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A motif-based approach to processes on networks: Process motifs for the differential entropy of the Ornstein–Uhlenbeck process

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Abstract:
A challenge in neuroscience and many other fields is the inference of a network's structure from observations of dynamics on the network. Understanding the relationship between network structure and dynamics on a network can help improve methods for network inference. We consider "process motifs" on a network as building blocks of processes on networks and propose to distinguish process motifs and graphlets as two different types of network motifs. We demonstrate that the analysis of process motifs can yield insights into the mechanisms by which processes and network structure contribute to differential entropy and other information-based properties of stochastic processes on networks, and we discuss the relationship between process motifs and graphlets.

Keywords: network inference, dynamics on networks, network motifs, differential entropy, stochastic dynamics

1. INTRODUCTION

In neuroscience and many other fields, researchers are interested in inferring the structure of a network from observations of dynamics on the network (Kramer et al., 2009). Popular approaches to this challenge have used concepts such as transfer entropy (Vicente et al., 2011), Granger causality (Bressler and Seth, 2011), and information-theoretic measures (Schaub et al., 2019), or graphical models Pearl (2014); Koller and Friedman (2009) to infer edges in a network from time series and other data. Understanding how motifs in a network's structure affect measurements help identify the network structures that one can (or cannot) infer using a given method.

The study of motifs in networks has advanced the understanding of various systems in biology (Alon, 2007), ecology (Rip et al., 2010), economics (Takes et al., 2018), computational social science (Hong-Lin et al., 2014), and other areas. Traditionally, network scientists have considered graphlets (i.e., small subgraphs in a network) and identified them as motifs in a network's structure if empirical data (Alon, 2007) or mathematical models (Kim et al., 2008) indicate their importance to a system's function.

We propose to consider "process motifs" (i.e., structured sets of walks; shown in Fig. 1) on a network as building blocks of processes on networks, and to distinguish the notion of process motifs from the common notion of motifs as graphlets. We define a process motif on a graph G to be a directed weighted multigraph in which each edge corresponds to a walk in G and each edge's weight corresponds to the length of the associated walk. We demonstrate how one can derive process motifs for a property of noisy linear dynamics on networks using the steady-state differential entropy of the Ornstein–Uhlenbeck process as an example, and use the results of this derivation to identify graphlets that contribute most to differential entropy.

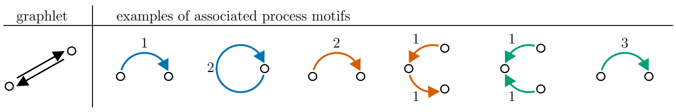

Fig. 1. **Graphlets and process motifs.** We show a graphlet and examples of associated process motifs. Numerical labels indicate the length of an edge in a process motif. The process motifs with blue edges are examples of process motifs that use each edge in the graphlet at most once and in which each node corresponds to a different node in the graphlet. The process motifs with orange edges are examples of process motifs in which two nodes correspond to the same node in the graphlet. The process motifs with green edges are examples of process motifs that use edges in the graphlet more than once.

2. DIFFERENTIAL ENTROPY OF THE ORNSTEIN–UHLENBECK PROCESSES ON A NETWORK

The Ornstein–Uhlenbeck process (Uhlenbeck and Ornstein, 1930) is a simple and popular model for noisy coupled systems (Aalen and Gjessing, 2004). For example, it has been used as a model for the dynamics for neuronal systems (Tononi et al., 1994), stock prices (Liang et al., 2011), and gene expression (Rohlfs et al., 2013). The differential entropy of the Ornstein–Uhlenbeck process at steady state is the basis for several properties of dynamical processes on networks. Examples of such properties are neural complexity (Tononi et al., 1994), redundancy and degeneracy (Tononi et al., 1999), and robustness to small perturbations in node states (Demetrius and Manke, 2005). Across many domains, researchers have linked functions of entropy and differential entropy to a network's ability to robustly perform a desired function (Demetrius and Manke, 2005; West et al., 2012; Schieber et al., 2016). Deriving process motifs that contribute to differential entropy of the Ornstein–Uhlenbeck process at steady state is a step towards identifying the process motifs that contribute to steady-state transfer entropy, redundancy, degeneracy, and other properties of processes on networks.

3. PROCESS MOTIFS AND EMERGENCE OF PROCESS PROPERTIES

For several properties of processes on networks, one can calculate the contribution of a process motif to the property. Studying the contributions of process motifs can further understanding of processes on networks in several ways. For example, it can help one understand how properties of processes on networks "emerge" from the superposition of small subprocesses. One can also calculate a graphlet's contribution to a property of a process from the contributions of process motifs that can occur on the graphlet.

In Fig. 2, we show a graphlet and examples of associated process motifs. We also show several process motifs that contribute to steady-state differential entropy of the Ornstein–Uhlenbeck process and the respective contributions. We find that the process motifs that contribute to steady-state differential entropy are circular [1]. By considering different network structures, we find structures on which cyclic process motifs contribute most to differential entropy and structures on which acyclic process motifs contribute most to differential entropy.

4. SUMMARY

The analysis of process motifs and their contribution to differential entropy demonstrates that it is important to consider processes on a network (instead of just a network's structure) as a composite entity that one can decompose into many small parts. Such a decomposition of processes on networks provides a framework for studying the mechanisms by which processes and network structure contribute to differential entropy, transfer entropy, redundancy, and other properties of processes on networks. A knowledge of these mechanisms lead to crucial insights into the limitations of methods for the inference of network structure from observations of dynamical processes on networks.

[1] A *circular* process motif is a process motif such that if one replaces each directed edge by an undirected edge, the resulting graph is an undirected cycle.

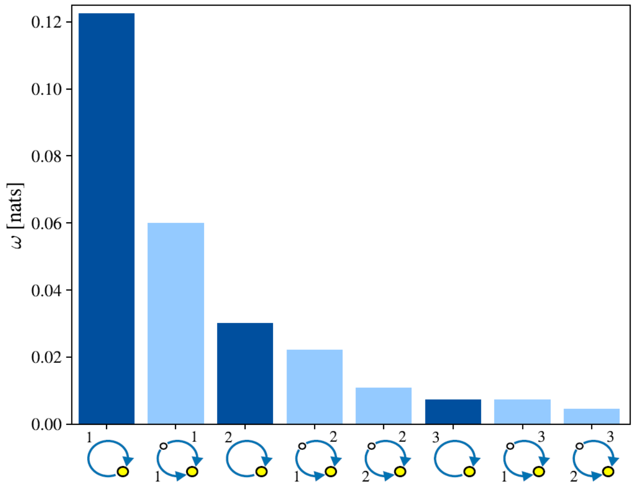

Fig. 2. **Process-motif contributions to differential entropy.** We show contributions of process motifs to the steady-state differential entropy of the Ornstein–Uhlenbeck process. Bars are light blue when the corresponding process motif is acyclic and dark blue when the corresponding process motif is cyclic.

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