beta_{ij}(1-z_i) + \beta_{\min} z_i) y_j.$$

Obviously, for some sufficiently small $\epsilon_i > 0$, we then have

$$\dot{x}_i \geq \epsilon_i\phi y_i - \epsilon_i^2 y_i \sum_{j\in\mathcal{N}_i} (\beta_{ij}(1-z_i) + \beta_{\min} z_i) y_j > 0.$$

By choosing $\bar{\epsilon} = \min_i \epsilon_i$, we conclude that Ξ_ϵ for every $\epsilon \in (0, \bar{\epsilon}]$ is a positive invariant set of Eq. (3).

Since Ξ_ϵ for $\epsilon \in (0, \bar{\epsilon}]$ is compact and convex, and Eq. (3) is Lipschitz smooth in Ξ_ϵ, the result of (Lajmanovich and Yorke, 1976, Lemma 4.1) immediately establishes that Eq. (3) has at least one equilibrium in Ξ_ϵ. Taking ϵ to be arbitrarily small, and recalling Lemma 7 establishes that Eq. (3) has at least one equilibrium in $\mathrm{Int}([0,1]^{2n})$. ■

4. CONCLUSIONS

In this paper, we have proposed a novel network SIS model coupled with opinion dynamics. We have analyzed the system's limiting behavior, equilibria, and their stability. Simulations suggest that when $\rho(D_{\min}^{-1}B) > 1$, the system has a unique endemic state which is globally asymptotically stable except when the initial state is zero. The complete analysis of the proposed system is more challenging than traditional continuous-time network SIS models like the one by Lajmanovich and Yorke (1976). An approach using the Poincaré–Hopf Theorem from differential topology, as by Ye et al. (2020), will be explored in the future.

REFERENCES

Berman, A. and Plemmons, R.J. (1979). Nonnegative Matrices. *The Mathematical Sciences, Classics in Applied Mathematics*, 9.

Eshghi, S., Sarkar, S., and Venkatesh, S. (2017). Visibility-aware optimal contagion of malware epidemics. *IEEE Transactions on Automatic Control*, 62(10), 5205–5212.

Fall, A., Iggidr, A., Sallet, G., and Tewa, J.J. (2007). Epidemiological Models and Lyapunov Functions. *Mathematical Modelling of Natural Phenomena*, 2(1), 62–83.

Funk, S., Gilad, E., Watkins, C., and Jansen, V.A.A. (2009). The spread of awareness and its impact on epidemic outbreaks. *Proceedings of the National Academy of Sciences*, 106(16), 6872–6877.

Funk, S., Salathé, M., and Jansen, V.A.A. (2010). Modelling the influence of human behaviour on the spread of infectious diseases: a review. *Journal of the Royal Society Interface*, 7(50), 1247–1256.

Glanz, K., Rimer, B.K., and Viswanath, K. (2008). *Health Behavior and Health Education: Theory, Research, and Practice.* Jossey-Bass.

Horn, R.A. and Johnson, C.R. (2012). *Matrix Analysis.* Cambridge University Press, New York.

Kermack, W.O. and McKendrick, A.G. (1927). A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society A*, 115(772), 700–721.

Khalil, H. (2002). *Nonlinear Systems.* Prentice Hall.

Khanafer, A., Başar, T., and Gharesifard, B. (2016). Stability of epidemic models over directed graphs: A positive systems approach. *Automatica*, 74, 126–134.

Lajmanovich, A. and Yorke, J.A. (1976). A deterministic model for gonorrhea in a nonhomogeneous population. *Mathematical Biosciences*, 28(3-4), 221–236.

Liu, J., Paré, P.E., Nedić, A., Beck, C.L., and Başar, T. (2017). On a continuous-time multi-group bi-virus model with human awareness. In *Proceedings of the 56th IEEE Conference on Decision and Control*, 4124–4129.

Liu, J., Paré, P., Nedić, A., Tang, C., Beck, C., and Başar, T. (2019). Analysis and control of a continuous-time bi-virus model. *IEEE Transactions on Automatic Control*, 64(12), 4891–4906.

Mei, W., Mohagheghi, S., Zampieri, S., and Bullo, F. (2017). On the dynamics of deterministic epidemic propagation over networks. *Annual Reviews in Control.*

Mieghem, P.V., Omic, J., and Kooij, R. (2009). Virus spread in networks. *IEEE/ACM Transactions on Networking*, 17(1), 1–14.

Ni, S., Weng, W., and Zhang, H. (2011). Modeling the effects of social impact on epidemic spreading in complex networks. *Physica A: Statistical Mechanics and its Applications*, 390(23-24), 4528–4534.

Nowzari, C., Preciado, V.M., and Pappas, G.J. (2016). Analysis and Control of Epidemics: A Survey of Spreading Processes on Complex Networks. *IEEE Control Systems*, 36(1), 26–46.

Paarporn, K., Eksin, C., Weitz, J.S., and Shamma, J.S. (2017). Networked SIS epidemics with awareness. *IEEE Transactions on Computational Social Systems*, 4(3), 93–103.

Qu, Z. (2009). *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles.* Springer Science & Business Media.

Rizzo, A., Frasca, M., and Porfiri, M. (2014). Effect of individual behavior on epidemic spreading in activity-driven networks. *Physical Review E*, 90(4), 042801.

Rosenstock, I.M. (1974). Historical origins of the health belief model. *Health education monographs*, 2(4), 328–335.

Ruf, S.R., Paarporn, K., and Paré, P.E. (2019). Going viral: Stability of consensus-driven adoptive spread. *IEEE Transactions on Network Science and Engineering.* doi: 10.1109/TNSE.2019.2952986.

Ruf, S.R., Paarporn, K., Paré, P.E., and Egerstedt, M. (2017). Exploring opinion-dependent product spread. In *Proceedings of the 56th IEEE Conference on Decision and Control*, 2935–2940.

Varga, R.S. (2009). *Matrix Iterative Analysis*, volume 27. Springer Science & Business Media.

Ye, M., Liu, J., Anderson, B., and Cao, M. (2020). Distributed feedback control on the SIS network model: An impossibility result. In *Proceedings of the 21st IFAC World Congress.* To appear.