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Koopman Operator Methods for Global Phase Space Exploration of Equivariant Dynamical Systems ⋆

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Abstract: In this paper, we develop the Koopman operator theory for dynamical systems with symmetry. In particular, we investigate how the Koopman operator and eigenfunctions behave under the action of the symmetry group of the underlying dynamical system. Further, exploring the underlying symmetry, we propose an algorithm to construct a global Koopman operator from local Koopman operators. In particular, we show, by exploiting the symmetry, data from all the invariant sets are not required for constructing the global Koopman operator; that is, local knowledge of the system is enough to infer the global dynamics.

Keywords: Dynamic systems, Operators, Learning algorithms, Equivariant systems, Koopman operators, Data-driven analysis.

1. INTRODUCTION

Dynamical systems theory is one of the most important branches of mathematics in the sense that it has applications in almost all fields of science and engineering. Any system which changes with time is a dynamical system and hence, they are ubiquitous in nature. As such, both theoretical and numerical analysis of dynamical systems is important. An important class of dynamical systems are the ones which have a symmetry in the sense that there exists some transformations on the state space which carries one solution of the dynamical system to another solution of the dynamical system Field (1970); Field (1980); Golubitsky et al. (2012) and the symmetries manifest themselves in asymptotic dynamics, bifurcation, attractor structures etc. Chossat and Golubitsky (1988); Sparrow (2012); Mesbahi et al. (2019); Salova et al. (2019). Moreover, symmetries play an important role in synchronization, pattern formation, quantum systems, etc. Mathematically, symmetry is specified by the action of some group on the state space and hence, for studying symmetric dynamical systems, elements of group theory and representation theory are used.

Traditionally, theoretical analysis of dynamical systems is performed by studying the evolution of trajectories in the phase space. However, more recently a different technique is being increasingly used to study dynamical systems, where instead of studying the trajectories in the phase space, the focus is, using transfer operators like Perron-Frobenius operator (P-F) and Koopman operator, on studying the evolution of measures or functions defined on the phase space Lasota and Mackey (1994); Vaidya and Mehta (2008); Mezić (2005); Budisic et al. (2012).

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The main advantage of this approach is the fact that the evolution of measures or functions is linear in the infinite-dimensional space. Moreover, the evolution of functions, which is governed by the Koopman operator, is tailor-made for data-driven analysis of dynamical systems. This is especially useful for analysis of higher dimensional systems like power networks, building systems, biological networks, etc. However, one drawback of using transfer operators is that these are typically infinite-dimensional operators. Hence, for data-driven analysis researchers have developed many different methods for computing the finite-dimensional approximations of these transfer operators and using the developed framework for analysis and control of dynamical systems Dellnitz and Junge (1999); Mezic and Banaszuk (2000); Mezić (2005); Vaidya and Mehta (2008); Raghunathan and Vaidya (2014); Budisic et al. (2012); Mauroy and Mezic (2013); Yeung et al. (2018); Yeung et al. (2017); Sinha et al. (2019b); Johnson and Yeung (2018); Sinha et al. (2018b); Sinha et al. (2018a); Sinha et al. (2019a).

In this paper, we use the Koopman operator framework to study dynamical systems with symmetry. In particular, we analyze some basic properties of symmetric dynamical systems and their symmetry group and investigate how these properties are reflected on the infinite-dimensional Koopman operator for the corresponding symmetric dynamical systems. In particular, we analyze how the symmetry of the underlying system affects the evolution of functions in the function space under the action of the Koopman operator. Moreover, as mentioned before, Koopman operator techniques facilitate the data-driven analysis of dynamical systems. To this end, in this paper, we use the construction technique proposed in Nandanoori et al. (2019) to provide a method for constructing the global Koopman operator (defined on the entire phase space) from local Koopman operators (defined on locally invariant sets). In particular, we show that using the symmetry of the underlying system, one does not need to train the local Koopman operators on all the different invariant spaces and hence, one *does not need the data from all the invariant subspaces* for constructing the global operator.

2. PRELIMINARIES

In this section, we discuss the preliminaries of equivariant systems and transfer operators.

2.1 Equivariant Dynamical Systems

Consider a dynamical system $\dot{x} = f(x)$, where $x \in M \subseteq \mathbb{R}^n$ and $f : M \to M$ is assumed to be at least $\mathcal{C}^1$. A symmetry of the dynamical system is a transformation that maps solutions of the system to other solutions of the system. A dynamical system with such a transformation is known as an equivariant dynamical system and is defined as follows:

Definition 1. (Equivariant Dynamical System). Consider the dynamical system $\dot{x} = f(x)$ and let G be a group acting on M. The system is called G-equivariant if

$$f(g \cdot x) = g \cdot f(x), \text{ for } g \in G, x \in M.$$

The definition for an equivariant discrete-time dynamical system $x_{t+1} = T(x_t)$ is defined analogously. In particular, a discrete-time dynamical system $x_{t+1} = T(x_t)$ is G-equivariant if

$$T(g \cdot x) = g \cdot T(x) \text{ for } g \in G.$$

Example 1. Consider the Lorenz system given by

$$\dot{x} = \sigma(y - x); \quad \dot{y} = x(\rho - z) - y; \quad \dot{z} = xy - \beta z \tag{1}$$

where σ, ρ and β are constants. The system equations remain invariant under the transformation $(x, y, z) \mapsto (-x, -y, z)$ and hence the Lorenz system is invariant under the transformation matrix $\gamma = \text{diag}(-1, -1, 1)$.

In this paper, we assume G to be a finite subgroup of $O(n)$ and $M \subset \mathbb{R}^n$ to be a compact G-invariant set. In general, a symmetry group can be any subgroup of the group of isometries of the Euclidean space $\mathbb{E}^n$, but in this work, we consider finite subgroups of the group of point symmetries of $\mathbb{E}^n$. Further, we consider discrete-time systems of the form $x_{t+1} = T(x_t)$.

Remark 1. We consider discrete-time systems because Koopman operators are tailor-made for data-driven analysis of dynamical systems and data (from a simulation or from an experiment) is always in the form of a discrete time-series.

Also, given an abstract group G, let Γ be the n-dimensional representation of the group G in $\mathbb{R}^n$, such that $g \mapsto \gamma_g$, where $g \in G$ and $\gamma_g \in \Gamma$. Note that, γ_g has the matrix representation $\gamma_g \in \mathbb{R}^{n\times n}$ and the action of the abstract group G on the state space $\mathbb{R}^n$ is specified by the action of the representation group Γ acting on $\mathbb{R}^n$, where the action is by matrix multiplication *.

Definition 2. (Isotropy Set). Let $x_0(t)$ be a solution (trajectory) of G-equivariant dynamical system from the initial condition x_0. Then the isotropy set is defined as

$$\Sigma_{x_0(t)} = \{g \in G | g \cdot x_0(t) = x_0(t)\}.$$

With the notion of isotropy set, we have the following.

Lemma 1. The isotropy set corresponding to a solution $x_0(t)$ is a subgroup of the symmetry group G.

* For notational convenience we will use g throughout the paper. However, it should be kept in mind that the action of g is through appropriate representations of G on the concerned spaces.

Proof. Proof is omitted due to space constraints. For proof see Sinha et al. (2020).

Proposition 1. Let $\Sigma_{x_0(t)}$ and $\Sigma_{gx_0(t)}$ be the isotropy groups of $x_0(t)$ and $gx_0(t)$ respectively. Then we have

$$\Sigma_{gx_0(t)} = g\Sigma_{x_0(t)}g^{-1}.$$

Proof. Proof is omitted due to space constraints. For proof see Sinha et al. (2020).

2.2 Transfer Operators

In this subsection, we briefly discuss the transfer operators, namely the Perron-Frobenius (P-F) and Koopman operator. Consider a discrete-time dynamical system

$$x_{t+1} = T(x_t) \tag{2}$$

where $T : M \subset \mathbb{R}^N \to M$ is assumed to be at least $\mathcal{C}^1$. Associated with the dynamical system (2) is the Borel-σ algebra $\mathcal{B}(M)$ on M and the vector space $\mathcal{M}(X)$ of bounded complex valued measures on M. With this, two linear operators, namely, Perron-Frobenius (P-F) and Koopman operator, can be defined as follows Lasota and Mackey (1994):

Definition 3. The P-F operator $\mathbb{P} : \mathcal{M}(X) \to \mathcal{M}(X)$ is given by

$$[\mathbb{P}\mu](A) = \int_M \delta_{T(x)}(A) d\mu(x) = \mu(T^{-1}(A))$$

$\delta_{T(x)}(A)$ is stochastic transition function which measure the probability that point x will reach the set A in one time step under the system mapping T.

Definition 4. Given any $h \in \mathcal{F}$, the Koopman operator $\mathbb{U} : \mathcal{F} \to \mathcal{F}$ is defined as $[\mathbb{U}h](x) = h(T(x))$, where $\mathcal{F}$ is the space of functions (observables) invariant under the action of the Koopman operator.

Both the Perron-Frobenius and the Koopman operators are linear operators, even if the underlying system is nonlinear. But while analysis is made tractable by linearity, the trade-off is that these operators are typically infinite-dimensional. In particular, the P-F operator and Koopman operator often will lift a dynamical system from a finite-dimensional space to generate an infinite-dimensional linear system.

3. KOOPMAN OPERATOR AND EQUIVARIANT DYNAMICAL SYSTEMS

We begin this section with the analysis of group action on a Koopman operator.

3.1 Group Action and Koopman Operator

Suppose

$$x_{t+1} = T(x_t) \tag{3}$$

be a dynamical system defined on the state space $M \subset \mathbb{R}^n$, which is symmetric with respect to a group G and let $\mathbb{U}$ be the associated Koopman operator. The Koopman operator, $\mathbb{U}$ is a linear operator on the space of functions $(\mathcal{F}(M))$ on M. We define a map

$$\begin{aligned} \varphi :&G \times \mathcal{F}(M) \to \mathcal{F}(M) \\ &(g \star f)(x) \mapsto f(g^{-1} \cdot x). \end{aligned} \tag{4}$$

Lemma 2. The map φ, defined in Eq. (4) defines a group action on the space $\mathcal{F}(M)$.

Proof. The proof is omitted for space constraints. For proof see Sinha et al. (2020).

From the action of the symmetry group G on $\mathcal{F}(M)$, we have the following theorem.

Theorem 1. Let $\mathbb{U}$ be the Koopman operator associated with a G-equivariant system $x_{t+1} = T(x_t)$. Then

$$[g \star (\mathbb{U}f)](x) = [\mathbb{U}(g \star f)](x). \tag{5}$$

for all $g \in G$ and $f \in \mathcal{F}(M)$.

Proof. For the dynamical system $x_{t+1} = T(x_t)$ and any function $f \in \mathcal{F}(M)$, the Koopman operator $\mathbb{U}$ is defined as $[\mathbb{U}f](x) = f(T(x))$. Hence, for $g \in G$ we have,

$$\begin{aligned} g \star (\mathbb{U}f)(x) &= g \star f(T(x)) = f(g^{-1} \cdot T(x)) \\ &= f(T(g^{-1} \cdot x)) = \mathbb{U}f(g^{-1} \cdot x) = [\mathbb{U}(g \star f)](x), \end{aligned}$$

where the third equality follows from the definition of G-equivariant systems.

The above theorem essentially says that the Koopman operator commutes with the elements of the symmetry group G.

Associated with a Koopman operator is its eigenspectrum, that is, the eigenvalues λ, and their corresponding eigenfunctions $\phi_\lambda(x)$, such that $[\mathbb{U}\phi_\lambda](x) = \lambda\phi_\lambda(x)$. and the eigenspectrum (especially eigenfunctions corresponding to dominant eigenvalues) of a Koopman operator dictates the evolution of the functions $f \in \mathcal{F}(M)$, under the map T (refer Eq. 3).

Definition 5. Let $\mathbb{U}$ be a Koopman operator and let $\phi_\lambda^i(x)$ be eigenfunctions of $\mathbb{U}$ corresponding to the eigenvalue λ, that is, $\mathbb{U}\phi_\lambda^i(x) = \lambda\phi_\lambda^i(x)$. Then the eigenspace $\mathcal{E}_\lambda$ is defined as $\mathcal{E}_\lambda = \text{span}\{\phi_\lambda^i(x)\}$.

The following result establishes that the eigenspace is left invariant under group action.

Proposition 2. Let Eq. (3) be a G-equivariant discrete-time dynamical system and $\mathbb{U}$ be the associated Koopman operator. If λ is an eigenvalue of the Koopman operator $\mathbb{U}$ and $\mathcal{E}_\lambda$ is the corresponding eigenspace, then the eigenspace remains invariant under the action of the symmetry group G.

Proof. Proof is omitted due to space constraints. For proof see Sinha et al. (2020).

Note that a Koopman operator is a linear operator which gives the evolution of functions which are defined on the state space. Let $x \in M$ and $g \cdot x \in M$ for a G-equivariant system and let $f \in L_2(M)$. Let $\hat{f} = g \star f$. Then the following proposition relates the representation (analogous to a matrix representation of a linear transformation) of the Koopman operator when the functions f and $\hat{f}$ are evaluated at x and $g \cdot x$ respectively.

Proposition 3. Let (3) be a G-equivariant dynamical system and its associated Koopman operator be $\mathbb{U} : \mathcal{F}(M) \to \mathcal{F}(M)$. Suppose $f \in \mathcal{F}(M)$ and let $\mathbb{U}_f$ be the representation of $\mathbb{U}$ with respect to f. For $g \in G$, let $\hat{f} = g \star f$ and $\mathbb{U}_{\hat{f}}$ be the representation of $\mathbb{U}$ with respect to $\hat{f}$. Then for $x \in M$, we have

$$\mathbb{U}_{\hat{f}}\hat{f}(g \cdot x) = \mathbb{U}_f f(x) \tag{6}$$

Proof. We have

$$\begin{aligned} \mathbb{U}_{\hat{f}}\hat{f}(g \cdot x) =& \hat{f}(T(g \cdot x)) = \hat{f}(g \cdot T(x)) = g^{-1} \star \hat{f}(T(x)) \\ =& f(T(x)) = \mathbb{U}_f f(x). \end{aligned}$$

3.2 Group Action and Invariant Spaces

Definition 6. For a dynamical system $x_{t+1} = T(x_t)$, defined on $M \subseteq \mathbb{R}^n$, a subset $\mathcal{M} \subset M$ is an invariant set if for every trajectory $x_0(t)$,

$$x_0(t) \in \mathcal{M} \implies x_0(\tau) \in \mathcal{M}, \forall \tau \geq t. \tag{7}$$

Note that an orbit from an initial condition x_0 is an invariant set.

For a measure preserving transformation T, all the eigenvalues of the associated Koopman operator $\mathbb{U}$ lie on the unit circle Budisic et al. (2012). Moreover, when T is an ergodic transformation, then all eigenvalues of $\mathbb{U}$ are simple Petersen (1989); Budisic et al. (2012). However, if T is not ergodic, then the state space can be partitioned into subsets $\mathcal{M}_i$ (minimal invariant subspaces) such that the restriction $T|_{\mathcal{M}_i} : \mathcal{M}_i \to \mathcal{M}_i$ is ergodic. A partition of the state space into invariant sets is called an ergodic partition or stationary partition. Hence, for any transformation T, defined on $M \subseteq \mathbb{R}^n$, the state space M can be expressed as

$$M = \cup_{i=1}^m \mathcal{M}_i \text{ (modulo measure zero sets)}, \tag{8}$$

where each $\mathcal{M}_i$ is an invariant set and $\mathcal{M}_i$ and $\mathcal{M}_j$ are disjoint for $i \neq j$. Hence, all ergodic partitions are disjoint and they support mutually singular functions from $L_2(M)$ Budisic et al. (2012). Therefore, the number of linearly independent eigenfunctions of $\mathbb{U}$ corresponding to an eigenvalue λ is bounded above by the number of ergodic sets in the state space Budisic et al. (2012). The dynamics of the system dictates the number of ergodic partitions (invariant sets) in the state space.

Definition 7. Let $\mathcal{M}_i$ be an invariant set of the G-equivariant dynamical system (3). Then for $g \in G$, define the set $g \cdot \mathcal{M}_i$ as

$$g \cdot \mathcal{M}_i := \{\tilde{x} \in M | \tilde{x} = g \cdot x \text{ for } x \in \mathcal{M}_i\}$$

Proposition 4. If $\mathcal{M}_i$ is an invariant set for a G-equivariant dynamical system $x_{t+1} = T(x_t)$, then $g \cdot \mathcal{M}_i$ is also an invariant set for $g \in G$.

Proof. The proof is omitted for space constraints. For proof see Sinha et al. (2020).

Corollary 1. For any invariant set $\mathcal{M}_i$, $G \cdot \mathcal{M}_i$ is invariant, where

$$G \cdot \mathcal{M}_i = \{\tilde{x} \in M | \tilde{x} = g \cdot x, \text{ for } x \in \mathcal{M}_i \text{ and } g \in G\}.$$

4. GLOBAL PHASE SPACE RECONSTRUCTION FROM DATA

In this section, we develop the data-driven tools for analysis of equivariant dynamical systems.

4.1 Finite Dimensional Approximation of Koopman Operator

Let

$$X_p = [x_1, x_2, \ldots, x_M], \quad X_f = [y_1, y_2, \ldots, y_M] \tag{9}$$

be snapshots of data obtained from simulating a dynamical system $x \mapsto T(x)$, or from an experiment, where $x_i \in M$ and $y_i \in M$, $M \subset \mathbb{R}^n$. The two pairs of data sets are assumed to be two consecutive snapshots i.e., $y_i = T(x_i)$. Let $\mathcal{D} = \{\psi_1, \psi_2, \ldots, \psi_K\}$ be the set of observables, where $\psi_i : M \to \mathbb{R}$ and $\psi_i \in L_2(M)$. Let $\mathcal{G}_\mathcal{D}$ denote the span of $\mathcal{D}$. Let $\mathbf{\Psi} : X \to \mathbb{R}^K$ be a vector valued function, such that

$$\mathbf{\Psi}(\boldsymbol{x}) := [\psi_1(x) \ \psi_2(x) \ \cdots \ \psi_K(x)]^\top .$$

Here $\mathbf{\Psi}$ is the mapping from physical space to feature space. The finite dimensional approximate Koopman operator $\mathbf{K} \in \mathbb{R}^{\mathbf{K}\times\mathbf{K}}$ is computed as:

$$\mathbf{K} = Y_f {Y_p}^\dagger \tag{10}$$

where

$$Y_p = \mathbf{\Psi}(X_p) = [\mathbf{\Psi}(x_1), \mathbf{\Psi}(x_2), \cdots, \mathbf{\Psi}(x_M)] \tag{11}$$

$$Y_f = \mathbf{\Psi}(X_f) = [\mathbf{\Psi}(y_1), \mathbf{\Psi}(y_2), \cdots, \mathbf{\Psi}(y_M)], \tag{12}$$

and ${Y_p}^\dagger$ is the pseudo-inverse of matrix Y_p. DMD is a special case of EDMD algorithm with $\mathbf{\Psi}(x) = x$.

4.2 Global Koopman Operator from Local Koopman Operators

As mentioned earlier, any phase space M can be decomposed into disjoint invariant sets $\mathcal{M}_i$. Let

$$\mathbf{\Psi}_i = \{\psi_1^i, \psi_2^i, \cdots, \psi_{K_i}^i\} \tag{13}$$

be the dictionary functions (observables) on each $\mathcal{M}_i$ and let $\mathbf{K}_i$ be the corresponding Koopman operator on $\mathcal{M}_i$. Note that, in general, on each $\mathcal{M}_i$, the dictionary functions are different and hence, the finite-dimensional matrix representation of each of the local Koopman operators $\mathbf{K}_i$ is different. This is because given any linear transformation $\mathcal{O} : V \to W$, where V and W are vector spaces, the matrix representation of $\mathcal{O}$ depends on the choice of the basis vectors of V and W. For the self-containment of the paper, in this subsection, we briefly review the results of Nandanoori et al. (2019) where we had proposed a systematic method to construct the global Koopman operator $\mathbf{K}$, which describes the evolution of the system on the entire phase space M, from the local Koopman operators $\mathbf{K}_i$.

Let $\mathbf{\Psi}_i$ be the dictionary functions on each invariant set $\mathcal{M}_i$, $i = 1, 2, \cdots, m$ and let $\mathbf{K}_i$ be the corresponding Koopman operator which describes the evolution of the system in each $\mathcal{M}_i$. We define the set of dictionary functions on the entire state space M as $\mathbf{\Psi} = \bigsqcup_{i=1}^m \mathbf{\Psi}_i$. Then if $\mathbf{K}$ is the global Koopman operator on the entire state space M with dictionary function $\mathbf{\Psi}$, then $\mathbf{K}$ can be expressed as

$$\mathbf{K} = \text{diag}(\mathbf{K}_1, \mathbf{K}_2, \cdots, \mathbf{K}_m). \tag{14}$$

4.3 Global Koopman Operator for Equivariant Systems

Consider the G-equivariant system (3), defined on the state space $M \in \mathbb{R}^n$ with disjoint invariant sets $\mathcal{M}_i$, as in Eq. (8).

From proposition 4 we have that $g \cdot \mathcal{M}_i$ is also invariant for all $g \in G$. Hence, $g \cdot \mathcal{M}_i \subset \mathcal{M}_j$ for some $j \in \{1, 2, \cdots, m\}$.

Assumption 1. We assume there exists some $g \in G$, such that $g \cdot \mathcal{M}_i \subset \mathcal{M}_j$ and $i \neq j$.

Let $\mathbf{\Psi}_i$ be the dictionary functions defined on $\mathcal{M}_i$ and let $\mathbf{K}_i$ be the local Koopman operator on $\mathcal{M}_i$. Let

$$X = [x_1, x_2, \cdots, x_{M+1}]$$

be points in $\mathcal{M}_i$, such that $x_{k+1} = T(x_k)$ and thus

$$[\mathbf{K}_i \mathbf{\Psi}_i](x_k) = \mathbf{\Psi}_i(x_{k+1}). \tag{15}$$

Now, $\mathbf{K}_j$ governs the evolution of dictionary functions on $\mathcal{M}_j$ and the goal is to compute $\mathbf{K}_j$. Since in computing $\mathbf{K}_i$, $\mathbf{\Psi}_i(x_k)$ are already computed, we would like to use this information for computation of $\mathbf{K}_j$. This can be done in two different ways.

Case I. We use the same dictionary function $\mathbf{\Psi}_i$ on $\mathcal{M}_j$, that is

$$\mathbf{\Psi}_j = \mathbf{\Psi}_i.$$

Theorem 2. Let $\mathcal{M}_i$ be an invariant set of the G-equivariant system (3) and let $\mathbf{K}_i \in \mathbb{R}^{K_i \times K_i}$ be the local Koopman operator on $\mathcal{M}_i$ with dictionary function $\mathbf{\Psi}_i(x)$, $x \in \mathcal{M}_i$. Let, for $g \in G$, $g \cdot \mathcal{M}_i \subset \mathcal{M}_j$. Let $\mathbf{K}_j$ be the local Koopman operator on $\mathcal{M}_j$ with dictionary functions $\mathbf{\Psi}_j = \mathbf{\Psi}_i$. Then

$$\mathbf{K}_i = \gamma \mathbf{K}_j \gamma^{-1},$$

where $g \mapsto \gamma \in \Gamma$ and Γ is the K_i dimensional matrix representation of G in $\mathbb{R}^{K_i}$.

Proof. Since $\mathbf{\Psi}_j = \mathbf{\Psi}_i$, $\mathbf{K}_j \in \mathbb{R}^{K_i \times K_i}$. Now consider $x_k \in \mathcal{M}_i$ and $g \cdot x_k \in \mathcal{M}_j$. Then we have, $\mathbf{K}_j \mathbf{\Psi}_i(g \cdot x_k) = \mathbf{\Psi}_i(g \cdot x_{k+1}) = \gamma^{-1}\mathbf{\Psi}_i(x_{k+1}) = \gamma^{-1}\mathbf{K}_i\mathbf{\Psi}_i(x_k)$.. Again, $\mathbf{K}_j \mathbf{\Psi}_i(g \cdot x_k) = \mathbf{K}_j \gamma^{-1} \mathbf{\Psi}_i(x_k)$. Hence,

$$\mathbf{K}_i \mathbf{\Psi}_i(x_k) = \gamma \mathbf{K}_j \gamma^{-1} \mathbf{\Psi}_i(x_k). \tag{16}$$

Since Eq. (16) is true for all $x_k \in \mathcal{M}_i$, we obtain

$$\mathbf{K}_i = \gamma \mathbf{K}_j \gamma^{-1}.$$

Corollary 2. Let $x_0(t)$ be a trajectory of a G-equivariant system $x_{t+1} = T(x_t)$ and let $g \cdot x_0(t)$ be the image of $x_0(t)$ under the action of $g \in G$. Let $\mathbf{\Psi} \in L_2(M)$ be a set of dictionary functions of cardinality K. Let $\mathbf{K}_{x_0}$ and $\mathbf{K}_{g \cdot x_0}$ be the finite-dimensional representation of Koopman operator which governs the evolution of $\mathbf{\Psi}$ on $x_0(t)$ and $g \cdot x_0(t)$ respectively. Then

$$\mathbf{K}_{x_0} = \gamma \mathbf{K}_{g \cdot x_0} \gamma^{-1}, \tag{17}$$

where $g \mapsto \gamma \in \Gamma$ and Γ is the K dimensional matrix representation of G in $\mathbb{R}^K$.

Proof. Similar to the proof in theorem 2.

Note that the local Koopman operators obtained using the DMD algorithm satisfy theorem 2.

Case II. We define the dictionary function on $g \cdot \mathcal{M}_i$ and hence on $\mathcal{M}_j$, $j \neq i$ as

$$\mathbf{\Psi}_j = g \star \mathbf{\Psi}_i.$$

In this case, from proposition 3, we have

$$\mathbf{K}_i = \mathbf{K}_j.$$

Hence, starting with an invariant set $\mathcal{M}_i$, with a local Koopman operator $\mathbf{K}_i$, if $g_j \cdot \mathcal{M}_i \subset \mathcal{M}_j$ for $j \in \{1, 2, \cdots, m\}$, $j \neq i$, we can obtain all the local Koopman operators $\mathbf{K}_j$, $j = 1, 2, \cdots, m$. Hence, using the procedure of Nandanoori et al. (2019), we can obtain the global Koopman operator, defined on the entire state space M for the G-equivariant system (3).

5. SIMULATION RESULTS

In this section, by applying the symmetry in the system, we identify the global Koopman operator starting from an invariant subspace. We begin with the discussion on systems with reflective symmetry. In all the examples, we use the same dictionary functions on two different invariant spaces (or two different trajectories), which are related by the symmetry group (as in theorem 2).

5.1 Reflection Symmetry: Bistable Toggle Switch

Consider the bistable toggle switch system, first introduced in Gardner et al. (2000). The governing equations of motion are:

$$\dot{x}_1 = \frac{\alpha_1}{1 + x_2^\beta} - \kappa_1 x_1; \quad \dot{x}_2 = \frac{\alpha_2}{1 + x_1^\theta} - \kappa_2 x_2 \tag{18}$$

where the states $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ indicate the concentration of the repressor 1 and 2; the effective rate of synthesis of repressor 1 and 2 are denoted by α_1 and α_2; the self decay rates of concentration of repressor 1 and 2 are given by $\kappa_1 > 0$ and $\kappa_2 > 0$; the cooperativity of repression of promoter 2 and 1 are respectively denoted by β and θ. The system has two invariant sets and the line $x_1 = x_2$ is the separatrix that separates the two invariant sets. The phase portrait of this system is shown in Fig. 1.

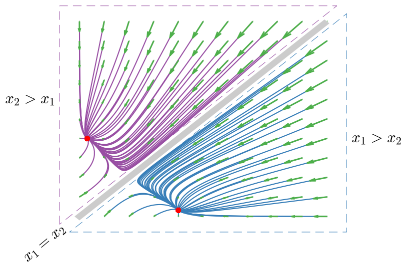

Fig. 1. Phase portrait of bistable toggle switch.

With the given values of the parameters, the system equations are symmetric under $\mathbb{Z}_2$-action, where the action is given by the transformation $(x_1, x_2)^\top \overset{\gamma}{\mapsto} (x_2, x_1)^\top$. In the phase space, this corresponds to a reflection about the $x_1 = x_2$ line and the 2-dimensional representation of the non-identity element of the symmetry group is $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The goal is to construct the global Koopman operator using only the time-series data from any one of the invariant sets. To demonstrate the proposed framework, we collected time-series data from only the invariant given by the region $x_1 > x_2$. The local Koopman operator, obtained using the DMD algorithm, is given by

$$\mathbf{K}_{right} = \begin{pmatrix} 0.6039 & 0.0313 \\ -0.4784 & 1.0375 \end{pmatrix}.$$

Hence, from theorem 2, the Koopman operator corresponding to the region $x_2 > x_1$ can be identified as

$$\mathbf{K}_{left} = \gamma^{-1}\mathbf{K}_{right}\gamma = \begin{pmatrix} 1.0375 & -0.4784 \\ 0.0313 & 0.6039 \end{pmatrix}.$$

Hence the global Koopman operator is

$$\mathbf{K}_{global} = \begin{pmatrix} \mathbf{K}_{left} & 0 \\ 0 & \mathbf{K}_{right} \end{pmatrix}.$$

The phase portrait corresponding to the two regions is shown in Fig. 1. Moreover, as a verification, we computed the Koopman operator using data from the region $x_2 > x_1$ and it was equal to

$$\mathbf{K}_{x_2 > x_1} = \begin{pmatrix} 1.0375 & -0.4784 \\ 0.0313 & 0.6039 \end{pmatrix} = \mathbf{K}_{left}.$$

5.2 Rotational Symmetry: Lorenz System

Consider the Lorenz system as shown in Eq. (1) The Lorenz system is symmetric under $\mathbb{Z}_2$ action given by

$$(x, y, z)^\top \overset{\gamma}{\mapsto} (-x, -y, z)^\top, \tag{19}$$

which corresponds to a rotation of 180° about the z-axis and matrix representation of γ is $\gamma = \text{diag}(-1, -1, 1)$. The phase portrait of the Lorenz system with $\rho = 28$, $\sigma = 10$ and $\beta = 8/3$, is shown in Fig. 2. The colours blue and magenta correspond to the symmetric components of the strange attractor. The Koopman operator computed, using DMD algorithm, from the blue region of the attractor is

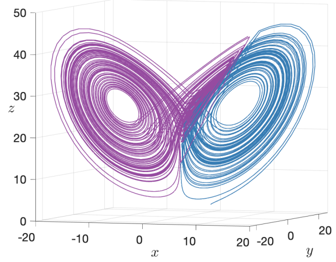

Fig. 2. Phase portrait of the Lorenz system.

$$\mathbf{K}_{blue} = \begin{pmatrix} 0.076 & 0.709 & 0.042 \\ -0.667 & 1.064 & 0.124 \\ -0.422 & 0.926 & 0.836 \end{pmatrix}.$$

Hence, from corollary 2, the Koopman on the symmetric counterpart of the blue region will be

$$\mathbf{K}_{magenta} = \gamma^{-1}\mathbf{K}_{blue}\gamma = \begin{pmatrix} 0.076 & 0.709 & -0.042 \\ -0.667 & 1.064 & -0.124 \\ 0.422 & -0.926 & 0.836 \end{pmatrix}$$

which is the same Koopman operator obtained with DMD algorithm with data points on the magenta region of the attractor.

5.3 Reflection and Rotational Symmetry: A Hamiltonian System

Consider a Hamiltonian system with a Hamiltonian

$$H(q, p) = \frac{1}{4}p^4 - \frac{9}{2}p^2 - \frac{1}{4}q^4 + \frac{9}{2}q^2.$$

Hence the equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = p^3 - 9p; \quad \dot{p} = -\frac{\partial H}{\partial q} = q^3 - 9q. \tag{20}$$

This system has 4 invariant sets and the corresponding phase portrait of the system is shown in Fig. 3 and the system is symmetric under the actions given by

$$\begin{pmatrix} q \\ p \end{pmatrix} \overset{\gamma_1}{\longmapsto} \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \overset{\gamma_2}{\longmapsto} \begin{pmatrix} -q \\ -p \end{pmatrix}, \begin{pmatrix} q \\ p \end{pmatrix} \overset{\gamma_3}{\longmapsto} \begin{pmatrix} -p \\ -q \end{pmatrix}. \tag{21}$$

From the action of γ_i's, we have

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = I_2, \text{ and } \gamma_1\gamma_2 = \gamma_3.$$

Hence the symmetry group of the system is the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which has the presentation

$$\Gamma = \langle \gamma_1, \gamma_2 | \gamma_1^2 = \gamma_2^2 = (\gamma_1\gamma_2)^2 = I_2 \rangle.$$

and the matrix representation of the group elements are

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The local Koopman operators obtained using data from each of the invariant subspaces are

$$\mathbf{K}_{IS-1} = \begin{pmatrix} 0.955 & 0.486 \\ -0.059 & 0.215 \end{pmatrix}; \mathbf{K}_{IS-2} = \begin{pmatrix} 0.215 & -0.059 \\ 0.486 & 0.955 \end{pmatrix}$$

$$\mathbf{K}_{IS-3} = \begin{pmatrix} 0.957 & 0.511 \\ -0.061 & 0.214 \end{pmatrix}; \mathbf{K}_{IS-4} = \begin{pmatrix} 0.214 & -0.061 \\ 0.511 & 0.957 \end{pmatrix}$$

and it can be seen that $\mathbf{K}_{IS-2} = \gamma_1^{-1}\mathbf{K}_{IS-1}\gamma_1$. Similar relations are found to hold true for the other local Koopman

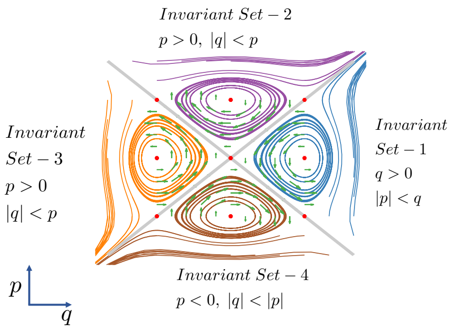

Fig. 3. Phase portrait of the Hamiltonian system.

operators. Hence if only one local Koopman operator is computed from data, all the other local Koopman operators can be computed using the relation of theorem 2, without using data from the other invariant subspaces and they are stitched together to obtain the global Koopman operator as described in Nandanoori et al. (2019).

6. CONCLUSIONS

In this paper, we developed Koopman operator theoretic based methods to study the global phase space in equivariant dynamical systems. In particular, we showed that the invariant subspaces are mapped to invariant subspaces and eigenspaces are left invariant under the group action of the symmetry group and established the properties of the Koopman operator for an equivariant dynamical system under group action. Assuming the knowledge of the type of symmetry in a dynamical system, it is shown that the global phase space can be studied based on data from any one invariant subspace only. The proposed framework is demonstrated on three different systems that possess various symmetries, such as reflective, rotational, or both. Future efforts focus on identifying the type of symmetries in a dynamical system, given the data for the global phase space.

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