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Online Control of Affine Systems in Stochastically Modeled Contexts

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Abstract: This paper proposes an algorithm for online controller synthesis for autonomous systems with LTI dynamics considering obstacle avoidance. The obstacles are assumed to be other systems with affine probabilistic dynamics. The initial state as well as the disturbances of these systems are Gaussian distributed. To guarantee that the probability of a collision is smaller than a predefined threshold, probabilistic reachable sets are used. Due to the Gaussian distribution, the probabilistic reachability procedure can use the principles of the ellipsoidal calculus. For the autonomous system, these time-varying reachable sets of the other systems are avoided by an approach, which is based on model predictive control and successive convexification of the constraints. Due to high computational times required for the computation of probabilistic reachable sets and the convexification, different techniques to reduce the computational time significantly are also proposed.

Keywords: multi-agent systems, distributed control, uncertain systems, constrained control, networked systems, predictive control.

1. INTRODUCTION

Multi-agent systems are systems composed of several individual subsystems, and typically, these subsystems solve a task collaboratively, see e.g. Tiang and Mahyuddin (2016); Zhang et al. (2018). But for some scenarios, the subsystems have individual objectives, and in this case, an important aspect is to guarantee collision avoidance between the subsystems and with other obstacles. In this paper, we propose a control scheme that is sample-free and predicts states which are probabilistically safe with respect to the states of the other subsystems. Mylvaganam et al. (2017) formulate a similar problem as a differential game with the assumption that the states of the subsystems are deterministic. But in many applications, uncertainties play a crucial role, and hence, the environment cannot be exactly modeled. One way to describe the obstacles is by using reachable sets of the subsystems, as in this paper. In most approaches, the uncertainties are specified in terms of bounded sets, e.g. HomChaudhuri et al. (2017), which is a major restriction, since in most cases the disturbances arise from measurement noise, where the state information is distributed around the true value. In Asselborn and Stursberg (2015); Vinod et al. (2018), the uncertainties are modeled by probability distributions, leading to probabilistic reachable sets with the interpretation that these sets contain only a certain (high) percentage of all reachable states of a system. The probabilistic reachable sets are over-approximated by ellipsoids, motivated by modeling the uncertainties as Gaussian distributions, and the confidence regions of a Gaussian distribution are ellipsoidal sets. But in Asselborn and Stursberg (2015), obstacle avoidance does not play a role, and in Vinod et al. (2018), reachable sets are computed with the assumption that the subsystems (robots) have translational motion only. Furthermore, they are not controlled to obtain probabilistic reachable sets which decrease over time in size, leading to infeasible problems as time advances. An alternatives is chance-constrained approaches, as e.g. in Blackmore et al. (2006, 2011), using disjunctive linear programming to formulate the probabilistic obstacle avoidance problem, or sample-based approaches, as e.g. in Prandini et al. (2012), Chiang et al. (2015). The chance-constrained approaches require polytopic obstacles, and they are computationally expensive. For the sample-based approach, in addition a high number of samples is necessary to achieve the desired confidence, particularly in higher dimensions. In terms of the real-time requirement, the computational time has to be kept as small as possible, and hence, it is necessary to choose an appropriate approach for obstacle avoidance, e.g. such that the optimization problem results in a convex program.

The main contribution of this paper is an online control scheme for a single subsystem/agent within a stochastically modeled environment. The environment is modeled by using probabilistic reachable sets for the other subsystems, since their uncertainties are assumed to be Gaussian distributions. The resulting non-convex obstacle avoidance problem is convexified via successive convexification (SC), as described by Mao et al. (2016); Dueri et al. (2017); Szmuk et al. (2017). A model predictive control (MPC) approach is used to formulate the resulting convex obstacle avoidance problem, and to achieve the desired computational time for online application, different modifications are proposed. In contrast to Vinod et al. (2018), the obstacles are also controlled to predict the behavior of the other subsystems in an optimal manner such that 1) the probabilistic reachable sets do not increase over time, and

2) they steer the subsystems towards their targets.
The paper is organized as follows: Sec. 2 describes the overall problem, introduces notation used afterwards, and it provides a description of the dynamic obstacles. In Sec. 3, the proposed synthesis techniques embedding a step of SC is described, followed in Sec. 4, by techniques to reduce computational time. A numerical example is presented in Sec. 5, and Sec. 6 concludes the paper.

2. PROBLEM STATEMENT AND CONTEXT MODELING

The considered environment is partitioned into $n_o + 1$ subsystems, of which one, called the autonomous system (AS), is to be controlled in context of the n_o others. The context cannot precisely modeled, since the dynamics of the other subsystems includes uncertainties. The AS has to be controlled such that the probability of harmful interaction with the other subsystems is lower than a given threshold. Therefore, bounded sets of uncertain states are computed for the n_o subsystems, which define forbidden regions (obstacles) for the AS, as illustrated in Fig. 1.

Assumption 1. For the AS, only the information of the other subsystems are known at time $\bar{k} \in \mathbb{N}_0$, and there is no communication between the subsystems. Furthermore, the other subsystems have the objective to be controlled, such that they reach their desired target regions.

Assumption 1 should be interpreted as follows: at time instant $\bar{k}$, the AS gets the information about the other subsystems (e.g. the expectation values and the covariance matrices of their states, target regions etc.), and from then on, it is assumed that no communication occurs until the state of the AS reaches its target region (e.g. for some reason, a communication network is not available). Hence, the bounded sets have to be computed online. From now on, assume that $\bar{k} = 0$ for ease of notation.

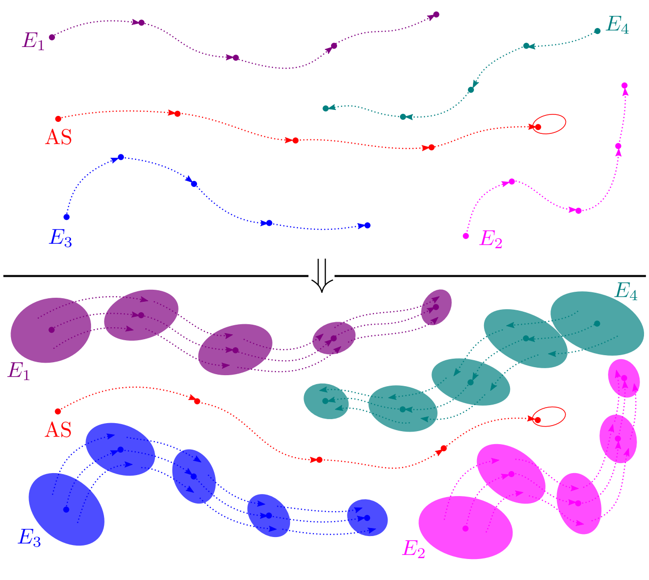

Fig. 1. Sets are computed which contain the uncertain states of the other subsystems (E_1 - E_4) to a specified confidence. These sets are obstacles for the AS, and the objective is to control the AS into the target set (red ellipse) without hitting the obstacles.

A. Preliminaries and System Definition

Let $\mathcal{S}^n_{++}$ be the set of all symmetric positive definite matrices of order $n \times n$, and $\mathcal{S}^n_{+}$ the set of all symmetric positive semi-definite matrices. $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote the minimum and maximum eigenvalues of the matrix M. The set $\mathcal{E}$ contains all ellipsoids in $\mathbb{R}^n$. An ellipsoid is defined by $\varepsilon(q,Q) = \{x \,|\, (x-q)^T Q^{-1}(x-q) \le 1\}$, where $q \in \mathbb{R}^n$ is the center point and $Q \in \mathcal{S}^n_{++}$ the shape matrix. A polytope P in half-space representation and parametrized by A and b is defined by $P_H(A,b) = \{x \,|\, Ax \le b, A \in \mathbb{R}^{n \times n_p}, b \in \mathbb{R}^{n_p}\}$. Given a compact set $\Omega \subset \mathbb{R}^n$, the boundary of this set is denoted by $\partial\Omega$ and its interior by $int(\Omega)$.

The dynamics of the AS is assumed to be discrete-time LTI with state and input constraints:

$$x_{k+1} = Ax_k + Bu_k,\; x_k \in \mathcal{X},\; u_k \in U, \tag{1}$$

where $\mathcal{X}$ and U are polytopic sets. Furthermore, let $\bar{x} \in \mathcal{X}$ be an equilibrium point, for which an input $\bar{u} \in U$ exists.

B. Specification of the Obstacles

In this subsection, the notion of probabilistic reachable sets as defined in Asselborn and Stursberg (2015) are used. This procedure is sketched as it plays an important role in the overall optimization to be presented later. In the following, the quantities for the i-th subsystem are indicated by the superscript (i). The dynamics of each other subsystem $i \in \{1, \dots, n_o\}$ is assumed to be affine probabilistic:

$$x^{(i)}_{k+1} = A^{(i)}x^{(i)}_k + B^{(i)}u^{(i)}_k + G^{(i)}v^{(i)}_k,\; u^{(i)}_k \in U^{(i)}, \tag{2a}$$

$$x^{(i)}_0 \sim \mathcal{N}(q^{(i)}_0, Q^{(i)}_0),\; v^{(i)}_k \sim \mathcal{N}(q^{(i)}_v, Q^{(i)}_v), \tag{2b}$$

where $q^{(i)}_0, q^{(i)}_v \in \mathbb{R}^n$ are the mean vectors of the probabilistic distributions of the initial states and the disturbances, and $Q^{(i)}_0, Q^{(i)}_v \in \mathcal{S}^n_{++}$ are the corresponding covariance matrices. Furthermore, the input is bounded by the polytope $U^{(i)} = P_H(R^{(i)}_u, b^{(i)}_u)$, where $R^{(i)}_u \in \mathbb{R}^{n^{(i)}_u \times m}$ and $b^{(i)}_u \in \mathbb{R}^{n^{(i)}_u}$.

Assumption 2. Let $\bar{x}^{(i)}$ be an equilibrium point for the system (2), i.e., a unique input $\bar{u}^{(i)} \in U^{(i)}$ exists with $\bar{x}^{(i)} = (I - A^{(i)})^{-1}(B^{(i)}\bar{u}^{(i)} + G^{(i)}q^{(i)}_v)$.

Assumption 2 is necessary to ensure the solvability of the following reachability problem. The reachable set $X^{(i)}_{k+1} \subseteq \mathbb{R}^n$ of a system describes the set of all states $x^{(i)}_k$, which are reachable from the previous set $X^{(i)}_k$ for at least one $u^{(i)}_k \in U^{(i)}$. It can be expressed as a set valued mapping:

$$X^{(i)}_{k+1} = A^{(i)}X^{(i)}_k \oplus B^{(i)}U^{(i)} \oplus G^{(i)}V^{(i)}, \tag{3}$$

where $X^{(i)}_k \in \mathcal{E}$ and $V^{(i)} = \varepsilon(q^{(i)}_v, Q^{(i)}_v) \in \mathcal{E}$ are two bounded ellipsoidal sets. Since the Minkowski sum of two ellipsoids is in general not an ellipsoid anymore, the Minkowski sum is over-approximated by an ellipsoid: $X^{(i)}_{k+1} \subseteq \hat{X}^{(i)}_{k+1} = \varepsilon(q^{(i)}_{k+1}, Q^{(i)}_{k+1})$. Due to the stochasticity of the considered subsystems, probabilistic reachable sets are introduced with the interpretation, that a percentage $\delta^{(i)}$ of the possible states for the probabilistic system are contained within these sets. A probabilistic reachable set with confidence δ is denoted by $\hat{X}_k^{\delta^{(i)}}$. Given that only normal distributions are considered, the confidence regions correspond to ellipsoidal sets. A confidence ellipsoid is computed with a scaling parameter $c^{(i)}$:

$$\hat{X}_k^{\delta^{(i)}} = \varepsilon(q_k^{(i)}, Q_k^{\delta^{(i)}}), \quad Q_k^{\delta^{(i)}} = c^{(i)} \cdot Q_k^{(i)}, \tag{4}$$

and $c^{(i)}$ is the solution of the inverse cumulative distribution function of the χ^2-distribution: $c^{(i)} = (F_{\chi^2})^{-1}(\delta^{(i)}, n)$, see Asselborn and Stursberg (2015).

Assuming that the subsystem is under effect of an affine control law $u_k^{(i)} = -K_k^{(i)} x_k^{(i)} + d_k^{(i)}$ in order to drive the i-th subsystem to $\bar{x}^{(i)}$, (3) can be rewritten to:

$$\begin{aligned} X_{k+1}^{\delta^{(i)}} \subseteq\ & \varepsilon(A_{cl,k}^{(i)} q_k^{(i)}, A_{cl,k}^{(i)} Q_k^{\delta^{(i)}} A_{cl,k}^{(i)T}) \oplus \dots \\ & \dots \oplus \varepsilon(G^{(i)} q_v^{(i)}, G^{(i)} Q_v^{\delta^{(i)}} G^{(i)T}) + B^{(i)} d_k^{(i)} \subseteq \hat{X}_{k+1}^{\delta^{(i)}}, \end{aligned} \tag{5}$$

where $A_{cl,k}^{(i)} = A^{(i)} - B^{(i)} K_k^{(i)}$. Now, the over-approximation of the next probabilistic reachable set is computed as follows: $\hat{X}_{k+1}^{\delta^{(i)}} = \varepsilon(q_{k+1}^{(i)}, Q_{k+1}^{\delta^{(i)}}) \supseteq X_{k+1}^{\delta^{(i)}}$, where $q_{k+1}^{(i)} = A_{cl,k}^{(i)} q_k^{(i)} + B^{(i)} d_k^{(i)} + G^{(i)} q_v^{(i)}$, and for $s^{(i)} \in \mathbb{R}^{>0}$: $Q_{k+1}^{\delta^{(i)}} = (1+s^{(i)^{-1}}) A_{cl,k}^{(i)} Q_k^{\delta^{(i)}} A_{cl,k}^{(i)T} + (1+s^{(i)}) G^{(i)} Q_v^{\delta^{(i)}} G^{(i)T}$. For all $s^{(i)} > 0$, $\hat{X}_{k+1}^{\delta^{(i)}}$ is an over-approximation of the Minkowski sum, but there exists a unique $s^{(i)}$, such that $\hat{X}_{k+1}^{\delta^{(i)}}$ becomes a Loewner-John-ellipsoid, Asselborn and Stursberg (2015).

The determination of the control tuple $(K_k^{(i)}, d_k^{(i)})$ to solve the reachability problem is cast into a solution of a semi-definite program (SDP). To guarantee stability, a quadratic time-invariant Lyapunov function $V(q_k^{(i)}) = q_k^{(i)T} M^{(i)} q_k^{(i)}$ is used, where $M^{(i)} \in \mathcal{S}_{++}^n$. The inequality $q_{k+1}^{(i)T} M^{(i)} q_{k+1}^{(i)} - \rho^{(i)} q_k^{(i)T} M^{(i)} q_k^{(i)} \leq 0$ with $\rho^{(i)} \in (0,1]$ has to be fulfilled to ensure stability with respect to the origin. For the convergence of the shape matrix $Q_k^{\delta^{(i)}}$, an over-approximated matrix $S_k^{(i)} \in \mathcal{S}_{++}^n$ is introduced, and it must hold that $\mathrm{tr}(S_{k+1}^{(i)}) \leq \mathrm{tr}(Q_k^{\delta^{(i)}})$ and $S_{k+1} \geq Q_{k+1}^{\delta^{(i)}}$. This can be expressed by the following linear matrix inequality (LMI):

$$\begin{bmatrix} S_{k+1}^{(i)} & A_{cl,k}^{(i)} Q_k^{\delta^{(i)}} & G^{(i)} Q_v^{\delta^{(i)}} \\ Q_k^{\delta^{(i)}} A_{cl,k}^{(i)T} & (1-\nu^{(i)}) Q_k^{\delta^{(i)}} & 0 \\ Q_v^{\delta^{(i)}} G^{(i)T} & 0 & \nu^{(i)} Q_v^{\delta^{(i)}} \end{bmatrix} \geq 0, \quad \nu^{(i)} \in (0,1), \tag{6}$$

with $\nu^{(i)} = 1/(1+s^{(i)})$. To ensure, that the input constraints are satisfied in each time step, the control law is inserted to the half-space representation of the polytope $U^{(i)}$:

$$R_u^{(i)}(-K_k^{(i)} x_k^{(i)} + d_k^{(i)}) \leq b_u^{(i)},\ \forall x_k^{(i)} \in \hat{X}_k^{\delta^{(i)}}. \tag{7}$$

By row-wise maximization, the condition (7) can be expressed as:

$$\max_{w^{(i)} \in W^{(i)}} r_{u,p}^{(i)} w \leq b_{u,p}^{(i)}, \quad \forall p^{(i)} \in \{1, \dots, n_u^{(i)}\}, \tag{8a}$$

$$W^{(i)} = \{w^{(i)} \in \mathbb{R}^m \mid w^{(i)} = -K_k^{(i)} x_k^{(i)} + d_k^{(i)},\ x_k^{(i)} \in X_k^{(i)}\}. \tag{8b}$$

Here, $r_{u,p}^{(i)}$ is the p-th row of $R_u^{(i)}$, and $b_{u,p}^{(i)}$ the corresponding $p^{(i)}$-th entry of the vector $b_u^{(i)}$. The ellipsoid $\hat{X}_k^{\delta^{(i)}}$ is now transformed into a unit hyper-ball by $\theta^{(i)} = Q_k^{\delta^{(i)^{-1/2}}}(x_k^{(i)} - q_k^{(i)})$, $||\theta^{(i)}||_2 \leq 1$. Now, the set (8b) can be expressed as:

$$W^{(i)} = \{w^{(i)} \mid w^{(i)} = -K_k^{(i)}(Q_k^{\delta^{(i)^{1/2}}} \theta^{(i)} + q_k^{(i)}) + d_k^{(i)}\}. \tag{9}$$

By use of (9), the maximization can be eliminated from (8a):

$$\begin{aligned} \max_{\substack{\theta^{(i)} \\ ||\theta^{(i)}||_2 \leq 1}} \ & r_{u,p}^{(i)}(-K_k^{(i)} Q_k^{\delta^{(i)^{1/2}}} \theta^{(i)}) - r_{u,p}^{(i)} K_k^{(i)} q_k^{(i)} + \dots \\ & \dots + r_{u,p}^{(i)} d_k^{(i)} \leq b_{u,p}^{(i)}, \end{aligned} \tag{10a}$$

$$|| - r_{u,p}^{(i)} K_k^{(i)} Q_k^{\delta^{(i)^{1/2}}} ||_2 \leq b_{u,p}^{(i)} - r_{u,p}^{(i)}(d_k^{(i)} - K_k^{(i)} q_k^{(i)}). \tag{10b}$$

Finally, the inequality (10b) can be expressed by the following LMI:

$$\begin{bmatrix} \varphi_{u,p,k}^{(i)} & -r_{u,p}^{(i)} K_k^{(i)} Q_k^{\delta^{(i)^{1/2}}} \\ (-r_{u,p}^{(i)} K_k^{(i)} Q_k^{\delta^{(i)^{1/2}}})^T & \varphi_{u,p,k}^{(i)} I_n \end{bmatrix} \geq 0, \quad \forall p^{(i)} \in \{1, \dots, n_u^{(i)}\}, \tag{11}$$

where $\varphi_{u,p,k}^{(i)} = b_{u,p}^{(i)} - r_{u,p}^{(i)}(d_k^{(i)} - K_k^{(i)} q_k^{(i)})$. The derived constraints together with an appropriate objective function yields the following optimization to determine a control strategy for the i-th subsystem:

$$\min_{S_k^{(i)}, K_k^{(i)}, d_k^{(i)}, \nu^{(i)}} \mu_0^{(i)}\, \mathrm{tr}(S_{k+1}^{(i)}) + \mu_1^{(i)} ||q_{k+1}^{(i)}||_2 + \mu_2^{(i)} ||u_k^{(i)}||_2, \tag{12a}$$

$$\text{s.t.:} \quad q_{k+1}^{(i)} = A_{cl,k}^{(i)} q_k^{(i)} + B^{(i)} d_k^{(i)} + G^{(i)} q_v^{(i)}, \tag{12b}$$

$$q_{k+1}^{(i)T} M^{(i)} q_{k+1}^{(i)} - \rho^{(i)} q_k^{(i)T} M^{(i)} q_k^{(i)} \leq 0, \tag{12c}$$

$$\mathrm{tr}(S_{k+1}^{(i)}) \leq \mathrm{tr}(Q_k^{\delta^{(i)}}), \tag{12d}$$

$$(6) \text{ and } (11)\ \forall p^{(i)} \in \{1, \dots, n_u^{(i)}\}. \tag{12e}$$

The weights μ_0, μ_1, and μ_2 are user-specific degrees of freedom and can be used to adjust priorities for the size of the over-approximated ellipsoid, the shift of the center point to the origin, and the magnitude of the input. The solution provides a control tuple $(K_k^{(i)}, d_k^{(i)})$ to compute the next over-approximated probabilistic reachable set. If this is repeated for each subsystem over a certain time horizon, the computed reachable sets determine the obstacles for the AS. Compared to Asselborn and Stursberg (2015), the reachable sets are executed online in this paper.

C. Control Problem for the AS

The objective is to find an input sequence $\bar{\pi}$, which steers the AS into a target region within a finite time N, such that no collision with any obstacle occurs:

Problem 1. Given is the AS (1) with an initial state x_0 and a target set $\mathbb{T} \subset \mathbb{R}^n$ with $\bar{x} \in \mathbb{T}$. Furthermore, let $\mathcal{O}_k^{\delta} = \{X_k^{\delta^{(1)}}, \dots, X_k^{\delta^{(n_o)}}\}$ be a set of obstacles, with states modeled as random variables as in the previous subsection. Find a control sequence $\bar{\pi} = \{u_0, u_1, \dots, u_{N-1}\}$ for the AS (1) such that:

- the solution fulfills the difference equations, i.e. $x_{k+1} = Ax_k + Bu_k\ \forall k \in \{0, 1, \dots, N-1\}$,
- a finite $N \in \mathbb{N}_0$ is determined (if existing) such that $x_N \in \mathbb{T}$ holds,

- the probability of collision avoidance with obstacle i is at least $\delta^{(i)}$, i.e. $x_k \notin X_k^{\delta^{(i)}}$ with $Pr(x_k^{(i)} \in X_k^{\delta^{(i)}}) = \delta^{(i)}$, $\forall k \in \{1, \dots, N\}$ and $\forall i \in \{1, \dots, n_o\}$,
- the input and the state constraints are satisfied, i.e. $u_k \in U$ and $x_k \in \mathcal{X}$ $\forall k \in \{0, 1, \dots, N-1\}$.

For solving this problem online, an MPC scheme with a quadratic objective function is chosen:

$$\min_{u_{k|k},\dots,u_{k+H-1|k}} ||x_{k+H|k} - x_{ref}||_P^2 + \sum_{j=0}^{H-1} ||x_{k+j|k} - x_{ref}||_{Q_x}^2 + ||u_{k+j|k}||_R^2 \tag{13a}$$

$$\text{s.t.:} \quad x_{k+j|k} = Ax_{k+j-1|k} + Bu_{k+j-1|k}, \tag{13b}$$

$$x_{k+j|k} \in \mathcal{X}, \quad u_{k+j-1|k} \in U, \tag{13c}$$

$$x_{k+j|k} \notin X_{k+j|k}^{\delta^{(i)}} \text{ with } Pr(x_{k+j|k}^{(i)} \in X_{k+j|k}^{\delta^{(i)}}) = \delta^{(i)}, \quad \forall i \in \{i, \dots, n_o\}. \tag{13d}$$

Here, $Q_x \in \mathcal{S}_+^n$, $R \in \mathcal{S}_{++}^n$ are weighting matrices, $P \in \mathcal{S}_+^n$ is the unique solution to the discrete algebraic Riccati equation for the infinite horizon unconstrained optimal control problem, and H the prediction horizon. Due to the non-convexity of the constraints (13d), the optimization problem is non-convex. In this paper, only convex optimization is considered to reduce the computational time, and to guarantee an optimal solution. Hence, the constraints (13d) are convexified such that the optimization problem (13) results in a substitute quadratic program (QP).

3. ALGORITHM FOR ONLINE CONTROL OF AS

The first objective is to convert the non-convex optimization problem into a QP to be able to solve the problem fast (thus to meet real-time requirements), and to guarantee a global solution.

A. Successive Convexification

To convexify the non-convex constraints (13d), the method of successive convexification (SC) according to Mao et al. (2016, 2017) and Vinod et al. (2018) is tailored to the problem at hand. These constraints can be reformulated into:

$$(x_{k+j|k} - q_{k+j|k}^{(i)})^T \left(Q_{k+j|k}^{\delta^{(i)}}\right)^{-1} (x_{k+j|k} - q_{k+j|k}^{(i)}) - 1 \geq 0. \tag{14}$$

Let $\phi_{k+j|k}^{(i)} : \mathbb{R}^n \to \mathbb{R}$ be a convex quadratic function of the following form:

$$\phi_{k+j|k}^{(i)}(z) := (z - q_{k+j|k}^{(i)})^T \left(Q_{k+j|k}^{\delta^{(i)}}\right)^{-1} (z - q_{k+j|k}^{(i)}) - 1. \tag{15}$$

Obviously, $\phi_{k+j|k}^{(i)}(z)$ is continuously differentiable and describes the left hand side of (14) for $z = x_{k+j|k}$. Let further $\mathcal{M}_{k+j|k} = \{x \in \mathcal{X} \mid x \notin \varepsilon(q_{k+j|k}^{(i)}, Q_{k+j|k}^{\delta^{(i)}})\ \forall i \in \{1, \dots, n_o\}\}$ be the time-varying set for which the non-convex constraint (13d) is valid. For every $x \in \mathcal{M}_{k+j|k}$ and every set $\hat{X}_{k+j|k}^{\delta^{(i)}}$, there exists a unique point $z \in \hat{X}_{k+j|k}^{\delta^{(i)}}$, such that $||x - z||_2$ is minimized over $\hat{X}_{k+j|k}^{\delta^{(i)}}$:

$$z_{k+j|k}^{*^{(i)}} = \arg\min_z ||x - z||_2 \quad \text{s.t.: } z \in \hat{X}_{k+j|k}^{\delta^{(i)}}. \tag{16}$$

The solution $z_{k+j|k}^{*^{(i)}}$ is called the projection of x onto $\hat{X}_{k+j|k}^{\delta^{(i)}}$. Now, the functions $\phi_{k+j|k}^{(i)}(z)$ are linearly approximated in $z_{k+j|k}^{*^{(i)}}$:

$$t_{k+j|k}^{(i)}(z, x) := \nabla_z^T \phi_{k+j|k}^{(i)}(z_{k+j|k}^{*^{(i)}})(z - z_{k+j|k}^{*^{(i)}}). \tag{17}$$

Note that the term of zero-th order is omitted, since the projection point lies always on the boundary of $\hat{X}_{k+j|k}^{\delta^{(i)}}$, and it holds that $\phi_{k+j|k}^{(i)}(z_{k+j|k}^{*^{(i)}}) = 0$. A feasible convex set is now given by the following condition:

$$t_{k+j|k}^{(i)}(z, x) \geq 0 \,\big|_{z = x_{k+j|k}}, \tag{18a}$$

$$\underbrace{-\nabla_z^T \phi_{k+j|k}^{(i)}(z_{k+j|k}^{*^{(i)}})}_{=:C_{k+j|k}^{(i)T}} x_{k+j|k} \leq \underbrace{-\nabla_z^T \phi_{k+j|k}^{(i)}(z_{k+j|k}^{*^{(i)}}) z_{k+j|k}^{*^{(i)}}}_{=:b_{k+j|k}^{(i)}}. \tag{18b}$$

The vector x which is needed to compute the projection point in (16), is now replaced by the predicted states of the previous time step $x_{k+j|k-1}$. For the first time step, no predicted states are available, and to handle this problem, (13) is solved without considering the non-convex constraint (13d). Now, for every predicted state over the entire horizon, it is checked whether there exists a point which is contained within the corresponding obstacle. If true, then the nearest point on the boundary of that obstacle is taken by solving the following SDP:

$$\min_{x_{k+j|0,new}} ||x_{k+j|0,old} - x_{k+j|0,new}||_2 \tag{19a}$$

$$\text{s.t.:} \quad x_{k+j|0,new} \in \partial\varepsilon(q_{k+j|0}^{(i)}, Q_{k+j|0}^{\delta^{(i)}}). \tag{19b}$$

This procedure constructs a polytopic subset as new constraint dividing the feasible set from the obstacles.

B. MPC Algorithm Based on SC and Reachable Set Computation

The non-convex optimization (13) is now converted into a convex one:

$$\min_{u_{k|k},\dots,u_{k+H-1|k}} \text{(13a)} \tag{20a}$$

$$\text{s.t.:} \quad \text{(13b), (13c) and (18b)} \ \forall i \in \{1, \dots, n_o\}. \tag{20b}$$

The objective function is a common quadratic function of the input and state for an MPC. The first constraint (13b) ensures that the solution satisfies the system equation, and the second one (13c) ensures the satisfaction of the state and input constraints. The constraints (18b) guarantee that no collision with any other subsystem occurs with a predefined probability. These constraints define time-varying (but not necessarily closed) polytopic sets. To compute these polytopic sets, the SDP (16) has to be solved for every $i \in \{1, \dots, n_o\}$ and over the entire horizon. In this optimization, the previous predicted states are included, which means that the predicted states have to be stored for the next time step. The constraints of this optimization make use of the probabilistic reachable sets

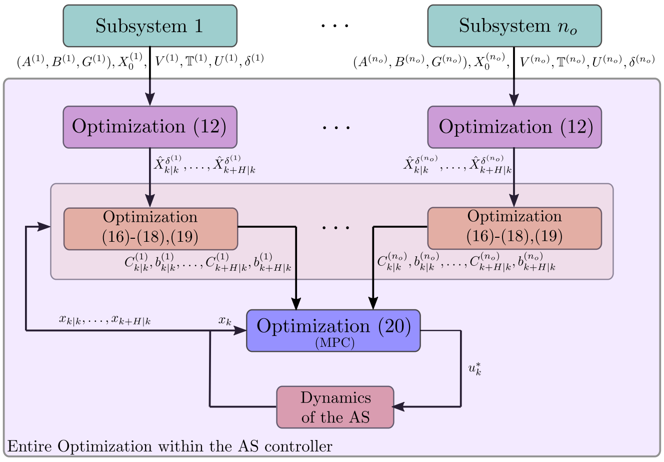

Fig. 2. Flowchart of the complete optimization procedure.

of the i-th subsystem. To get them, another SDP (12) has to be solved over the horizon. Fig. 2 illustrates the entire optimization procedure as a flowchart. The information of each subsystem is known at time $\bar{k} = 0$ with uncertainty, because of Assumption 1. First, the optimization (12) computes the reachable sets for each subsystem, defining the obstacles for the AS. Afterwards, the optimization (16)-(18) convexifies the originally non-convex feasible set into a convex polytope for each point of time, which is used in the optimization (20). The AS optimizes over the horizon up to the goal in each time step. Algorithm 1 solves the Problem 1 iteratively and terminates successfully, if the state of the system (1) reaches the target region eventually.

Algorithm 1 MPC Algorithm with Obstacle Avoidance

Require: System (1), x_0, $\mathbb{T}$, H, Q_x, R, P, k_{max}, and a set of obstacles $\mathcal{O}_0^\delta = \{X_0^{\delta^{(1)}}, \ldots, X_0^{\delta^{(n_o)}}\}$ with $\mathbb{T}^{(i)}$, $\delta^{(i)}$, $\rho^{(i)}$, $\mu_0^{(i)}$, $\mu_1^{(i)}$, $\mu_2^{(i)}$, $M^{(i)}$, and $U^{(i)}$, $i \in \{1, \ldots, n_o\}$.

```
k := 0
while x_k ∉ T and k ≤ k_max do
    for every i and over the horizon do
        solve the SDP (12)
    end for
    if k = 0 then
        solve (13) without the constraints (13d)
        if x_{k+j|k} ∈ int(ε(q^{(i)}_{k+j|k}, Q^{δ^{(i)}}_{k+j|k})) then
            solve (19)
        end if
    end if
    for every i and over the horizon do
        solve (16) and compute C^{(i)}_{k+j|k}, b^{(i)}_{k+j|k}
    end for
    solve the QP (20)
    if no feasible solution is found then
        stop algorithm (synthesis failed)
    end if
    compute the next state & store the predicted states
    k ← k + 1
end while
```

Theorem 1. If Algorithm 1 terminates with $x_N \in \mathbb{T}$, Problem 1 is successfully solved and an input sequence exists, which transfers the initial state x_0 into a target region $\mathbb{T}$ in N steps. Furthermore, the probability of collision avoidance with the i-th subsystem is greater than $\delta^{(i)}$ for all $k \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, n_o\}$.

Proof. Successful termination of Algorithm 1 implies that $x_N \in \mathbb{T}$. Furthermore, the true reachable set of the i-th subsystem at each time step is over-approximated by the ellipsoid $\hat{X}_k^{\delta^{(i)}}$, and it holds that $Pr(x_k^{(i)} \in X_k^{\delta^{(i)}}) = \delta^{(i)}$. It follows, that the state of the i-th subsystem is contained within the reachable set $\hat{X}_k^{\delta^{(i)}}$ with a probability greater or equal than $\delta^{(i)}$ for all $k \in \{1, \ldots, N\}$. Furthermore, let $\mathcal{F}_{k+j|k} = \{x \in \mathbb{R}^n \mid C_{k+j|k}^{(i)^T} x \le b_{k+j|k}^{(i)},\ \forall i \in \{1, \ldots, n_o\}\}$ be the time-varying convex polytope at time $k+j$ predicted at time k after the SC procedure. Since the projection point $z_{k+j|k}^{*^{(i)}}$ is the closest point to the previous predicted state $x_{k+j|k-1}$, the vector $(x_{k+j|k-1} - z_{k+j|k}^{*^{(i)}})$ is the normal vector of the corresponding i-th ellipsoid at point $z_{k+j|k}^{*^{(i)}}$. Now, if the inner product of the normal vector and $(x_{k+j|k} - z_{k+j|k}^{*^{(i)}})$ for all $i \in \{1, \ldots, n_o\}$ is non-negative, as ensured by the convex constraint (18), it follows that $x_{k+j|k} \in \mathcal{F}_{k+j|k}$. According to Sec. 2, every obstacle is represented by a convex set. Thus, it can be expressed by the convex function $\phi_{k+j|k}^{(i)}(z)$, and for all $z \in \mathcal{F}_{k+j|k}$ holds:

$$\phi_{k+j|k}^{(i)}(z) \ge t_{k+j|k}^{(i)}(z, x_{k+j|k-1}),\ \forall i \in \{1, \ldots, n_o\}. \quad (21)$$

Thus,

$$\phi_{k+j|k}^{(i)}(z) \ge 0, \quad \forall i \in \{1, \ldots, n_o\}, \quad (22)$$

which implies that $z \in \mathcal{M}_{k+j|k}$ and hence, $\mathcal{F}_{k+j|k} \subseteq \mathcal{M}_{k+j|k}$ and $x_{k+j|k} \notin \varepsilon(q_{k+j|k}^{(i)}, Q_{k+j|k}^{\delta^{(i)}}) = X_{k+j|k}^{\delta^{(i)}}$, $\forall i \in \{1, \ldots, n_o\}$ and $k \in \{1, \ldots, N\}$. □

In general, feasibility of Algorithm 1 cannot be provided, even with additional assumptions/conditions (e.g. only a small number of subsystems or a small δ). But Theorem 1 ensures that the solution is collision-free in a probabilistic manner if a feasible solution is found.

4. MEANS TO IMPROVE COMPUTATIONAL EFFICIENCY

In order to be able to run the procedure in real time, modifications of Algorithm 1 are made to improve computational efficiency: Once the reachable sets are computed, they are stored for the next time step, such that only the reachable set at predicted time $k + H$ has to be computed in every iteration, except of the first one. In the first time step, the reachable sets are computed over the entire horizon. Because of Assumption 1, the optimizations (12) for the subsystems can be accomplished efficiently by *parallel programming*. The optimization of each subsystem is assigned to one computational node, assuming that as many computational nodes are available as subsystems. Furthermore, only relevant parts of the context are considered: First, (13) is solved without the non-convex constraints (13d). Then, it is checked whether a predicted state is contained within an obstacle. If not, the next state of the AS is computed directly, and the predicted states are stored for the next time step. Secondly, obstacles are neglected, if they are far away, or if they move away from the path of the AS. In the first case, the obstacles are omitted temporarily in the convexification procedure, if the following condition is valid:

$$||x_{k+1} - q_{k+1}^{(i)}||_2 > \sqrt{\lambda_{max}(Q_{k+1}^{\delta^{(i)}})} + \dots \\ \dots + ||q_{k+1}^{(i)} - q_k^{(i)}||_2 + \max_{u_k} ||x_{k+1} - x_k||_2. \quad (23)$$

In the second case, the optimization of the reachable sets is omitted with the additional assumption, that the target regions of the subsystems are static. This criterion is expressed by:

$$||x_{k+j|k-1} - q_{k+j}^{(i)}||_2 \geq ||x_{k+j-1|k-1} - q_{k+j-1}^{(i)}||_2, \quad (24a)$$

$$||x_{k+j|k-1} - q_T^{(i)}||_2 \geq ||x_{k+j-1|k-1} - q_T^{(i)}||_2, \quad (24b)$$

$$\forall j \in \{1, \dots, H\}.$$

Note that once an obstacle is eliminated, the optimization does not consider this obstacle anymore for the remaining time.

Thirdly, the ellipsoids can be over-approximated by hyperspheres:

$$\frac{\lambda_{min}(Q_{k+j|k}^{\delta^{(i)}})}{\lambda_{max}(Q_{k+j|k}^{\delta^{(i)}})} > \alpha, \quad 0 < \alpha \leq 1. \quad (25)$$

The advantage is now that the projection point can be computed analytically, and thus, (16) can be omitted. The parameter α establishes a compromise between computational time and optimality of the solution.

The concept of a dynamic horizon, see Liu and Guan (2011), to reduce the computational time further is applied in addition. The idea is to adapt the horizon to the local environment: For a complicated environment with several obstacles, a large horizon is particularly advantageous, since a feasible path can be planned early, while a short horizon could lead to infeasibility (e.g. if the AS steers to a dead end). For a simple environment (e.g. obstacles moving in parallel to AS), a short horizon may be sufficient. To detect which case applies, the norms of the difference between the center point of an obstacle and the state of the AS may be compared.

5. NUMERICAL EXAMPLE

To illustrate the capability of the Algorithm 1, a numerical example is used. The chosen AS has four states, while the other subsystems/obstacles have only two. Only the first two states of the AS are considered to avoid the obstacles (interpreted e.g. as the position coordinates of a vehicle which should not coincide with an obstacle). The dynamics of the AS is given by:

$$A = \begin{bmatrix} 1 & 0 & 0.15 & 0 \\ 0 & 1 & 0 & 0.15 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = 10^{-2} \begin{bmatrix} 1.125 & 0 \\ 0 & 1.125 \\ 15 & 0 \\ 0 & 15 \end{bmatrix},$$

with initial state $x_0 = [-5\ 45\ 0\ 0]^T$, and target set $\mathbb{T} = \varepsilon([-8\ -15\ 0\ 0]^T, I)$. The inputs are bounded to $-25 \leq u_{1,k} \leq 25$, $-25 \leq u_{2,k} \leq 25$. The horizon is chosen to $H = 20$. The dynamics of the other subsystems are modeled arbitrarily as:

$$A^{(1)} = A^{(2)} = \frac{1}{10} \begin{bmatrix} 9.22 & 0.19 \\ -0.58 & 10.4 \end{bmatrix}, A^{(3)} = \frac{1}{10} \begin{bmatrix} 9.2 & -0.31 \\ 0.72 & 9.79 \end{bmatrix},$$

$$B^{(1)} = B^{(2)} = \frac{1}{10} \begin{bmatrix} 1.96 & 0.02 \\ 4.02 & 2.04 \end{bmatrix}, B^{(3)} = \frac{1}{10} \begin{bmatrix} 2.12 & -1.7 \\ 0.4 & 2.5 \end{bmatrix},$$

$$G^{(1)} = G^{(2)} = G^{(3)} = \frac{1}{10} \begin{bmatrix} 1 & 0.5 \\ 0.8 & 2 \end{bmatrix},$$

with the following information at $\bar{k} = 0$:

$$q_0^{(1)} = \begin{bmatrix} -11 \\ 50 \end{bmatrix}, q_0^{(2)} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}, q_0^{(3)} = \begin{bmatrix} -20 \\ 10 \end{bmatrix},$$

and $Q_0^{(i)} = I, i \in \{1, 2, 3\}$. The disturbances are assumed to be white noise with the following covariance matrices:

$$Q_v^{(i)} = 10^{-2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \ i \in \{1, 2, 3\}.$$

The target sets are given by:

$$\mathbb{T}^{(i)} = \varepsilon(q_T^{(i)}, Q_T), \ Q_T = \begin{bmatrix} 0.96 & 0.64 \\ 0.64 & 0.8 \end{bmatrix},$$

with $q_T^{(1)} = [0\ 0]^T$, $q_T^{(2)} = [-5\ -10]^T$, $q_T^{(3)} = [2\ 30]^T$. The remaining parameters are chosen for $M^{(i)} = 10^{-3}I$, $\rho^{(i)} = 1$, $\mu_0^{(i)} = 1$, $\mu_2^{(i)} = 0\ \forall i = \{1, 2, 3\}$, and $\mu_1^{(1)} = \mu_1^{(2)} = 5$, $\mu_1^{(3)} = 10$. The probabilities of the confidence ellipsoids are selected to: $\delta^{(1)} = \delta^{(2)} = 99\,\%$, $\delta^{(3)} = 95\,\%$.

By applying the proposed control scheme, the system is transferred into the target region in $N = 28$ time steps with a total time of 149.07 sec, and an average time per iteration of 5.32 sec. Fig. 3 shows the results exemplarily at certain time instants.

By implementation of the presented techniques for reducing the computational effort according to Sec. 4 with a dynamic horizon with lower bound $H_{min} = 7$, the average computational time is reduced to 286.4 ms per iteration. Tab. 1 shows the time and call reduction of every optimization. In particular, the optimization (16) is less often called, which results in a significant time reduction. Since the reduction techniques relate to the optimization for the obstacles and convexification, the convergence of the AS to the target $\mathbb{T}$ remains assured.

Table 1. Total computational times and optimizations for the chosen example: 1) without time reduction techniques (TRT), and 2) with TRT.

	without TRT		with TRT	
Algorithm	total time	calls	total time	calls
Optimization (12)	13.86 sec	141	1.92 sec	71
Optimization (16)	124.13 sec	1680	1.88 sec	90
Optimization (20)	10.38 sec	28	2.62 sec	31

6. CONCLUSION

An MPC-like synthesis procedure based on obstacle avoidance has been proposed, which computes a feasible input sequence for the AS while guaranteeing obstacle avoidance in a probabilistic manner. Due to the stochasticity of the other systems, probabilistic reachable sets were used to describe the dynamic obstacles. The non-convex optimal control problem was then convexified via successive convexification. Since the computational time was initially high, different techniques have been presented to reduce the required time. While the computational time

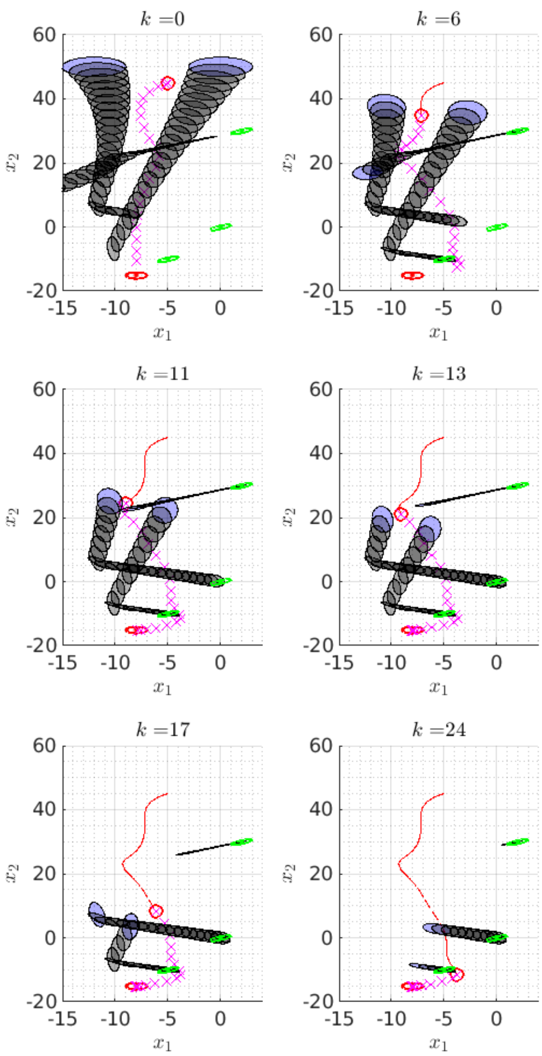

Fig. 3. Control result for the chosen example: The little red circle represents the state at time k, and the ellipse with the same color the target region $\mathbb{T}$ of the AS. The magenta crosses are the predicted states. The blue ellipses are the reachable sets of the obstacles in k, and the grey ones the corresponding predicted reachable sets. The target sets of the obstacles are the green ellipses.

is significantly reduced, the guarantee of solvability of the overall problem may be lost due to: 1) a bad choice of the parameterization (e.g. δ close to 1), 2) too many agents/subsystems are operating in the same environment, or 3) the AS steers towards a dead end.

Future research will consider additional uncertainties for the autonomous system. Also, the investigation of the feasibility of (20), and the time efficiency of the convexification methods are possible points for improvement.

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